# A syntactic approach to unification in transitive reflexive modal logics

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June 10, 2016

**Abstract**

This paper contains a proof-theoretic account of unification in transitive reflexive modal logics, which means that the reasoning is syntactic and uses as little semantics as possible. New proofs of theorems on unification types are presented and these results are extended to negationless fragments. In particular, a syntactic proof of Ghilardi's result that $\mathsf{S4}$ has finitary unification is provided. In this approach the relation between classical valuations, projective unifiers and admissible rules is clarified.

*Keywords:* unification, admissible rules, modal logic, valuations, fragments.

*MSC:* 03F03, 03B70, 03B45.

## 1 Introduction

When restricted to propositional logic, unification theory is concerned with the problem whether a given formula can become derivable under a substitution. In general, a unification problem asks for the unifier of a pair of terms, or collection of pairs of terms, which in the context of a logic is a substitution under which two formulas become equivalent in the logic. This, however, can be reformulated as the problem of finding a substitution under which a formula becomes derivable. Such substitutions are called the *unifiers* of a formula.

In classical propositional logic every consistent formula has a unifier, because every satisfying valuation corresponds to a *ground unifier* that replaces the atoms in the formula by $\top$ or $\bot$. A substitution is a *maximal unifier (mu)* of a formula if among the unifiers of the formula it is maximal in the following ordering:

$$\tau \leqslant \sigma \equiv_{def} \exists \pi (\tau =_{\mathsf{L}} \pi\sigma),$$

and it is a *most general unifier (mgu)* if it is also unique modulo $=$, which is the intersection of $\leqslant$ and $\geqslant$. Here $=_{\mathsf{L}}$ is the equivalence relation on substitutions

*Support by the Netherlands Organisation for Scientific Research under grant 639.032.918 is gratefully acknowledged

associated with the logic: $\sigma =_{\mathsf{L}} \tau$ if and only if $\sigma(p) \leftrightarrow \tau(p)$ is derivable for all atoms $p$. If $\tau \leqslant \sigma$ we say that $\tau$ is *less general* than $\sigma$.

Mgus generate all unifiers of a formula, which is the reason that they play an important role in unification theory. In classical propositional logic every unifiable formula has a mgu, but this no longer holds for intermediate and modal logics, as was first observed by Ghilardi [8, 9]. For modal logics, which will be the logics this paper is concerned with, the formula $\Box p \vee \Box \neg p$ is an example of a formula that has two unifiers such that neither one is less general than the other, namely $\sigma_0(p) = \top$ and $\sigma_1(p) = \bot$. Thus this formula has no mgu. But, as Ghilardi showed in [9], for many transitive modal logics, something almost as good holds: instead of unitary unification these logics have finitary unification, which is defined as follows.

A *complete* set of unifiers for a formula is a set of unifiers such that every unifier of the formula is less general than a unifier in the set. It is *minimal* if no two unifiers in the set are comparable with respect to $\leqslant$. A logic has *unification type*

- *unitary* if every unifiable formula has a mgu,
- *finitary* if every unifiable formula has a finite complete set of mus,
- *infinitary* if every unifiable formula has a (in)finite complete set of mus,
- *nullary* if none of the above.

The classes are meant to be disjunct. For example, in a logic of unification type infinitary there exists at least one formula that has no finite complete set of mus. As was mentioned above, classical logic has unitary unification type, and several transitive modal logics, including the well-known logics $\mathsf{K4}$, $\mathsf{S4}$, and $\mathsf{GL}$, have finitary unification. For example, in the example above $\{\sigma_0, \sigma_1\}$ is a finite complete set of mus for $\Box p \vee \Box \neg p$ in $\mathsf{K4}$, $\mathsf{S4}$, as well as $\mathsf{GL}$.

In this paper we extend these results to the negationless fragment of $\mathsf{S4}$. However, our aim is not so much to extend Ghilardi's results to this fragment, an extension that is not terribly interesting and might have been obtained from existing work on $\mathsf{S4}$ anyway, but rather to give a proof-theoretic analysis of unification in transitive modal logics.

Let us first describe how Ghilardi proves that $\mathsf{S4}$ and several other modal logics have finitary unification. In [9] it is shown that if $A$ satisfies a certain semantical property (the extension property), it has a mgu. Then it is proved that for every formula $A$ there exists a finite set of formulas with the extension property, forming the *projective approximation* $\Pi_A$ of $A$, such that every unifier of $A$ is less general than one of the mgus of the formulas in $\Pi_A$. These two theorems then establish the finitary unification of $\mathsf{S4}$.

Ghilardi uses semantics in the form of Kripke models to prove these theorems (in fact, his results stem from a categorical approach to unification in logic). Our Theorems 1 and 3 and Lemma 8 can be viewed as proof-theoretic analogues of these theorems. They provide a syntactic closure condition on formulas which is sufficient for having a mgu. And they show that in $\mathsf{S4}$ and its negationless fragment, there is for every formula $A$ a finite set of formulas that satisfy the closure condition and such that every unifier of $A$ is less general than one of the mgus of the formulas in that set. Observe that this indeed proves that these logics have finitary unification (Corollary 2).

Besides providing a proof-theoretic treatment of unification, another aim is to clarify the relation between unifiers and valuations. The mgus that play an important role in unification in modal logic often are *projective*, where a unifier $\sigma$ of a formula $A$ is called projective if $A \vdash \sigma(p) \leftrightarrow p$ for all atoms $p$, that is, if $A$ implies that the substitution is the identity. The projective unifiers that Ghilardi introduced in [9] are compositions of substitutions of the form

$$\sigma_I(p) \equiv_{def} \begin{cases} A \wedge p & \text{if } p \notin I \\ A \to p & \text{if } p \in I, \end{cases}$$

where $I$ is a set of atoms. It is not difficult to see that $\sigma_I$ is a projective unifier of $A$ in classical propositional logic whenever $A$ is valid under the valuation

$$v_I(p) \equiv_{def} \begin{cases} 0 & \text{if } p \notin I \\ 1 & \text{if } p \in I. \end{cases}$$

One could view Theorem 1 below as an analogue of this fact for modal logic.

At the end of the paper we apply these results to admissible rules, which are the rules under which a logic is closed. Jeřábek proved in [19] that the modal rules $\mathsf{V}^\circ$ (definition in Section 6) form a basis for the admissible rules of any extension of $\mathsf{S4}$ in which they are admissible. In Theorem 4 we show that this result can be obtained via syntactic methods as well and extend it to the negationless fragment of $\mathsf{S4}$.

The restriction in this paper to reflexive logics is, we think, not essential for a proof-theoretic treatment of unification, but it seems to simplify the reasoning at some points, and we therefore leave the general case (fragments of $\mathsf{K4}$) for future work.

Finally, let us briefly discuss other work on unification and admissible rules in modal logic. We restrict ourselves to the results that are relevant for this paper, and will therefore not discuss intermediate logics or multimodal logics. Rybakov was the first to prove the decidability of admissibility for various modal logics, including $\mathsf{S4}$. Chagrov constructed a decidable modal logic which admissibility problem is undecidable [2], and Wolter and Zakharyaschev did the same for the unification problem [28]. As mentioned above, Ghilardi introduced the notion of projectivity for formulas and unifiers, proved that various modal and intermediate logics have finitary unification and showed that projective approximations can be found effectively [9]. The latter also holds for the irreducible projective approximations from [14] that we use in this paper. Ghilardi also provided an elegant algorithm for deciding admissibility of several modal logics. Jeřábek in [19] gave a basis for the admissible rules of various modal logics, including $\mathsf{S4}$. In [20] he showed that the admissibility problem of $\mathsf{S4}$ and various other logics is *coNEXP*-complete. Iemhoff and Metcalfe in [14, 15] developed proof systems for admissibility for K4, S4, and GL.

