Provability Logic and Admissible Rules

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Chapter 1

Introduction

Het leven is een toovertuin.
L.E.J. Brouwer
(uit een brief aan C.S. Adama van Scheltema)

In this thesis we wander through two different landscapes: intuitionistic propositional logic and intuitionistic provability logic. Both theories are based on intuitionistic logic, the logic of constructive reasoning. Although constructive tendencies are as old as mathematics, they only became explicit at the beginning of the twentieth century through the work of the Dutch mathematician L. E. J. Brouwer. Constructivism in itself is a conception of mathematical truth, but one can also consider it from the outside by studying the properties *of* constructive theories. This is the path we follow in this thesis. In particular, we are interested in the formal aspects of constructive theories. Thus we find ourselves in the realm of mathematical logic.

We consider two aspects of constructive proofs, one in the context of propositional logic and the other in the context of provability logic. Although the results we prove in the different areas are related, we treat them independently in part I and part II of the thesis. In the following we summarize what we have (and have not) done in these fields. More extensive and technical introductions will follow in Chapters 2 and 6.

Since we are interested in what is true about constructive theories, and not necessarily in what is *constructively* true about them, we consider these theories from the classical point of view. However, we have some evidence that many of our results are constructively valid as well, and therefore, also the constructivists will probably find something to their taste in the following chapters.

1.1 Intuitionistic provability logic

Ever since Gödel (1931) we know that arithmetical theories can reason about themselves, by encoding properties *about* the theory *in* the theory itself. In particular, such theories T allow a formalization of 'being provable in T': one can define a predicate $\mathrm{Proof}_T(x, y)$ in the language of T which is a formalization of the statement that y is the code of a proof in T of the formula with code x. If we let $\ulcorner\varphi\urcorner$ denote the code[1] (Gödelnumber) of the formula φ, then $\exists y \mathrm{Proof}_T(\ulcorner\varphi\urcorner, y)$ denotes the statement that φ is provable in T. We write $\Box_T\varphi$ for $\exists y \mathrm{Proof}_T(\ulcorner\varphi\urcorner, y)$ and call it the *formalized provability predicate of* T.

We can ensure that the proof predicate has natural properties like

$$\text{for all } n, \text{ for all } \varphi\text{: } \mathbb{N} \models \mathrm{Proof}_T(\ulcorner\varphi\urcorner, n) \text{ iff } T \vdash \mathrm{Proof}_T(\ulcorner\varphi\urcorner, n).$$

However, Gödel's well-known second incompleteness result shows that T cannot prove everything that is true about its provability predicate. Namely, if T is consistent, then T does not prove the formalized version of this statement, i.e. if $\neg\Box_T\bot$ holds in $\mathbb{N}$ then we have $T \nvdash \neg\Box_T\bot$ ($\Box_T\bot$ is the formalization of 'T derives falsum', thus $\neg\Box_T\bot$ expresses that T is consistent). Given this difference, one can compare what $\mathbb{N}$ and T say about $\Box_T$. This question becomes more interesting when one abstracts from particular sentences. For example, instead of asking if T proves $(\Box_T\varphi \to \varphi)$ for a specific formula φ, we want to know if T proves the principle $(\Box_T\varphi \to \varphi)$ for *all* φ. That is, whether it proves soundness (it does not, as the previous example implies). Thus intuitively we want to ask questions like, does T prove all substitution instances of $(p \to \Box_T p)$, where we replace p by arithmetical sentences.

This is how one arrives at provability logic. Intuitively, the provability logic of a theory consists of the (propositional) schemes that the theory can prove about its provability predicate. The idea behind provability logic can be applied to any predicate which can be encoded in arithmetical theories. We will see an example of this below: the notion of preservativity.

Provability logics of classical theories are well-investigated. One remarkable thing is their stability; many theories, like for example Peano Arithmetic PA and set theory ZF, have the same provability logic, GL (Solovay 1976)(Visser 1984)(Beklemishev 1990). Until recently, provability logics of intuitionistic theories have hardly been considered. This is probably due to the fact that the logic on which these theories are based is already rather complicated. On the classical side, the first theory studied in the context of provability logic was PA, for it is strong enough to allow the formalization of provability notions in an easy way. For the same reason the questions in intuitionistic provability logic focus on Heyting Arithmetic HA, the intuitionistic counterpart of PA.

In Part I of the thesis we discuss what is known about the provability logic of HA and present our own contributions to the field. We will see that in combination

[1] We will not distinguish between a number and its numeral.

with results by Visser (1994)(1998), these results lead to a fragment of the provability logic of HA which captures the main features of HA that are expressible in provability logic. Whence for the first time there is a reasonable conjecture concerning the provability logic of HA, namely that it is equal to this fragment. The rest of the section is a summary of the results treated in part I.

It is not difficult to see that the axioms of the provability logic of PA are also part of the provability logic of HA. In contrast to the stability of classical provability logics mentioned above, one expects more valid schemes in the case of HA. This is due to the fact that for some of its properties, the statement that says that HA has this property is expressible in provability logic. Consider for example Markov's Rule, which reads

Markov's Rule for all $\varphi \in \Sigma_1$: if $\mathsf{HA} \vdash \neg\neg\varphi$, then $\mathsf{HA} \vdash \varphi$.

The principle $\Box\neg\neg\Box A \to \Box\Box A$ partly expresses Markov's Rule, since formulas of the form $\Box A$ are Σ_1-formulas (as we will see in Chapter 2, more of Markov's Rule can be expressed in provability logic, but we do not want to complicate matters in this informal exposition). To describe the provability logic of HA one has to make sure whether HA proves this statement or not. If it does, then the principle belongs to the provability logic and otherwise it does not.