Dzik in several papers studied the lattice of transitive reflexive modal logics. In [6] he showed that one can split the lattice in two parts in such a way that one part, those logics that contain S4.2, contains all extensions of S4 that have unitary unification, and that the other part contains all extensions of S4 that have finitary unification. Dzik and Wojtylak showed in [7] that every logic containing S4 has projective unification if and only if it contains S4.3, where a logic has *projective unification* if every unifiable formula has a projective unifier. In the same paper they also showed that among the extensions of S4.3 those that are extensions of S4.1 are exactly those that are structurally complete.

The above provides but a short description of some of the literature on unification in modal logic. For further references, see [1].

The inspiration for this paper is the proof-theoretic approach to unification in intuitionistic logic as developed by Rozière in [26]. In [16] we have extended these results to intermediate logics. I thank Emil Jeřábek, George Metcalfe, and Paul Rozière for helpful remarks along the way, and an anonymous referee for many comments that helped improve the paper.

## 2 The logics

The logics we consider are normal transitive modal logics that contain S4, as well as the negationless fragments of such logics, which means those fragments that do not contain $\bot$ and $\neg$ but do contain all other connectives. The results in this paper are proved for the full logics, but the extension to the negationless fragments is straightforward: inspection of the proofs shows that only implication and conjunction are explicitly used.

$\mathcal{P} = \{p_1, p_2, \dots\}$ is the set of propositional variables (also called atoms) and $p, q, r, s$ denote arbitrary elements of $\mathcal{P}$. In the case that $\bot$ is part of the language, $p, q, r, s$ range over $\mathcal{P} \cup \{\bot\}$. $A, B, C$ denote formulas. $\mathcal{F}(p_1, \dots, p_n)$ is the set of formulas in which only atoms in $\{p_1, \dots, p_n\}$ occur. We use $\Gamma, \Delta$ to denote finite sets of formulas. *Sequents* are expressions $\Gamma \Rightarrow \Delta$, thus pairs of finite sets of formulas. In the case that $\bot$ and negation do not belong to the language, we require that $\Delta$ is not empty. $S$ ranges over sequents. A sequent is *irreducible* if it only contains atoms, boxed atoms ($\Box p$ for an atom $p$), and $\bot$. A formula is *irredicuble* if it is of the form $I(S)$ for an irreducible sequent $S$. $\mathcal{S}, \mathcal{G}, \mathcal{H}$ range over finite sets of sequents.

$\mathcal{P}_\mathcal{G}$ denotes the set of atoms that occur in $\mathcal{G}$, and if $\bot$ is present and occurs in $\mathcal{G}$, $\mathcal{P}_\mathcal{G}$ also contains $\bot$. $n_\mathcal{G}$ is the minimal $n$ for which all atoms in $\mathcal{G}$ are among $p_1, \dots, p_n$.

We need the following notation, where $v$ stands for variable, $b$ for box, $i$ for interior, $a$ for assumption, and $c$ for conclusion:

$$\Gamma^v \equiv_{def} \{p \mid p \in \Gamma\} \quad \Gamma^b \equiv_{def} \{\Box p \mid \Box p \in \Gamma\} \quad \Gamma^i \equiv_{def} \{p \mid \Box p \in \Gamma)\}$$

$$(\Gamma \Rightarrow \Delta)^a \equiv_{def} \Gamma \quad (\Gamma \Rightarrow \Delta)^c \equiv_{def} \Delta$$
$$S^{kl} \equiv_{def} (S^k)^l \quad k \in \{a, c\} \;\; l \in \{a, c, v, b, i\}.$$

For example, $S^{ab}$ is the set of boxed atoms in the antecedent of $S$. Sequents are interpreted as formulas in the usual way: $I(S) = (\bigwedge S^a \to \bigvee S^c)$. For notational convenience we sometimes write $S$ for $I(S)$, for example in $\vdash S$, which thus should be read as $\vdash \bigwedge S^a \to \bigvee S^c$. Also expressions like "$S$ is derivable" are short for "$I(S)$ is derivable". The following sets play an important role in what follows.

$$\mathcal{B}_{\mathcal{G}} \equiv_{def} \bigcup \{S^{ab} \mid S \in \mathcal{G}\} \quad \Sigma^{\mathcal{G}}_{\mathcal{S}} \equiv_{def} \{p \mid \mathcal{G} \vdash I(\mathcal{B}_{\mathcal{S}} \Rightarrow p)\}.$$

Sets of sequents are interpreted as conjunctions and we sometimes use the non-calligraphic version of a letter to denote the corresponding boxed formula:

$$I(\mathcal{G}) \equiv_{def} \bigwedge_{S \in \mathcal{G}} I(S) \quad G \equiv_{def} \Box I(\mathcal{G}).$$

When we speak of the unifiability of $\mathcal{G}$, we mean the unifiability of $G$. Note that reflexivity implies that $\vdash G \to I(\mathcal{G})$.

We assume that the logics are given by consequence relations. In the setting of rules it is convenient to consider *multi-conclusion finitary structural consequence relations*, which are relations $\vdash$ between finite sets of formulas satisfying

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|---|---|
| *reflexivity* | $A \vdash A$, |
| *weakening* | if $\Gamma \vdash \Delta$, then $\Gamma', \Gamma \vdash \Delta, \Delta'$, |
| *transitivity* | if $\Gamma \vdash \Delta, A$ and $\Gamma', A \vdash \Delta'$, then $\Gamma', \Gamma \vdash \Delta, \Delta'$, |
| *structurality* | if $\Gamma \vdash \Delta$, then $\sigma\Gamma \vdash \sigma\Delta$ for all substitutions $\sigma$. |

A *finitary single-conclusion consequence relation* is a relation between finite sets of formulas and formulas satisfying the single-conclusion variants of the four properties above. Thus for single-conclusion consequence relations the conclusion of a rule cannot be empty.

The *theorems* of (the logic given by) a consequence relation $\vdash$ are those $A$ for which $\emptyset \vdash A$, which we denote by $\vdash A$, holds. There are many consequence relations that correspond to a single set of theorems. Here we do not require much of the consequence relation, except that $A \vdash_{\mathsf{L}} \Box A$ holds for all $A$, and if $\bigwedge \Gamma \to A$ holds in the logic, then $\Gamma \vdash_{\mathsf{L}} A$ holds for the consequence relation $\vdash_{\mathsf{L}}$.

A *(multi-conclusion) rule* is an expression of the form $\Gamma/\Delta$. It is *derivable* in a logic given by consequence relation $\vdash_{\mathsf{L}}$ if $\Gamma \vdash_{\mathsf{L}} A$ for some $A \in \Delta$, and *admissible*, written $\Gamma \mid\!\sim_{\mathsf{L}} \Delta$, if for all substitutions $\sigma$, if $\sigma\Gamma$ consists of theorems of $\mathsf{L}$, then $\sigma\Delta$ contains a theorem of $\mathsf{L}$. Note that a logic has the modal disjunction property ($\vdash_{\mathsf{L}} \Box A \vee \Box B$ implies $\vdash_{\mathsf{L}} A$ or $\vdash_{\mathsf{L}} B$) if and only if $\{\Box p \vee \Box q\}/\{p, q\}$ is admissible. Given a set of rules $\mathcal{R}$, $\vdash_{\mathsf{L}}^{\mathcal{R}}$ is the smallest finitary structural multi-conclusion consequence relation that extends $\vdash_{\mathsf{L}}$ in which all rules in $\mathcal{R}$ are derivable. For more on consequence relations in this setting, see [18].

## 3 Proof sketch

Given a formula $A$ and a subset $I$ of the atoms in $A$, consider the valuation $v_I$ and substitution $\sigma_I^A$ given in the introduction:

$$v_I(p) \equiv_{def} \begin{cases} 1 & \text{if } p \in I \\ 0 & \text{if } p \notin I \end{cases} \qquad \sigma_I^A(p) \equiv_{def} \begin{cases} A \to p & \text{if } p \in I \\ A \wedge p & \text{if } p \notin I. \end{cases}$$

It is not difficult to see that if $S$ consists of atoms, then for $A = I(S)$, if $v_I(A) = 1$, then $\vdash_{\mathsf{L}} \sigma_I^A(A)$. Also, $A \vdash_{\mathsf{L}} \sigma_I^A(B) \leftrightarrow B$ for all $B$. Therefore, in case $v_I$ satisfies $A$, $\sigma_I^A$ is a most general unifier of $A$ in $\mathsf{L}$. For if $\vdash_{\mathsf{L}} \tau A$, then as $\tau A \vdash_{\mathsf{L}} \tau\sigma_I^A(B) \leftrightarrow \tau B$, also $\vdash_{\mathsf{L}} \tau\sigma_I^A(B) \leftrightarrow \tau B$. That is, $\tau \leqslant \sigma_I^A$.

Because the logics contain (the negationless fragment of) $\mathsf{S4}$, the above arguments extends in the following way to irreducible sequents $S$: if $v_I(S^{av} \cup S^{ai} \Rightarrow S^{cv}) = 1$, then $\sigma_I^{I(S)}$ is a most general unifier of $I(S)$.

One of the key observations in the results below, Corollary 1, states that a set $\mathcal{G}$ of irreducible sequents closed under the rules $\mathsf{V}^\circ$ is projective. The projective unifier of the formula $G$, where $G = \Box I(\mathcal{G})$, is a composition of substitutions of the form $\sigma_I^G$, for some $I$. The main part of the proof of Corollary 1 is to show that such a composition is a unifier for the formula, because the argument above implies that if so, it is a most general one.

The proof that a certain composition $\sigma = \sigma_n \dots \sigma_1$ of substitutions of the form $\sigma_I^G$ is a unifier for $G$ is based on the following simple observation. Writing $\overline{\sigma}_i$ for $\sigma_n \dots \sigma_i$, to prove $\vdash_{\mathsf{L}} \sigma G$, one has to show that $\vdash_{\mathsf{L}} \overline{\sigma}_1 S_1^a \Rightarrow \overline{\sigma}_1 S_1^c$ for all $S_1 \in \mathcal{G}$. For this it suffices to show that for some $i_2 \geq 1$ and for all $S_2 \in \mathcal{G}$:

$$\vdash_{\mathsf{L}} \overline{\sigma}_1 S_1^{ab} \Rightarrow I(\overline{\sigma}_{i_2} S_2). \tag{1}$$

This would namely imply that $\vdash_{\mathsf{L}} \overline{\sigma}_1 S_1^{ab} \Rightarrow \overline{\sigma}_{i_2} I(\mathcal{G})$ and thus that $\vdash_{\mathsf{L}} \overline{\sigma}_1 S_1^{ab} \Rightarrow \overline{\sigma}_{i_2} G$. And as the $\sigma_j$ are such that $\vdash_{\mathsf{L}} G \to \sigma_{i_2-1} \dots \sigma_1 G$, an application of $\overline{\sigma}_{i_2}$ gives $\vdash_{\mathsf{L}} \overline{\sigma}_{i_2} G \Rightarrow \overline{\sigma}_1 G$. Thus $\vdash_{\mathsf{L}} \overline{\sigma}_1 S_1^{ab} \Rightarrow \overline{\sigma}_1 G$, which implies $\vdash_{\mathsf{L}} \overline{\sigma}_1 S_1^{ab} \Rightarrow \overline{\sigma}_1 S_1$, as $S_1 \in \mathcal{G}$. And thus $\vdash_{\mathsf{L}} \overline{\sigma}_1 S_1^a \Rightarrow \overline{\sigma}_1 S_1^c$.

Repeating this argument shows that to prove (1) it suffices to show that for some $i_3 \geq i_2$ and for all $S_3 \in \mathcal{G}$:

$$\vdash_{\mathsf{L}} \overline{\sigma}_1 S_1^{ab}, \overline{\sigma}_{i_2} S_2^{ab} \Rightarrow I(\overline{\sigma}_{i_3} S_3). \tag{2}$$

Continuing this argument, one sees that in order to prove $\vdash_{\mathsf{L}} \sigma G$ it suffices to show that for all possible sequences $S_1, \dots, S_m$ of sequents from $\mathcal{G}$ and all numbers $1 \leq i_2 \leq i_3 \leq \dots \leq i_m$ there is an $j \geq i_m$ such that for all $S \in \mathcal{G}$:

$$\vdash_{\mathsf{L}} \overline{\sigma}_1 S_1^{ab}, \overline{\sigma}_{i_2} S_2^{ab}, \dots, \overline{\sigma}_{i_m} S_{i_m}^{ab} \Rightarrow I(\overline{\sigma}_j S). \tag{3}$$

Reasoning as above in the simpler case, one sees that if for $\mathcal{S} = \{S_1, \dots, S_i\}$, $I$ would be such that $\sigma_j = \sigma_I^G$ and $v_I$ satisfies $I(S^{av} \cup S^{ai} \Rightarrow S^{cv} \cup (S^{ci} \cap \Sigma_{\mathcal{S}}^{\mathcal{G}}))$, then (3) holds. This explains the notion of strong satisfiability introduced below, which requires that $v_I$ satifsies $I(S^{av} \cup S^{ai} \Rightarrow S^{cv} \cup (S^{ci} \cap \Sigma_{\mathcal{S}}^{\mathcal{G}}))$ for all $S \in \mathcal{S} \subseteq \mathcal{G}$.

The proof of Corollary 1 therefore consists of two parts: Lemma 8 stating that closure under the rules $\mathsf{V}^\circ$ implies strong satisfiability and Theorem 1 stating that strong satisfiability implies projectivity. The rest of the paper shows how to apply Corollary 1 to prove that certain (fragments of) logics have finitary unification type and $\mathsf{V}^\circ$ as a basis for admissibility.