In general, for any such property of HA, similar questions have to be answered in order to be able to describe the provability logic of HA. There are three properties involved: Markov's Rule, the

Disjunction Property if $\mathsf{HA} \vdash \varphi \vee \psi$, then $\mathsf{HA} \vdash \varphi$ or $\mathsf{HA} \vdash \psi$,

and the admissible rules of HA. The admissible rules of a theory are the rules under which the theory is closed. Visser (1982) showed that HA proves that it has Markov's Rule, i.e. the statement that expresses Markov's Rule is part of the provability logic of HA. The second problem was settled by Friedman (1975) and Leivant (1975): Friedman proved that although HA has the Disjunction Property, the statement that expresses this fact is not contained in the provability logic of HA, and Leivant showed that a slightly weaker version does belong to the provability logic of HA. As we will show in Chapter 2, the third problem is solved through results in Chapter 7 and (Visser 1994)(1998). There we show that HA recognizes its admissible rules, i.e. that for any admissible rule A/B, the principle $(\Box A \to \Box B)$ belongs to the provability logic of HA.

Visser (1994) proposed an extension of provability logic in which many principles of the provability logic of HA have a more elegant formulation. This extension is called preservativity logic and is based on the the notion of Σ_1-preservativity, which for classical theories is equivalent to Π_1-conservativity. In fact, Σ_1-preservativity is the constructive analogue of interpretability. We will see (Chapter 2) that preservativity logic captures the principles of the interpretability logic of PA (and more). Moreover, the fact that many of these axioms have a simple formulation in preservativity logic, implies that preservativity probably is the more natural approach of the two. For example, in Chapter 2 we show that there is a natural strengthening of Löb's Principle (Löb's Preservativity Principle) that is directly expressible in preservativity logic. The principles of the preservativity logic of HA given by Visser (1994) captured all the principles of the provability logic of HA known at that time, as we will prove in this thesis (Section 3.3).

However, the most important feature of this extension of provability logic is that besides the characteristic axioms of the provability logic of PA it captures exactly the three main properties of HA that are expressible in provability logic: the Disjunction Property (the weaker version by Leivant), Markov's Rule and the propositional admissible rules. Together with the fact that it does so in such a natural way, this leads us to the conjecture that it axiomatizes the preservativity logic of HA. Clearly, if this would be true we have a characterization of the provability logic of HA as well.

The first step in showing that a logic is the provability logic of some theory is to prove that the provability logic is complete with respect to a modal semantics. Moreover, if this semantics is simple it can be used to determine in an easy manner whether an expression belongs to the provability logic or not. In Chapter 5 we prove such a completeness theorem for the conjectured preservativity logic of HA. This is the heart of part I of the thesis. In this proof we use the results of Chapter 4 where we consider the principles separately.

We also prove (Chapter 5) the completeness of the fragment of the provability logic of HA obtained by omitting the principles that correspond to the propositional admissible rules. This was the first known part of the provability logic of HA, and therefore the first logic we worked on. In contrast to the (preservativity and provability) logic we know now, this logic still has the finite model property. However, its completeness proof is much more complicated than that of the former logic. This strengthens our expectation that the principles we know now are a complete axiomatization of the preservativity and provability logic of HA.

Although our first aim was to prove the modal completeness of preservativity and intuitionistic provability logic, part I of the thesis could also be viewed as a study in intuitionistic modal logic. The characterization of the principles required many technical tools from modal logic. Moreover, these logics deviate considerably from the logics that are regularly studied in intuitionistic modal logic. Therefore, some surprising properties and problems come to light, and many proofs are quite different from the ones for the modal logics one usually encounters. Therefore, also from the modal point of view these logics are interesting.

As explained above, the modal completeness of the preservativity logic could be seen as a first step to prove that it is the preservativity logic of HA. Clearly, as long as no proof has been provided, the question remains whether the preservativity logic we know at the moment really axiomatizes all of the preservativity logic of HA. However, we have gained some insight in the provability (and preservativity) logic of HA during our journey.

1.2 Intuitionistic propositional logic

One of the most salient differences between intuitionistic and classical propositional logic is that the former has nonderivable admissible rules while the latter has not. The admissible rules of a theory are the rules under which the theory is closed. Hence a description of the admissible rules of intuitionistic propositional logic IPC sheds light on the nature of constructive (propositional) inference.

It is often not very difficult to prove that some specific rule is a nonderivable admissible rule of intuitionistic propositional logic, i.e. an admissible rule for which the corresponding implication is constructively invalid. The following is a well-known example of such a rule:

$$\text{if } \mathsf{IPC} \vdash (\neg\neg A \to A) \to (A \vee \neg A), \text{ then } \mathsf{IPC} \vdash \neg\neg A \vee \neg A.$$

On the other hand, more general questions concerning admissible rules are more difficult to answer. One of the first things one would like to know is whether it is decidable if a rule is admissible or not and whether there exists a transparent axiomatization of the admissible rules.

Rybakov (1992) gave an algorithm that decides whether a rule is admissible or not, and that settled the first part of the problem. In this thesis we provide an answer to the second part. Namely, some ten years ago both de Jongh and Visser isolated a simple computably enumerable (c.e.) set of rules $\mathcal{V}$ which they conjectured to be a basis for the admissible rules of IPC. This means that they conjectured that this set of rules generates all the admissible rules of intuitionistic propositional logic. In Chapter 7 we prove this conjecture. This provides us with a perspicuous description of the admissible rules of IPC, and that settles the second part of the question mentioned above. This result is the heart of Part II of the thesis.

We also define a proof system for the admissible rules and characterize them in a semantical way. The simplicity of $\mathcal{V}$ and the proof system make them very useful for further research on admissible rules. We will see that the basis $\mathcal{V}$ is infinite. This cannot be improved, because IPC does not have a finite basis, as was shown by Rybakov (1997).

This description of the admissible rules of IPC also leads to a characterization of IPC. Many intuitionistic theories, for example IPC, have the Disjunction Property, which says that

Disjunction Property if $\mathsf{IPC} \vdash \varphi \vee \psi$, then $\mathsf{IPC} \vdash \varphi$ or $\mathsf{IPC} \vdash \psi$.