## 4 Substitutions and valuations

The discussion above serves as a background for the definitions given below. In this and the next section we consider an arbitrary finite set $\mathcal{G}$ of *irreducible* sequents, and corresponding boxed formula $G = \Box I(\mathcal{G})$. Without loss of generality we assume the set of atoms that occur in $\mathcal{G}$ to be $\mathcal{P}_\mathcal{G} = \{p_1, \ldots, p_{n_\mathcal{G}}\}$. Most definitions are relative to $\mathcal{G}$ but for simplicity we do not always indicate this in our notation. Observe that $G$ derives $\Box G$ and $I(\mathcal{G})$ by transitivity and reflexivity.

We fix an arbitrary enumeration $J_1, \ldots, J_{2^{n_\mathcal{G}}}$ of all subsets of $\mathcal{P}_\mathcal{G}$ and $I$ ranges over arbitrary subsets of $\mathcal{P}_\mathcal{G}$. Valuation of the form $v_I$ have been defined at the beginning of Section 3. We extend these to valuations for sequents $S$ relative to a set of sequents $\mathcal{S} \subseteq \mathcal{G}$: $S$ is *strongly satisfiable* with respect to $\mathcal{S}$ if

$$\tilde{v}_I(S \mid \mathcal{S}) \equiv_{\mathit{def}} v_I\big(S^{av} \cup S^{ai} \Rightarrow S^{cv} \cup (S^{ci} \cap \Sigma_\mathcal{S}^\mathcal{G})\big) = 1.$$

The empty sequent is interpreted as $\perp$ and thus has no satisfying valuation. The valuations are extended to sets of sequents in the usual way: $\tilde{v}_I(\mathcal{S}' \mid \mathcal{S}) = 1$ if and only if $\tilde{v}_I(S \mid \mathcal{S}) = 1$ for all $S \in \mathcal{S}'$. We write $\tilde{v}_I(\mathcal{S})$ for $\tilde{v}_I(\mathcal{S} \mid \mathcal{S})$. $\mathcal{G}$ is *strongly satisfiable* if for all $\mathcal{S} \subseteq \mathcal{G}$ there is an $I$ such that $\tilde{v}_I(\mathcal{S}) = 1$.

$\sigma$ and $\tau$ range over substitutions that assign propositional formulas in the language of $\mathsf{L}$ to atoms, and $\iota$ is the identity substitution. As usual, $\tau\Gamma = \{\tau A \mid A \in \Gamma\}$ and $\tau S = (\tau S^a \Rightarrow \tau S^c)$. The substitutions that we consider have finite domains, where $\mathit{dom}(\sigma)$ denotes the domain of $\sigma$. We use the following notation:

$$\sigma \leftrightarrow \tau \equiv_{\mathit{def}} \bigwedge_{p \in \mathit{dom}(\sigma) \cup \mathit{dom}(\tau)} \big(\sigma(p) \leftrightarrow \tau(p)\big).$$

Observe that $\sigma \leftrightarrow \tau$ is a propositional formula, and that

$$\vdash \sigma \leftrightarrow \tau \text{ implies } \vdash \sigma A \leftrightarrow \tau A.$$

Given a set of atoms $I$, the substitutions $\sigma_I$, $\overline{\sigma}$ and $\sigma_G$ are defined as

$$\sigma_I(p) \equiv_{\mathit{def}} \begin{cases} G \to p & \text{if } p \in I \\ G \wedge p & \text{if } p \notin I. \end{cases}$$

$$\overline{\sigma} \equiv_{\mathit{def}} \sigma_{J_g} \ldots \sigma_{J_1} \qquad \sigma_G \equiv_{\mathit{def}} \overline{\sigma}^{|\mathcal{G}|+1},$$

where $g$ is short for $2^{n_\mathcal{G}}$. Thus $\sigma_G$ is the composition of $g(|\mathcal{G}|+1)$ substitutions. The $i$-th substitution in $\sigma_G$ (reading from right to left) is denoted by $\sigma_i$ and for

$i < j$, $\sigma_j \dots \sigma_i$ is denoted by $\sigma_{j,i}$. We denote $\sigma_{g(|\mathcal{G}|+1),i} = \sigma_{g(|\mathcal{G}|+1)} \dots \sigma_i$ by $\overline{\sigma}_i$. For example, $\sigma_2 = \sigma_{g+2} = \dots = \sigma_{g|\mathcal{G}|+2} = \sigma_{J_2}$, $\overline{\sigma}_1 = \sigma_G$, and $\overline{\sigma}_{g+1} = \overline{\sigma}^{|\mathcal{G}|}$. We denote by $I_i$ the set $J_j$ such that $\sigma_i = \sigma_{J_j}$. Thus $i = kg + j$ for some $k$. For valuations we define:

$$v_i \equiv_{def} v_{I_i}.$$

The rest of this section contains technical lemmas that we will need later on.

**Lemma 1** For all $m$ and $i < j$: $\vdash G \to \Box\big(\iota \leftrightarrow \sigma_i \leftrightarrow \sigma_{j,i}\big)$ and $\vdash \overline{\sigma}_j G \to \overline{\sigma}_i G$.

**Proof** Observe that $\vdash G \to \Box G$ holds because the logic is transitive. The first equivalence in the first statement immediately follows from this. The second equivalence follows from this and the fact that $\vdash \Box(B \leftrightarrow C) \to (A[B/p] \leftrightarrow A[C/p])$ for any atom $p$.
The first statement implies that $\vdash G \to \sigma_{j-1,i} G$, which implies $\vdash \overline{\sigma}_j G \to \overline{\sigma}_i G$. $\Box$

**Lemma 2** For all $S \in \mathcal{G}$ for which $(S^{cv} \cap I)$ or $(S^{av} \cup S^{ai}) \backslash I$ is not empty, $\vdash \sigma_I S$.

**Proof** We treat the case that $S^{ai} \backslash I$ is not empty, say it contains the atom $p$. Thus $\Box p$ belongs to $S^a$ and since the logic is reflexive, $S^a$ implies $p$. $p$ is under $\sigma_I$ replaced by $G \wedge p$. Thus $\sigma_I S^a$ implies $G$, and Lemma 1 and the fact that $S \in \mathcal{G}$ prove that it implies $S^c$, and $\sigma_I S^c$ as well, which gives the result. $\Box$
For numbers $i_1, \dots, i_j$, sequents $S_1, \dots, S_j$, and formula $A$ we define

$$F(i_1, \dots, i_j, S_1, \dots, S_j, A) \equiv_{def} I(\overline{\sigma}_{i_1} S_1^{ab}, \overline{\sigma}_{i_2} S_2^{ab}, \dots, \overline{\sigma}_{i_j} S_j^{ab} \Rightarrow A). \qquad (4)$$

We write $F(i_1, \dots, i_j, S_1, \dots, S_j, S)$ for $F(i_1, \dots, i_j, S_1, \dots, S_j, I(S))$ when convenient. Recall that $g$ is short for $2^{n_{\mathcal{G}}}$.

**Lemma 3** For all $\mathcal{S} = \{S_1, \dots, S_j\} \subseteq \mathcal{G}$ and $1 \leq i_1, \dots, i_j < h \leq g(|\mathcal{G}| + 1)$, if $\tilde{v}_h(\mathcal{S}) = 1$, then $\vdash F(i_1, \dots, i_j, S_1, \dots, S_j, I(\sigma_h S) \wedge I(\overline{\sigma}_h S))$ for all $S \in \mathcal{S}$.

**Proof** Suppose $\tilde{v}_h(\mathcal{S}) = 1$ and consider $S \in \mathcal{S}$. If $(S^{cv} \cap I_h)$ or $(S^{av} \cup S^{ai}) \backslash I_h$ is not empty, the previous lemma implies that $\sigma_h S$ and thus $\overline{\sigma}_h S$ is derivable, which implies what has to be shown. The case remains that $\Box p \in S^c$ for some $p \in \Sigma_{\mathcal{S}}^{\mathcal{G}} \cap I_h$. Since $G \vdash I(B_{\mathcal{S}} \Rightarrow p)$, $B_{\mathcal{S}} = \{S_1^{ab}, \dots, S_j^{ab}\}$, and $p$ is under $\sigma_h$ replaced by $G \to p$, it follows that $\vdash I(\overline{\sigma}_{i_1} S_1^{ab}, \dots, \overline{\sigma}_{i_j} S_j^{ab} \Rightarrow \sigma_h p)$ by Lemma 1. Hence $\vdash I(\tau_1 S_1^{ab}, \dots, \tau_j S_j^{ab} \Rightarrow \sigma_h S)$. Writing $\tau_k$ for $\sigma_{h,i_k}$, one readily sees that also $\vdash I(\tau_1 S_1^{ab}, \dots, \tau_j S_j^{ab} \Rightarrow \sigma_h p)$. Thus $\vdash I(\overline{\sigma}_{i_1} S_1^{ab}, \dots, \overline{\sigma}_{i_j} S_j^{ab} \Rightarrow \Box \sigma_h p)$ and $\vdash I(\tau_1 S_1^{ab}, \dots, \tau_j S_j^{ab} \Rightarrow \Box \sigma_h p)$ as well. As $\Box p \in S^c$, $\vdash I(\tau_1 S_1^{ab}, \dots, \tau_j S_j^{ab} \Rightarrow \Box \sigma_h S)$. An application of $\overline{\sigma}_{h+1}$ gives $\vdash F(i_1, \dots, i_j, S_1, \dots, S_j, \overline{\sigma}_h S)$. $\Box$

# 5 Unifiers

In this section we show that strong satisfiability implies projectivity. The proof of this fact is syntactic and does not use models. The definitions below are relative to $\mathcal{G}$, but we do not indicate this in our notation. Substitutions $\sigma$ and $\sigma_i$ have been defined in the previous section. For the intuition behind the notions defined below we refer the reader to Section 3.

A sequence of $m$ numbers followed by $m$ sequents $i_1, \ldots, i_m, S_1, \ldots, S_m$ is *appropriate* if $m \leq |\mathcal{G}|$,

$$1 = i_1 \leq g < i_2 \leq 2g \leq \cdots < i_m \leq mg,$$

and the sequents are distinct and belong to $\mathcal{G}$. It is *G-sufficient* if for all numbers $j$ such that $mg < j \leq (m+1)g$ and $\tilde{v}_j(\{S_1, \ldots, S_m\}) = 1$, the formula $F(i_1, \ldots, i_m, S_1, \ldots, S_m, \overline{\sigma}_j G)$ is derivable, where $F$ is defined in (4).

**Lemma 4** If $\mathcal{G}$ is strongly satisfiable, then for any number $k \geq 0$ and every appropriate sequence $i_1, \ldots, i_m, S_1, \ldots, S_m$ there exists a natural number $h$ such that $kg < h \leq (k+1)g$ and $\tilde{v}_h(\{S_1, \ldots, S_m\}) = 1$.

**Proof** As $\mathcal{G}$ is strongly satisfiable, there is a $1 \leq j \leq g$ such that $\tilde{v}_j(\{S_1, \ldots, S_m\})$ equals 1. Since $v_j = v_{kg+j}$, the lemma follows. □

**Lemma 5** If $\mathcal{G}$ is strongly satisfiable then for all $m \leq |\mathcal{G}|$: if all appropriate sequences of length $2m$ are $G$-sufficient, then so are all appropriate sequences of length $2m-2$.

**Proof** Consider an appropriate $i_1, \ldots, i_{m-1}, S_1, \ldots, S_{m-1}$ and let $j$ be such that $(m-1)g < j \leq mg$ and $\tilde{v}_j(\{S_1, \ldots, S_{m-1}\}) = 1$. It suffices to show that for all $S \in \mathcal{G}$:

$$\vdash F(i_1, \ldots, i_{m-1}, S_1, \ldots, S_{m-1}, \overline{\sigma}_j S). \tag{5}$$

If $S \in \{S_1, \ldots, S_{m-1}\}$, then (5) follows from Lemma 3. If, on the other hand, $S \notin \{S_1, \ldots, S_{m-1}\}$, then $i_1, \ldots, i_{m-1}, j, S_1, \ldots, S_{m-1}, S$ is an appropriate sequence of length $2m$. By Lemma 4 there exists a number $h$ such that $mg < h \leq (m+1)g$ and $\tilde{v}_h(\{S_1, \ldots, S_{m-1}, S\}) = 1$. Therefore by $G$-sufficiency

$$\vdash F(i_1, \ldots, i_{m-1}, j, S_1, \ldots, S_{m-1}, S, \overline{\sigma}_h G).$$

Since $\vdash \overline{\sigma}_h G \to \overline{\sigma}_j G$ and $S \in \mathcal{G}$, this implies that

$$\vdash F(i_1, \ldots, i_{m-1}, j, S_1, \ldots, S_{m-1}, S, \overline{\sigma}_j S).$$

Hence $\vdash F(i_1, \ldots, i_{m-1}, S_1, \ldots, S_{m-1}, \overline{\sigma}_j S)$, which is what we had to show. □

**Lemma 6** If $S \in \mathcal{G}$ and $1, S$ is $G$-sufficient, then $\vdash \overline{\sigma} S$.

**Proof** By Lemma 4 there exists an $1 \leq i \leq g$ such that $\tilde{v}_i(\{S\}) = 1$. Therefore $\vdash \bigwedge \overline{\sigma}_1 S^a \to \overline{\sigma}_i G$. Since $\vdash \overline{\sigma}_i G \to \overline{\sigma}_1 G$ by Lemma 1, this gives $\vdash \bigwedge \overline{\sigma}_1 S^a \to \overline{\sigma}_1 G$. As $S \in \mathcal{G}$, $\vdash \overline{\sigma}_1 S$ follows, that is, $\vdash \sigma_G S$. □

**Lemma 7** Every appropriate sequence of length $2|\mathcal{G}|$ is $G$-sufficient.

**Proof** Let $|\mathcal{G}| = m$ and consider an appropriate sequence $i_1, \ldots, i_m, S_1, \ldots, S_m$ and let $j$ be such that $mg < j \leq (m+1)g$ and $\tilde{v}_j(\{S_1, \ldots, S_m\}) = 1$. Because $m = |\mathcal{G}|$ and the $S_i$ are distinct, $\{S_1, \ldots, S_m\} = \mathcal{G}$. Therefore by Lemma 3, $\vdash F(i_1, \ldots, i_m, S_1, \ldots, S_m, \overline{\sigma}_j S)$ for all $S \in \mathcal{G}$. This implies that the sequence is $G$-sufficient. □

**Theorem 1** If $\mathcal{G}$ is strongly satisfiable, then $\vdash \sigma_G G$.

**Proof** By Lemma 7 every appropriate sequence of length $2|\mathcal{G}|$ is $G$-sufficient. By repeated application of Lemma 5 it follows that $1, S$ is $G$-sufficient for every $S \in \mathcal{G}$. This implies $\vdash \sigma_G S$ by Lemma 6. Hence $\vdash \sigma_G G$. □

## 6 Rules and satisfiability

In the following we use $\Gamma, \Box A \equiv A \Rightarrow \Delta$ as an abbreviation for the sequence of two sequents $(\Gamma, \Box A, A \Rightarrow \Delta), (\Gamma \Rightarrow A, \Box A, \Delta)$, and $\Box\{A_1, \ldots, A_n\} \equiv \{A_1, \ldots, A_n\}$ for $\Box A_1 \equiv A_1, \ldots, \Box A_n \equiv A_n$. Furthermore, *resolution proofs* are sequent derivations in which every sequent contains only atoms, and every inference is a cut.