Kreisel and Putnam (1957) showed that IPC is not characterized by its Disjunction Property. This means that there exist intermediate logics, i.e. logics between IPC and classical propositional logic CPC, which are proper extensions of IPC that have the Disjunction Property. It is easy to see that IPC is not characterized by its admissible rules either. For example, all its admissible rules are admissible for CPC, and CPC clearly is an extension of IPC. However, in Chapter 8 we show that IPC is characterized by the Disjunction Property plus its admissible rules: IPC is the only intermediate logic with the Disjunction Property for which all admissible rules of IPC are admissible. In the proof of this fact we use the mentioned result that $\mathcal{V}$ is a basis for the admissible rules of IPC. We actually show that IPC is the only intermediate logic with the Disjunction Property for which all rules in $\mathcal{V}$ are admissible.

Thus for any intermediate logic which is a proper extension of IPC with the Disjunction Property, there is a rule in $\mathcal{V}$ which is not admissible. In Chapter 8 we show that for the well-known intermediate logics with the Disjunction Property from Gabbay and de Jongh (1974) we know which rules in $\mathcal{V}$ are admissible and which not. Moreover, we prove that like IPC these logics are also characterized by their admissible rules plus the Disjunction Property.

There even is a correspondence between the rules in $\mathcal{V}$ and the logics from (Gabbay and de Jongh 1974). The logics and the rules in $\mathcal{V}$ can be enumerated in a natural way. We will see that if a rule is admissible for a logic then so are all the rules that precede it in this enumeration. We will prove that for every rule there is a logic for which the rule is admissible and for which the rule that follows it is not admissible, and we will also prove the converse: for every logic there is a rule such that the next rule is not admissible while the rule itself is admissible for the logic.

The characterization of the admissible rules of IPC mentioned above, also leads to insights in the provability logic of Heyting Arithmetic HA. Namely, in combination with results by Visser (1994)(1998) they imply that HA recognizes its propositional admissible rules, as was explained in the previous section.

1.3 Overview

Part I of the thesis contains the material discussed in Section 1.1: Chapter 2 is the introduction, Chapters 4 and 5 contain the results, and Chapter 3 contains the tools for these results. Part II of the thesis contains the material discussed in Section 1.2: Chapter 6 is the introduction and Chapters 7 and 8 contain the results.

Chapters 4 and 5 are based on the articles (Iemhoff 1998) and (Iemhoff 2000b). Chapter 7 is more or less equal to the article (Iemhoff 1999) and Chapter 8 to (Iemhoff 2000a).

Part I

Intuitionistic Provability Logic

Chapter 2

Concepts

In this chapter we introduce the notions studied in part I of the thesis. Sections 2.1, 2.2 and 2.5 explain what provability, preservativity, and intuitionistic provability logic are. We introduce preservativity and intuitionistic provability logic in more detail than classical provability logic, for which there are many nice overview articles and books: (Boolos 1979)(Smoryński 1985)(Boolos 1993)(Japardize and de Jongh 1997) (Visser 1997). Section 2.6 briefly summarizes the literature on intuitionistic modal logic and explains how in the case of provability logic the presented results deviate from the regular literature. Section 2.8 gives an overview of the next chapters of part I.

2.1 Provability logic

In Chapter 1 we explained the idea behind provability logic. There we saw that the provability logic of an arithmetical theory T consists of all the propositional schemes that T can prove about its provability predicate. In this section we give the formal definition of provability logic. Recall that $\Box_T$ denotes the provability predicate of T and that a sentence $\Box_T\varphi$ expresses the statement that φ is provable in T.

Let $\mathcal{L}_\Box$ be the language of propositional logic extended with one modal operator $\Box$. The formulas in $\mathcal{L}_\Box$ are called *modal formulas*. Let T be an arithmetical theory that is strong enough to allow the formalization of its provability predicate[1]. An *arithmetical realization* of $\mathcal{L}_\Box$ into the language of T is a mapping $*$ from the formulas of $\mathcal{L}_\Box$ to sentences in the language of T that commutes with the propositional connectives and such that $(\Box A)^* = \Box_T(\ulcorner A^* \urcorner)$. The *provability logic of T* is the set of modal formulas A such that T proves A^* for any arithmetical realization $*$, i.e. the set $\{A \mid \forall^* \; T \vdash A^*\}$. The *truth provability logic of T* is the

[1] We will not discuss the minimal requirements that such a theory should satisfy, they can be found in (Smoryński 1985) or (Hájek and Pudlák 1991).

set of modal formulas A such that A^* is valid in the standard model $\mathbb{N}$ for any arithmetical realization $*$, i.e. the set $\{A \mid \forall^* \ \mathbb{N} \models A^*\}$.

Note that in general the provability logic of a theory T may depend on T as well as on the chosen formalization of the proof predicate Proof_T. We will be a bit ambiguous in this respect. When talking about 'the provability logic' of a certain theory, we will always assume that a not-to-unusual proof predicate is fixed in advance.

The famous article by Solovay (1976) may well be seen as the starting point of provability logic. In this paper Solovay proves that the the provability logic of Peano Arithmetic PA is the logic now known as L or GL, consisting of the principles $K, 4$ and L (Section 2.5), the tautologies of classical propositional logic and the rules Necessitation ($A/\Box A$) and Modus Ponens. Moreover, the proof gives a method to construct for any formula A which is not a principle of the provability logic of PA an actual counterexample A^*, that is, a realization such that PA does not prove A^*. The way A^* is obtained employs a modal completeness result for GL. First, it is shown that GL is complete with respect to the class of finite, transitive, conversely well-founded Kripke models. And second, it is shown that for every such Kripke model $\mathcal{K}$ there exists a realization * such that

for all nodes k of $\mathcal{K}$ (if $\mathcal{K}, k \not\Vdash A$, then $\neg A^*$ is consistent with PA).

This shows the usefulness of a semantical characterization of provability logics.

As mentioned before, provability logics of classical theories are well-investigated. One remarkable thing is their stability; many theories, like for example PA and ZF, have the same provability logic, GL (Solovay 1976)(Visser 1984)(Beklemishev 1990). Although for intuitionistic theories we know much less, we do know that there is in general no stability in going from a classical theory to its intuitionistic counterpart, as can be seen in the comparison of HA and PA. This will be explained in Section 2.5 on intuitionistic provability logic.

It is clear that the idea behind provability logic can be applied to any predicate which can be encoded in arithmetical theories. We will see examples of this in the next section.