Jeřábek in [19] showed that the following rule is a basis for the admissible rules of $\mathsf{S4}$, and obtained similar results for other modal logics.

$$\frac{\{\Box\Gamma \equiv \Gamma \Rightarrow \Box\Delta\}}{\{\Box\Gamma \Rightarrow p \mid p \in \Delta\}}\ \mathsf{V}^\circ$$

We provide another proof of this fact and extend it to the negationless fragment of $\mathsf{S4}$. We prove it by showing that closure under $\mathsf{V}^\circ$ is a sufficient condition for strong satisfiability. A set of irreducible sequents $\mathcal{G}$ is *closed under* $\mathsf{V}^\circ$ if for all instances of $\mathsf{V}^\circ$ with irreducible hypothesis $\Box\Gamma \equiv \Gamma \Rightarrow \Delta$ that only contains atoms that belong to $\mathcal{P}_\mathcal{G}$, if $G$ derives every (formula corresponding to a) sequent in $\Box\Gamma \equiv \Gamma \Rightarrow \Delta$, then there is a $p \in \Delta$ such that $G$ derives $\Box\Gamma \Rightarrow p$.

**Lemma 8** If $\mathcal{G}$ is a consistent set of irreducible sequents closed under $\mathsf{V}^\circ$, then $\mathcal{G}$ is strongly satisfiable.

**Proof** Arguing by contraposition, suppose that for some $\mathcal{S} \subseteq \mathcal{G}$, $\tilde{v}_I(\mathcal{S}) = 0$ for all $I$. Thus there exists a resolution proof from the set of sequents

$$\{S^{av} \cup S^{ai} \Rightarrow S^{cv} \cup (S^{ci} \cap \Sigma^{\mathcal{G}}_{\mathcal{S}}) \mid S \in \mathcal{S}\}$$

that ends in the empty sequent. For clarity we denote, in this proof, the sequents in the resolution proof by $C$ and call them clauses. We can assume that no atom in a clause belongs both to the antecedent and the succedent. We are going to associate with every clause $C$ in the refutation a sequent $S_C$ derivable from $\mathcal{G}$ such that

$$S_C^{av} \cup S_C^{ai} \subseteq C^a \qquad S_C^{cv} \cup (S_C^{ci} \cap \Sigma_{\mathcal{S}}^{\mathcal{G}}) \subseteq C^v.$$

The antecedent of such a sequent can contain atoms, boxed atoms, and formulas of the form $p \equiv \Box p$, and the succedent consists of atoms and boxed atoms only. For the initial clauses $C$, $S_C$ is the sequent to which $C$ corresponds. For a cut on clauses $C_1$ and $C_2$ with corresponding sequents $S_1$ and $S_2$ there are the following four possibilities. Let $C$ be the clause resulting from the cut. First, if $p \in S_1^{cv}$ and $p \in S_2^{av}$, then $S_C$ is the result of applying a cut to the sequents $S_1$ and $S_2$ with cutformula $p$. Second, if $p \in S_1^{ci}$ and $p \in S_2^{ai}$, then $S_C$ is the result of applying a cut to the sequents $S_1$ and $S_2$ with cutformula $\Box p$. Third, if $\Box p \in S_1^c$ and $p \in S_2^a$, then because of reflexivity, $\mathcal{G}$ derives $S_1' = (S_1^a \Rightarrow p, S_1^c \backslash \{\Box p\})$, and $S_C$ is the result of a cut on $S_1'$ and $S_2$ with cutformula $p$. In the remaining fourth case, $p \in S_1^c$ and $\Box p \in S_2^a$, we put

$$S_C = S_2^a \backslash \{\Box p\} \cup S_1^a \cup \{\Box p \equiv p\} \Rightarrow S_1^c \backslash \{p\} \cup S_2^c.$$

Note that $S_C$ is derivable from $\mathcal{G}$ if $S_1$ and $S_2$ are. Also note that for all $\Box p \equiv p$ that occur in $S_C$, $\Box p \in \mathcal{B}_{\mathcal{G}}$.
Now $S_\emptyset$ is of the form $\Box\Gamma \equiv \Gamma \Rightarrow \Box\Delta$, for which $\Delta \cap \Sigma_{\mathcal{S}}^{\mathcal{G}}$ is empty. If $\Delta$ is empty, then $\mathcal{G}$ derives all sequents in $\Box\Gamma \equiv \Gamma \Rightarrow$, which would make $\mathcal{G}$ inconsistent. Therefore $\Delta$ is not empty. As $\mathcal{G}$ is closed under $\mathsf{V}^\circ$ there exists a $p \in \Delta$ such that $\mathcal{G}$ derives $(\Box\Gamma \Rightarrow p)$. Hence $p \in \Delta \cap \Sigma_{\mathcal{S}}^{\mathcal{G}}$, contradicting $\Delta \cap \Sigma_{\mathcal{S}}^{\mathcal{G}} = \emptyset$. $\Box$
Combining the previous lemma with Theorem 1 gives a necessary condition for projectivity.

**Corollary 1** If $\mathcal{G}$ is a consistent set of irreducible sequents closed under $\mathsf{V}^\circ$, then $G$ is projective.

**Theorem 2** If $\mathsf{V}^\circ$ is admissible in $\mathsf{L}$ and $\mathcal{G}$ is a consistent set of irreducible sequents, then $\mathcal{G}$ is closed under $\mathsf{V}^\circ$ if and only if $G$ is projective if and only if $\sigma_G$ is a unifier of $G$ if and only if $\mathcal{G}$ is strongly satisfiable.

**Proof** We prove the first equivalence. The direction from left to right is Corollary 1. For the other direction, let $\sigma$ be a projective unifier of $G$ and suppose that $\mathcal{G}$ derives $(\Box\Gamma \equiv \Gamma \Rightarrow \Box\Delta)$, meaning the conjunction of all the sequents of the sequence $(\Box\Gamma \equiv \Gamma \Rightarrow \Box\Delta)$. Thus $\sigma(\Box\Gamma \equiv \Gamma \Rightarrow \Box\Delta)$ is derivable in $\mathsf{L}$. Hence so is $\sigma(\Box\Gamma \Rightarrow p)$ for some $p$ in $\Delta$. Therefore $\mathcal{G}$ derives $(\Box\Gamma \Rightarrow p)$. $\Box$

## 7 Unification types

In this section we use the previous results to show that in $\mathsf{S4}$ and its negationless fragment, as well as in all their extensions, admissibility of $\mathsf{V}^\circ$ implies finitary unification. For $\mathsf{S4}$ and its extensions this was first shown by Jeřábek in [19] by semantic means. The use of projective formmulas in this setting goes back to Ghilardi [9]. In our approach, that is also based on this key idea, we use a method developed in [14, 15] that first reduces a formula to a set of irreducble sequents and then to sets of irreducible sequents closed under $\mathsf{V}^\circ$. From the previous paragraph we know that thus the formulas corresponding to the last sets are projective. This then will prove the finitary unification of the logics, as we will see below.