2.2 Preservativity logic

In this section we introduce the notion of Σ_1-preservativity which is an extension of provability. This notion was invented by Visser (1994) and arose from the study of the admissible rules of Heyting Arithmetic HA, the constructive theory of the natural numbers (a definition of HA can be found in (Troelstra and van Dalen 1988)). It turns out that many principles of the provability logic of Heyting Arithmetic have an elegant formulation in this setting. Therefore, the questions in provability logic can better be studied in the context of preservativity.

The definition of preservativity

Let Σ_i and Π_i denote the well-known levels of the arithmetical hierarchy (Hájek and Pudlák 1991). For an arithmetical theory T and sentences φ and ψ in the language of T, φ is said to Σ_1-*preserve* ψ with respect to T, if for all Σ_1-sentences θ it holds that $T \vdash (\theta \to \varphi)$ implies $T \vdash (\theta \to \psi)$. We denote this with $\varphi \rhd_T \psi$. Since we will not consider any other forms of preservativity than Σ_1-preservativity we will, as in the title, always refer to preservativity instead.

On the modal side the notion of preservativity gives rise to a modal language $\mathcal{L}_{\rhd}$ with one binary modal operator, $\rhd$. Analogous to provability logic the preservativity logic of T is defined as the collection of $\mathcal{L}_{\rhd}$-formulas A such that $T \vdash A^*$ for any arithmetical realization *. In this context the definition of an 'arithmetical realization' is extended to cover formulas in which the preservativity symbol $\rhd$ occurs: an arithmetical realization * is a mapping from $\mathcal{L}_{\rhd}$-formulas to arithmetical formulas which commutes with the connectives and such that $(A \rhd B)^* = \mathrm{Pres}_T(\ulcorner A^* \urcorner, \ulcorner B^* \urcorner)$, where $\mathrm{Pres}_T(x, y)$ is a formula in the language of T that is the formalized version of the statement $A \rhd_T B$. Like in the case for $\mathcal{L}_{\Box}$, the formulas in $\mathcal{L}_{\rhd}$ are called modal formulas.

Clearly, preservativity is an extension of provability because we have

$$\Box_T \varphi \text{ iff } \top \rhd_T \varphi.$$

In Section 2.5 we will return to this relation with provability logic.

For *classical* theories T the notion of preservativity is equivalent to the notion of Π_1-conservativity: we have that φ Σ_1-preserves ψ if and only if $\neg\varphi$ is Π_1-conservative over $\neg\psi$. For many classical theories, for example PA, the notion of Π_1-conservativity is again equivalent to the well-investigated notion of interpretability. Therefore, for these theories the preservativity logic is known, although the notion is not studied directly but only via the equivalence with interpretability. In Section 2.4 we will discuss the connection with interpretability logic in more detail.

For constructive theories like HA, the situation is completely different. In the next section we will explain how in this setting the notion of preservativity arises in a natural way from the admissible rules and that the admissible rules play a prominent role in the provability and preservativity logic of HA. Moreover, we will see that the notion of preservativity seems to give the right view on questions in provability logic of constructive theories.

On the classical side, the first theory studied in the context of provability logic was Peano Arithmetic PA (Hájek and Pudlák 1991), the well-known theory of the natural numbers, for it is strong enough to allow the formalization of provability notions in an easy way. For the same reason the questions in intuitionistic provability logic focus on Heyting Arithmetic HA (Troelstra and van Dalen 1988), the intuitionistic counterpart of PA. In this thesis we will only consider preservativity with respect to HA.

2.3 Heyting Arithmetic

An (c.e.) axiomatization of the preservativity logic (or the provability logic) of HA is not known. However, Visser (1994) has given some principles of the preservativity logic of HA, which capture all principles of the provability logic of HA known before that time. In the following years the meaning of these principles became clear (Visser 1999)(Iemhoff 2000b) (Chapter 7). These insights in the system led us to the conjecture that it is the preservativity logic of HA. In this section we introduce the system, discuss its meaning and explain why we conjecture it to be the preservativity logic of HA.

To state the principles of the preservativity logic of HA known so far, we need the following notation. For formulas $A, B_1, \ldots, B_n$, the formula $(A)(B_1, \ldots, B_n)$ is inductively defined to be

$$
\begin{array}{lll}
(A)(B, C_1, \ldots, C_n) & \equiv_{def} & (A)(B) \vee (A)(C_1, \ldots, C_n) \\
(A)(\bot) & \equiv_{def} & \bot \\
(A)(B \wedge B') & \equiv_{def} & (A)(B) \wedge (A)(B') \\
(A)(\Box B) & \equiv_{def} & \Box B \\
(A)(B) & \equiv_{def} & (A \to B) \\
 & & B \text{ not of the form } \bot, (C \wedge C') \text{ or } \Box C.
\end{array}
$$

Note that we have $(A)(C_1, \ldots, C_n) = (A)(C_1) \vee \ldots \vee (A)(C_n)$, and that $(A)(\top) = (A \to \top)$, hence $(A)(\top) \leftrightarrow \top$.

The expression $(\cdot)(\cdot)$ is an abbreviation and not an operator, because applying it to equivalent formulas does not give equivalent results. For example, $\Box p$ is equivalent to $(\top \to \Box p)$, but $(A)(\top \to \Box p) = (A \to (\top \to \Box p))$ and $(A)(\Box p) = \Box p$. Hence the formulas $(A)(\top \to \Box p)$ and $(A)(\Box p)$ are in general not equivalent.

In (Visser 1994) the following principles of the preservativity logic of HA known so far are given. In fact, we give here a slightly different axiomatization then the one used by Visser. In Chapter 3 we will see that Visser's system is equivalent to the one introduced here. We denote intuitionistic propositional logic with IPC (Troelstra and van Dalen 1988). Recall that φ is provable if and only if $\top$ preserves φ. This accounts for the definition of $\Box$ in the system.

Principles of the preservativity logic of HA

	$\Box A \equiv_{def} \top \rhd A$
Taut	all tautologies of IPC
$P1$	$A \rhd B \wedge B \rhd C \to A \rhd C$
$P2$	$C \rhd A \wedge C \rhd B \to C \rhd (A \wedge B)$
Dp	$A \rhd C \wedge B \rhd C \to (A \vee B) \rhd C$ (Disjunctive Principle)
$4p$	$A \rhd \Box A$
Lp	$(\Box A \to A) \rhd A$ (Löb's Preservativity Principle)
Mp	$A \rhd B \to (\Box C \to A) \rhd (\Box C \to B)$ (Montagna's Principle)
Vp_n	$(\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_{n+1} \vee A_{n+2}) \rhd (\bigwedge_{i=1}^{n} A_i \to B_i)(A_1, \ldots, A_{n+2})$ (Visser's Principles)
Vp	$Vp_1, Vp_2, Vp_3, \ldots$ (Visser's Scheme)

We use the name iPH for the logic given by these principles and the rules Modus Ponens and the

Preservation Rule if $\vdash (A \to B)$ then $\vdash A \rhd B$.

In Sections 2.4 and 2.5 we discuss the relation between the logic iPH and interpretability and provability logic. In Section 2.4 we will see that all principles except the Disjunctive Principle hold for PA as well. In Section 2.7 we repeat the proofs by Visser (1994) that these principles and rules belong to the preservativity logic of HA. Visser's Scheme is a special and complicated scheme. In Section 3.2 we elaborate on the technical details of this scheme and show that our formulation of the scheme is equivalent to the one used by Visser (1994).

Here we discuss the meaning of the given principles. We will see that these principles form a natural fragment of the preservativity logic of HA. Namely, each of them corresponds to either a principle of the provability logic of PA or to one of the following characteristic properties of HA: its propositional admissible rules, Markov's Rule and the Disjunction Property.

The definition of $\Box$ and the first two principles are easily seen to be principles of the preservativity logic of HA. The principles $4p$ and Lp resemble the two characteristic axioms for the provability logic of PA, which are

4 $\quad \Box A \to \Box\Box A$

L $\quad \Box(\Box A \to A) \to \Box A.$

Since $A \rhd B$ implies $(\Box A \to \Box B)$ in the system (Section 3.1), the principles $4p$ and Lp imply their provability counterparts 4 and L. The principle 4 is derivable from L, but usually it is still included in the axioms. We will see that in the same way $4p$ is derivable from Lp (Section 4.3). The principle Mp is baptized after its classical counterpart in interpretability logic, which is discussed below. It is easy to see that it belongs to the preservativity logic of HA, using the fact that the arithmetical realization of a formula $\Box C$ is always Σ_1 (Section 2.7).

The Disjunctive Principle and the Disjunction Property

The Disjunctive Principle Dp is related to the Disjunction Property of HA, which reads

(*Disjunction Property*) if $\mathsf{HA} \vdash \varphi \vee \psi$, then $\mathsf{HA} \vdash \varphi$ or $\mathsf{HA} \vdash \psi$.

Friedman (1975) proved that HA does not prove its Disjunction Property, i.e. HA does not derive the true formula $\Box(\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)$. Leivant (1975) showed that HA does prove the weaker version

$$\mathsf{HA} \vdash \Box(\varphi \vee \psi) \to \Box(\varphi \vee \Box\psi).$$

Hence the so-called Leivant Principle $\Box(A \vee B) \to \Box(A \vee \Box B)$ is part of the provability logic of HA. In the preservativity logic of HA this principle occurs as a consequence of the two principles $4p$ and Dp. Note that the fact that Dp and $4p$ are in the preservativity logic of HA imply the following strengthening of Leivant's Principle:

$$\mathsf{HA} \vdash (\varphi \vee \psi) \rhd (\varphi \vee \Box\psi).$$

The arithmetical validity of the Disjunctive Principle was shown by Visser (1994) and will be treated in Section 2.7.

Visser's Scheme and the admissible rules

The scheme Vp is called after A. Visser who proved its arithmetical validity (Visser 1994). Note that it is not a principle but a collection of infinitely many principles. They describe (some) admissible rules of HA. For propositional formulas A, B we say that the rule A/B is a *propositional admissible rule of* HA if $\mathsf{HA} \vdash \sigma A$ implies $\mathsf{HA} \vdash \sigma B$, for all substitutions σ which replace the propositional variables by arithmetical formulas. Observe that if $(\Box A \to \Box B)$ is in the provability logic of HA this implies that A/B is an admissible rule of HA. Since $A \rhd B$ implies $(\Box A \to \Box B)$, it follows that if $A \rhd B$ is in the preservativity logic of HA, then A/B is an admissible rule for HA. The two most meaningful instances of Vp describe the propositional admissible rules and Markov's Rule for HA. We will discuss them briefly.

If one restricts Visser's Scheme to pure propositional formulas, i.e. without $\Box$ or $\rhd$, it characterizes the propositional admissible rules of HA, as will be proved in

Chapter 7. There we will see that if we let AR be the logic given by the principles *Taut*, *P*1, *P*2, *Dp*, *Vp* and the Preservation Rule, then we have

for propositional formulas A, B:
A/B is a propositional admissible rule of HA iff $\mathsf{AR} \vdash A \rhd B$.

This will be explained in more detail in Section 2.3.1. Here we consider one example. It is well-known that

$$\neg A \to B \vee C/(\neg A \to B) \vee (\neg A \to C)$$

is an admissible rule of HA (Harrop 1960). We show how we can derive the corresponding statement

$$(\neg A \to B \vee C) \rhd ((\neg A \to B) \vee (\neg A \to C))$$

in the system AR:

$$\begin{array}{lll}
\vdash_{\mathsf{AR}} & (\neg A \to B \vee C) \rhd (\neg A)(A, B, C) \quad (Vp) & (1) \\
& (\neg A)(A, B, C) \leftrightarrow (\neg A \to A) \vee (\neg A \to B) \vee (\neg A \to C) & (2) \\
& (\neg A \to A) \to \neg\neg A \qquad (Taut) & (3) \\
& \neg\neg A \to (\neg A \to B) \qquad (Taut) & (4) \\
& (\neg A \to A) \vee (\neg A \to B) \vee (\neg A \to C) \to & \\
& \quad (\neg A \to B) \vee (\neg A \to C) \qquad (3)(4)(Taut) & (5) \\
& (\neg A \to A) \vee (\neg A \to B) \vee (\neg A \to C) \rhd & \\
& \quad (\neg A \to B) \vee (\neg A \to C) \qquad (5)(Preservation Rule) & (6) \\
& (\neg A \to B \vee C) \rhd ((\neg A \to B) \vee (\neg A \to C)). \quad (1)(2)(6)(P1) &
\end{array}$$

This shows that the admissible rule given above is captured by Visser's Scheme.
Markov's Rule, a well-known rule for HA, reads

(*Markov's Rule*) for all $\varphi \in \Pi_2$: if $\mathsf{HA} \vdash \neg\neg\varphi$, then $\mathsf{HA} \vdash \varphi$.