Recall that an irreducible formula is a formula of the form $I(S)$, where $S$ is irreducible, and that $\mathcal{F}(p_1, \dots, p_n)$ is the set of formulas in which only atoms in $\{p_1, \dots, p_n\}$ occur.

**Lemma 9** For every $n$ and every set of formulas $\Gamma \subseteq \mathcal{F}(p_1, \dots, p_n)$, there exists a finite set of irreducible formulas $\Pi$ such that for every $\Delta \subseteq \mathcal{F}(p_1, \dots, p_n)$:

1. $\Box\Gamma \mathrel{|\!\sim} \Delta$ if and only if $\Box\Pi \mathrel{|\!\sim} \Delta$,
2. $\Box\Gamma \vdash^{\mathsf{V}^\circ}\Delta$ if and only if $\Box\Pi \vdash^{\mathsf{V}^\circ}\Delta$,
3. $\Gamma \vdash \bigwedge \sigma\Pi$ for some $\sigma$ that is the identity on $\mathcal{F}(p_1, \dots, p_n)$.

**Proof** It is easier to consider $\Gamma$ and $\Pi$ as sets of the form $\{I(S) \mid S \in \mathcal{H}\}$ for some set of sequents $\mathcal{H}$. We start for $\Gamma$ with $\mathcal{H} = \{(\Rightarrow A) \mid A \in \Gamma\}$. We follow the method of proof of a similar lemma in [4]. The *length* of a formula is the number of symbols occurring in it. Let $ml(\mathcal{H})$ be the multiset of the lengths of the formulas in the sequents in $\mathcal{H}$. We prove the lemma by induction on $ml(\mathcal{H})$, using the multiset ordering. At every step we construct a new set of sequents $\mathcal{H}'$ such that (1) and (2) hold and $ml(\mathcal{H}') < ml(\mathcal{H})$, untill $\mathcal{H}'$ is irreducible. This will prove the lemma by taking $\{I(S) \mid S \in \mathcal{H}'\}$ for $\Pi$.

If $ml(\mathcal{H}) \leq 1$, $\mathcal{H}$ consists of irreducible sequents, and we can take $\mathcal{H}$ for $\mathcal{H}'$. Therefore suppose $ml(\mathcal{H}) > 1$ and consider a formula $A$ in a sequent $S \in \mathcal{H}$ that has length greater than 1. Thus $A$ is not an atom or a boxed atom. If $A = (B \wedge C)$ and $A \in S^a$, we replace $S$ by $(S^a \backslash \{A\}, B, C \Rightarrow S^c)$, and if $A \in S^c$ we replace $S$ by $(S^a \Rightarrow S^c \backslash \{A\}, B)$ and $(S^a \Rightarrow S^c \backslash \{A\}, C)$. Similarly if $A$ is a disjunction or an implication. For $\mathcal{H}'$ being the result of applying this replacement, (1) and (2) clearly hold.

Suppose $A = \Box B$. If $A \in S^c$ we choose a fresh atom $p$ different from $p_1, \dots, p_n$ and replace $S$ by $S_1 = (S^a \Rightarrow S^c \backslash \{A\}, \Box p)$ and $S_2 = (p \Rightarrow B)$. If $A \in S^a$, $S$ is replaced by $S_1 = (S^a \backslash \{A\}, \Box p \Rightarrow S^c)$ and $S_2 = (B \Rightarrow p)$. In both cases call the result $\mathcal{H}'$ and note we have $\Box I(S_1) \wedge \Box I(S_2) \vdash \Box I(S)$ and therefore $\Box I(\mathcal{H}') \vdash \Box I(\mathcal{H})$. Note that there is a substitution $\sigma$ that is the identity on $p_1, \dots, p_n$ such that $I(\mathcal{H}) \vdash I(\sigma\mathcal{H}')$. Namely, all such substitutions for which $\sigma(p) = B$. This implies (3).

The direction from left to right of (1) and (2) holds as $\Box I(\mathcal{H}') \vdash \Box I(\mathcal{H})$. For the other direction of (1), consider a unifier $\tau$ of $\Box I(\mathcal{H})$. This can be extended to a unifier $\tau'$ of $\Box S_1$ and $\Box S_2$ by putting $\tau'(p) = B$. Thus $\vdash \tau' C$ for some

$C \in \Delta$. As $\tau$ equals $\tau'$ on $\Delta$, $\vdash \tau C$ follows, proving that $\Box I(\mathcal{H}) \mid\!\sim \Delta$. To prove the direction from right to left of (2), assume that $\Box I(\mathcal{H}') \vdash^{\mathsf{V}^\circ} \Delta$. For the substitution $\sigma$ defined in the previous paragraph $\Box I(\sigma\mathcal{H}') \vdash^{\mathsf{V}^\circ} \Delta$ holds by structurality and the fact that $\sigma$ is the identity on $\Delta$. As $I(\mathcal{H}) \vdash I(\sigma\mathcal{H}')$ and the logic is reflexive, $\Box I(\mathcal{H}) \vdash^{\mathsf{V}^\circ} \Delta$ follows. $\Box$

The following lemma has essentially been proved in [14].

**Lemma 10** For every set of irreducible formulas $\Pi$ there exist sets of irreducible formulas $\Pi_1, \ldots, \Pi_m$ such that the $\bigwedge \Pi_i$ are projective and for all $i$:

$$\bigwedge \Pi_i \vdash \bigwedge \Pi \vdash^{\mathsf{V}^\circ} \{\Pi_1, \ldots, \Pi_m\}.$$

**Proof** As in the previous proof, it is easier to consider $\Pi$ and $\Pi_i$ as sets of the form $\{I(S) \mid S \in \mathcal{H}\}$ for some set of sequents $\mathcal{H}$, starting with $\mathcal{H} = \{(\Rightarrow A) \mid A \in \Pi\}$ for $\Pi$. Define the following (rewrite) relation on finite sets of finite sets of irreducible sequents in $\mathcal{L}_{\mathcal{H}}$, where $X$ and $Y$ range over such sets:

$$X \cup \{\mathcal{G} \cup \{\Box\Gamma \equiv \Gamma \Rightarrow \Box\Delta\}\} \mapsto X \cup \{\mathcal{G} \cup \{\Box\Gamma \equiv \Gamma \Rightarrow \Box\Delta, \Box\Gamma \Rightarrow p\} \mid p \in \Delta\}.$$

Slightly ambiguous, we also use $\mapsto$ for the transitive closure of this relation. A set of sequents $\mathcal{G}$ is in $\mapsto$-*normal form* if there is no $\mathcal{H} \supset \mathcal{G}$ such that $\mathcal{G} \mapsto \mathcal{H}$. As the number of atoms in $\mathcal{H}$ is finite and all sequents involved are irreducible and contain no atoms than those in $\mathcal{H}$, there are $\mathcal{H}_1, \ldots, \mathcal{H}_n$ such that $\{\mathcal{H}\} \mapsto \{\mathcal{H}_1, \ldots, \mathcal{H}_n\}$ and the $\mathcal{H}_i$ are in $\mapsto$-normal form. Observe that the latter means that the $\mathcal{H}_i$ are closed under $\mathsf{V}^\circ$, and thus that $I(\mathcal{H}_i)$ is projective by Corollary 1. Let $\Pi_i = \{I(S) \mid S \in \mathcal{H}_i\}$. Thus $\bigwedge \Pi_i$ is projective. It is easy to see that they satisfy the other properties in the lemma as well. $\Box$

Combining the previous two lemmas gives the following theorem.