To see how Markov's Rule is captured by Visser's Scheme, observe that the following formula is one of the consequences of Visser's Scheme,

$$\neg\neg\Box A \rhd \Box A. \tag{2.1}$$

Namely, $\neg\neg\Box A$ is short for $((\Box A \to \bot) \to \bot)$, and by Visser's Scheme

$$\begin{aligned}
&((\Box A \to \bot) \to \bot) \rhd (\Box A \to \bot)(\Box A, \bot) = \\
&\quad (\Box A \to \bot)(\Box A) \vee (\Box A \to \bot)(\bot) = (\Box A \vee \bot) \leftrightarrow \Box A.
\end{aligned}$$

Now (2.1) implies that HA proves the arithmetical realizations of the formula $(\Box\neg\neg\Box A \to \Box\Box A)$, which is a partial formalization of Markov's Rule. Thus the fact that (2.1) is in the preservativity logic of HA implies that HA proves Markov's Rule: $\mathsf{HA} \vdash (\Box\neg\neg\Box A \to \Box\Box A)$.

We saw that Visser's Scheme describes admissible rules of HA and considered various consequences of it. In Section 4.6 we will discuss more instances of Visser's Scheme. We will return to the correspondence between preservativity logic and admissible rules in Section 2.3.1.

Summarizing we could say that the preservativity logic presented by Visser (1994) seems a very natural part (if not all) of the preservativity logic of HA. It contains three basic principles, $P1$, $P2$ and Montagna's Principle, which arithmetical validity is trivial. It contains the (preservativity form of the) two characteristic principles of the provability logic of PA, namely $4p$ and Lp. And it contains two axioms, the Disjunctive Principle and Visser's Scheme, which are directly related to three well-known properties of HA: the Disjunction Property, Markov's Rule and the propositional admissible rules.

2.3.1 Three fragments

Although the preservativity logic of HA is not known, for three of its fragments there exists a decent axiomatization: for its propositional fragment, for the closed fragment of the provability logic and for that part of the preservativity logic that is connected with the admissible rules of HA.

The characterization of the propositional fragment

Recall that σ ranges over substitutions which replace propositional variables by arithmetical formulas, and that IPC denotes intuitionistic propositional logic. It was shown by de Jongh (1982) that

for all propositional A: $\forall\sigma(\mathsf{HA} \vdash \sigma A)$ iff $\mathsf{IPC} \vdash A$.

Note that for propositional formulas, an arithmetical realization A^* is nothing more than a substitution instance σA, for some σ. Therefore, we have

for all propositional A: A is in the provability logic of HA iff $\mathsf{IPC} \vdash A$.

This means that the propositional fragment of the provability logic of HA is equivalent to IPC.

The characterization of the closed fragment

Visser (1994) described the closed fragment of the provability logic of HA. This is the fragment without propositional variables. He shows that for every formula φ in the closed fragment there exists a number $n > 0$ such that

$$\mathsf{HA} \vdash \Box\varphi \leftrightarrow \Box^n\bot.$$

This resembles the situation for PA, where every formula in the closed fragment is a boolean combination of formulas $\Box^n\bot$, $\top$ and $\bot$.

The characterization of the admissible rules

The other fragment of the preservativity logic of HA that is axiomatized describes its propositional admissible rules. As mentioned in Section 2.2, if $A \rhd B$ is in the provability logic of HA, then A/B is an admissible rule of HA. The combination of results by Visser (1994)(1998) and results in part II of this thesis imply that the converse holds too: for all propositional formulas A, B we have that

A/B is a propositional admissible rule of HA iff

$A \rhd B$ is in the preservativity logic of HA.

In part II (Chapter 7) of this thesis we give an axiomatization of the propositional admissible rules of HA. In particular, we construct a perspicuous preservativity logic AR such that

$$A/B \text{ is a propositional admissible rule of } \mathsf{HA} \text{ iff } \mathsf{AR} \vdash A \rhd B. \tag{2.2}$$

This logic is axiomatized by the preservativity principles (Section 2.2) $P1$, $P2$, Dp and all the instances $A \rhd B$ of Vp, where A and B are propositional formulas. In combination with Visser's (1994) result that states that all these principles belong to the preservativity logic of HA, we arrive at the following axiomatization:

for propositional A, B: (2.3)

$A \rhd B$ is in the preservativity logic of HA iff $\mathsf{AR} \vdash A \rhd B$.

This completes our discussion on the three fragments of the preservativity logic of HA for which we have a c.e. axiomatization.

There are two other aspects of (2.2) worth noting. First, it shows that HA *proves* the admissibility of every instance of its propositional admissible rules:

for propositional A, B:

$\forall\sigma(\mathsf{HA} \vdash \sigma A$ implies $\mathsf{HA} \vdash \sigma B)$ iff (by definition)

A/B is an admissible rule of HA iff (by (2.3) (2.2))

$(\Box A \to \Box B)$ is in the provability logic of HA iff (by definition)

$\forall\sigma(\mathsf{HA} \vdash \Box\sigma A \to \Box\sigma B)$.

And second, from (2.2) it follows that

for propositional A, B: $A \rhd B$ is in the preservativity logic of HA iff

$(\Box A \to \Box B)$ is in the provability logic of HA.

In Section 5.2 of part II we will see that this actually holds for all formulas $(\Box A \to \Box B)$ of which we know that they are in the preservativity logic of HA. Observe that for example for classical provability principles of the form $(\Box A \to \Box B)$, like Löb's Principle $\Box(\Box A \to A) \to \Box A$, we already saw that the stronger $A \rhd B$ holds as well. Of course, the rule does not hold for all arithmetical formulas, as we will see in Section 3.1.

2.4 Interpretability logic

In this section we explain the connection between preservativity logic and interpretability logic. A theory T is Π_1-conservative over T' if T' proves all the Π_1-formulas that T proves. From the definition of Σ_1-preservativity it follows that for classical theories Σ_1-preservativity is equivalent to Π_1-conservativity, in the sense that $\varphi \rhd_T \psi$ if and only if $\neg\varphi$ is Π_1-conservative over $\neg\psi$. For theories that are classical c.e. extensions of PA, this again is equivalent to interpretability (Orey 1961)(Hájek 1971, Hájek 1972). We will not define interpretability here, but remark only that intuitively, 'φ interprets ψ' means that we can define a model for (a translation of) the theory T plus ψ in the theory T plus φ. Thus for PA we have (in PA) that $\varphi \rhd_{\mathsf{PA}} \psi$ if and only if PA plus $\neg\psi$ interprets PA plus $\neg\varphi$.

In a similar manner as for preservativity one can define the *interpretability logic* of a theory. We denote 'A interprets B' by $A \rhd_i B$ (in the literature this is denoted by $A \rhd B$). Interpretability logic has been extensively studied (Shavrukov 1988)(Berarducci 1990)(de Jongh and Veltman 1990)(Zambella 1992) (Visser 1997). The interpretability logic of PA is known to be ILM (a definition follows below). Since for PA the notions of preservativity and interpretability are the same, it seems natural to ask which principles of ILM are inherited by HA. That is, if we reformulate ILM in terms of preservativity by replacing $\neg A \rhd_i \neg B$ by $A \rhd B$, which of the principles belong to the preservativity logic of HA?

As we will see, under this translation all axioms of ILM are provable in iPH. Here follow the axioms of ILM. With every axiom we give its preservativity translation. The diamond $\Diamond$ denotes $\neg\Box\neg$.

L	$\Box(\Box A \to A) \to \Box A$	L
$J1$	$\Box(A \to B) \to A \rhd_i B$	$\Box(A \to B) \to A \rhd B$
$J2$	$A \rhd_i B \wedge B \rhd_i C \to A \rhd_i C$	$P1$
$J3$	$A \rhd_i C \wedge B \rhd_i C \to (A \vee B) \rhd_i C$	$P2$
$J4$	$A \rhd_i B \to (\Diamond A \to \Diamond B)$	$A \rhd B \to (\Box A \to \Box B)$
$J5$	$\Diamond A \rhd_i A$	$4p$
M	$A \rhd_i B \to (A \wedge \Box C) \rhd_i (B \wedge \Box C)$	Mp

(The rules of ILM are Modus Ponens and Necessitation.) In Section 3.1 we will see that the translations of $J1$ and $J4$ belong to iPH. Therefore, clearly all translations of ILM belong to iPH.

The converse, i.e. the statement that all translation of axioms of iPH belong to ILM, does not hold. Namely, the translation of Dp, which is $C \rhd_i A \wedge C \rhd_i B \to C \rhd_i (A \wedge B)$, is not valid for PA. It is easy to see that the translations of Vp_n and Lp are derivable in PA. Therefore, the only axiom in iPH which does not hold for classical theories is the Disjunctive Principle.

2.5 Intuitionistic provability logic

In this section we explain what is known so far about intuitionistic provability logic and summarize its history.

On the classical side provability logics are well-investigated. On the intuitionistic side we know much less. As stated earlier, it is not known what the provability logic of HA is. The first principles known for this logic were

K	$\Box(A \to B) \to (\Box A \to \Box B)$	
4	$\Box A \to \Box\Box A$	
L	$\Box(\Box A \to A) \to \Box A$	(Löb's Principle)
Le	$\Box(A \vee B) \to \Box(A \vee \Box B)$	(Leivant's Principle)
Ma	$\Box\neg\neg(\Box A \to \bigvee \Box B_i) \to \Box(\Box A \to \bigvee \Box B_i)$	(Formalized Markov Scheme)

We use the name iH for the logic given by these principles and the rules Modus Ponens and the

Necessitation Rule if $\vdash A$ then $\vdash \Box A$.

In Section 3.1 we show that all these principles and rules are derivable in the preservativity logic iPH discussed in Section 2.3, and that the latter contains principles not captured by iH. This disproves the conjecture that iH is the provability logic of HA.

The first three principles axiomatize the provability logic GL of PA, discussed in Section 2.1. Recall (Section 2.2) that Leivant's Principle is related to the Disjunction Property of which the formalized version is not provable in HA.

For the Formalized Markov Scheme as such there is no proof in the literature of its arithmetical validity. Visser (1981) showed that $\Box\neg\neg\Box A \to \Box\Box A$ belongs to the provability logic of HA. From this proof it is not difficult to infer that then also the Formalized Markov Scheme is in the provability logic of HA. This scheme is the partial formalization of Markov's Rule for HA:

for all $\varphi \in \Pi_2$: if $\mathsf{HA} \vdash \neg\neg\varphi$, then $\mathsf{HA} \vdash \varphi$.

Clearly any arithmetical realization of formulas of the form $(\Box A \to \bigvee \Box B_i)$ is Π_2. Arithmetical realizations of formulas $(\bigvee \Box A_i \to \bigvee \Box B_i)$ are Π_2 too. Note that

$$\Box\neg\neg(\bigvee \Box A_i \to \bigvee \Box B_i) \to \Box(\bigvee \Box A_i \to \bigvee \Box B_i)$$

is derivable from the Formalized Markov Scheme. As all formulas not equivalent to a formula of the form $(\bigvee \Box A_i \to \bigvee \Box B_i)$ have arithmetical realizations that are not Π_2, the Formalized Markov Scheme is all we can capture of Markov's Rule in provability logic.