**Theorem 3** For every $n$ and every set of formulas $\Gamma \subseteq \mathcal{F}(p_1, \ldots, p_n)$, there exist sets of irreducible formulas $\Pi_1, \ldots, \Pi_m$ such that all $\bigwedge \Pi_i$ are projective and for every $\Delta \subseteq \mathcal{F}(p_1, \ldots, p_n)$:

1. $\Box\Gamma \vdash^{\mathsf{V}^\circ} \Delta$ if and only if $\Box\Pi_i \vdash^{\mathsf{V}^\circ} \Delta$ for all $i$.
2. $\Gamma \vdash^{\mathsf{V}^\circ} \{\bigwedge \sigma\Pi_1, \ldots, \bigwedge \sigma\Pi_m\}$ for some $\sigma$ that is the identity on $\mathcal{F}(p_1, \ldots, p_n)$.

**Proof** Given $\Gamma$, construct $\Pi$ and $\sigma$ as in Lemma 9 and then sets of irreducible formulas $\Pi_1, \ldots, \Pi_m$ as in Lemma 10. Using that the logics are reflexive and that $A \vdash \Box A$ for all $A$, it is easy to see that (1) holds. For (2), observe that by Lemma 10 and structurality we have $\sigma\Pi \vdash^{\mathsf{V}^\circ} \{\bigwedge \sigma\Pi_1, \ldots, \bigwedge \sigma\Pi_m\}$. As $\Gamma \vdash \bigwedge \sigma\Pi$, (2) follows. $\Box$

**Corollary 2** If $\mathsf{V}^\circ$ is admissible in $\mathsf{L}$, then every formula has a finite complete set of unifiers in $\mathsf{L}$.

**Proof** Given a formula $A$, let $\Pi_1, \ldots, \Pi_n$ be as in Theorem 3, where $\Gamma = \{A\}$, and let $\sigma_i'$ be the projective unifier of $\bigwedge \Pi_i$. Let $\sigma_i$ be equal to $\sigma_i'$ on the atoms in $A$ and the identity everywhere else. We verify that $\{\sigma_1, \ldots, \sigma_n\}$ is a complete set of unifiers for $A$. Therefore suppose that $\vdash_{\mathsf{L}} \tau A$. Then for $\sigma$ as in (2) of Theorem 3, $\tau\Gamma \vdash^{\mathsf{V}^\circ}\{\tau\sigma\Pi_1, \ldots, \tau\sigma\Pi_m\}$. Thus $\vdash \bigwedge \tau\sigma\Pi_i$ for at least one $i \leq n$ by the admissibility of $\mathsf{V}^\circ$. Hence $\tau\sigma \leqslant \sigma_i'$. Thus $\tau \leqslant \sigma_i$. $\square$

The previous corollary implies the following corollary, which for full $\mathsf{S4}$ has been proved by Ghilardi in [9].

**Corollary 3** $\mathsf{S4}$ and its negationless fragment have finitary unification.

## 8 Admissible rules

This last section of the paper contains some applications of the previous results to admissible rules. A set of rules $\mathcal{R}$ is a *basis* for the admissible rules of a logic $\mathsf{L}$ if

$$\Gamma \mathrel{|\!\sim}_{\mathsf{L}} \Delta \;\Leftrightarrow\; \Gamma \vdash_{\mathsf{L}}^{\mathcal{R}} \Delta.$$

Thus intuitively, $\mathcal{R}$ is a basis if all admissible rules can derived from those in $\mathcal{R}$.

In intermediate logics all consistent formulas are unifiable, but this is no longer the case in modal logic. This leads to the notion of *passive* admissible rules, which are admissible rules for which the hypothesis $(\bigwedge \Gamma)$ has no unifier. $\bot/A$ is a typical example of such a rule, and $(\Gamma \equiv \Box\Gamma \Rightarrow)/A$ is another example in reflexive logics.

A logic is *structurally complete* if all single-conclusion admissible rules are derivable, and *almost structurally complete* if all nonpassive single-conclusion admissible rules are derivable [7]. A logic is *hereditarily (almost) structurally complete* if all its extensions, including the logic itself, are (almost) structurally complete.

Jeřábek has proved the following theorem for full $\mathsf{S4}$ [19]. Using the techniques in this paper it can also be proved in the following way, also for the negationless fragment.

**Theorem 4** In any extension of $\mathsf{S4}$ or its negationless fragment, the rules $\mathsf{V}^\circ$ form a basis for the admissible rules once they are admissible.

**Proof** Assume that $\mathsf{V}^\circ$ is admissible and consider $\Gamma \mathrel{|\!\sim}_{\mathsf{L}} \Delta$. Then by Theorem 3 there are $\Pi_1, \ldots, \Pi_n$ such that $\bigwedge \Pi_i$ is projective and $\Pi_i \mathrel{|\!\sim} \Delta$ for all $i$, and $\Box\Gamma \vdash^{\mathsf{V}^\circ} \Delta$ if and only if $\Box\Pi_i \vdash^{\mathsf{V}^\circ} \Delta$ for all $i$. The projectivity of the $\Pi$ implies that for all $i$ there is an $A_i \in \Delta$ such that $\bigwedge \Pi_i \vdash A_i$, and therefore $\bigwedge \Box\Pi_i \vdash A_i$. Hence $\Box\Pi_i \vdash \Delta$, and thereby $\Box\Gamma \vdash^{\mathsf{V}^\circ} \Delta$. This proves that $\mathsf{V}^\circ$ is a basis for admissibility. $\square$

**Corollary 4** $\mathsf{V}^\circ$ is a basis for the admissible rules of $\mathsf{S4}$ as well as for its negationless fragment.

Dzik and Wojtylak prove in [7] that any extension of $\mathsf{S4}$ has projective unification if and only if it contains $\mathsf{S4.3}$, where $\mathsf{S4.3}$ is the logic $\mathsf{S4}$ extended by the principle $\Box(\Box A \to \Box B) \vee \Box(\Box B \to \Box A)$. This implies that $\mathsf{S4.3}$ is hereditarily almost structurally complete. Here we provide another proof of the last result and extend it to fragments.

**Theorem 5** $\mathsf{S4.3}$ and its negationless fragment are hereditarily almost structurally complete.

**Proof** Let $\mathsf{L}$ be an extension of $\mathsf{S4}$ or its negationless fragment. The fact that $\mathsf{S4.3}$ is complete with respect to transitive reflexive Kripke frames in which every two nodes are compatible ($xRy$ or $yRx$ holds) is easily seen to imply that all non passive instances of $\mathsf{V}^\circ$ are derivable in $\mathsf{L}$. Theorem 4 now shows that all non passive admissible rules are derivable. $\Box$

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