As long as we stay on the classical side (truth) provability logics are very stable; many arithmetical theories have the same provability logic, namely GL (Section 2.1). However, there is in general no stability in going from a classical theory to its intuitionistic counterpart, as can be seen in the comparison of HA and PA. For example, Leivant's Principle $\Box(A \vee B) \to \Box(A \vee \Box B)$ is not a provability principle of PA. This can be seen easily. If GL would derive the Leivant Principle it would also derive $\Box(\Box\bot \vee \Box\neg\Box\bot)$, as it clearly derives $\Box(\Box\bot \vee \neg\Box\bot)$. But then an application of L shows that it would derive $\Box\Box\bot$. Hence the provability logic of HA is not a part of GL. The converse is not true either. The principle $(p \vee \neg p)$ is a theorem of the provability logic of PA, but not of the corresponding logic of HA. Note that this also shows that there is no monotonicity (converse monotonicity) in provability logics; stronger theories do not necessarily have stronger (weaker) provability logics.

In the context of intuitionistic logic the notion of intuitionistic truth provability logic is less natural, because the intuitionistic notion of truth is much more complex. Therefore, we will in the sequel only discuss provability logic. But let us note in passing that $\Box(A \vee B) \to \Box A \vee \Box B$ is an example of a principle that is in the truth provability logic of HA but not in the provability logic of HA.

History

The history of intuitionistic provability logic does not reach far back. The first results in this area come from Friedman (1975) and Leivant (1975). As mentioned above, Friedman showed that HA does not prove the formalized version of its Disjunction Property, and Leivant showed that the slightly weaker version $\Box(\varphi \vee \psi) \to \Box(\varphi \vee \Box\psi)$ is part of the provability logic of HA. Another related result is from Gargov (1984). He has shown that if a c.e. extension of HA has the Disjunction Property then so does its provability logic. Sambin (1976) proved a fixed point theorem for the diagonalizable algebras of intuitionistic theories, which were also studied by Ursini (1979a).

Then there is some work on the algebraic and on the frame characterization of the principles K, 4 and L: Ursini (1979b) and Kirov (1984) both show the completeness and the finite model property of iL, and so do Božić and Dŏsen (1984) for iK, and Wolter and Zakharyaschev (1999b) for $\mathsf{iK4}$.

Of a more arithmetical nature is the paper by Visser (1982). Here he gives some principles of the provability logic of HA, among which the one on the cover of his thesis (Visser 1981): $\Box(\neg\neg\Box A \to \Box A) \to \Box\Box A$. All the principles mentioned there are derivable from principles he found later (Visser 1994); they belong to the logic iPH.

The closed fragment of GL and of the provability logic of HA have also been studied. Kirov (1990) shows that the closed fragment of GL is complex in the sense that any free Heyting algebra with countably many generators can be embedded in the algebra of this fragment. An inspection of the proof shows that this still holds for iPH. As mentioned before, Visser (1994) characterizes the closed fragment of the provability logic of HA by showing that for every formula φ in this fragment there exists a number $n > 0$ such that

$$\mathsf{HA} \vdash \Box\varphi \leftrightarrow \Box^n\bot.$$

Finally, there is the introduction to preservativity logic by Visser (1994)(1998) discussed in Section 2.2 and Chapter 6. Related work has been done by de Jongh and Visser (1996), who studied which c.e. Heyting algebras can be embedded in the Heyting algebras of IPC or HA.

2.6 The main roads in intuitionistic modal logic

In this section we introduce intuitionistic modal logic in an informal way, and refer briefly to the different ways in which it has been studied in the literature. The account here is only historical. In Section 3.4 we introduce the modal logics used in this thesis in more detail.

Intuitionistic modal logic is modal logic on an intuitionistic basis. This means that an intuitionistic modal logic is a logic in the language of propositional logic extended with modal operators, that contains IPC. Thus the provability and preservativity logics introduced in the previous sections are modal logics. Intuitionistic polymodal logics have hardly been considered in the literature. Probably this is due to the fact that in the presence of an intuitionistic basis a monomodal logic is almost a bimodal logic; compare the Gödel translation of intuitionistic logic into S4 (Gödel 1933). Most of the logics deal with $\Box$ as well as $\Diamond$, which in general are not interdefinable in an intuitionistic setting. From the point of view of provability logic it is still not clear what a natural interpretation of $\Diamond$ should be. Therefore, in our case we only consider $\Box$ (and $\rhd$).

In the literature on intuitionistic modal logic one often encounters logics of which it is claimed that they are the 'true' intuitionistic counterparts of some classical modal logic, for example Löb's logic. Of course, what one will accept as an intuitionistic counterpart of a given (classical) logic, will depend on the interpretation one has in mind for the modal operators, hence on the properties one wants it to have. In this thesis we always have the provability/preservativity interpretation in mind. A striking difference between this interpretation and most others is that it is in *itself* of a mathematical nature. Thus verification of the validity of principles can be executed in a *formal* rigorous way.

Different interpretations of $\Box$ lead to different modal logics. In the literature there have been three prominent perspectives. Prior (1957) first proposed an axiomatization of a modal logic which corresponds to the monadic fragment of intuitionistic predicate logic, by replacing $\Box$, $\Diamond$ and p_i by respectively $\forall x$, $\exists x$ and

$P_i(x)$ (Bull 1965)(Bull 1966)(Ono 1977)(Bezhanishvili 1998) (Bezhanishvili 1999). Then there are many studies on intuitionistic modal logics whose modal axioms are equivalent to that of a well-known classical system, like K, L, $\mathsf{S4}$ or $\mathsf{S5}$. They contain various possible proof systems (Bierman, Meré and de Paiva 1997) (Simpson 1993), or different possible semantics and completeness results (Fischer-Servi 1977)(Ursini 1979b)(Vakarelov 1981)(Božić and Dŏsen 1984)(Sotirov 1984) (Simpson 1993)(Wolter and Zakharyaschev 1997)(Wolter and Zakharyaschev 1999b). Fischer-Servi (1977) and Wolter and Zakharyaschev (1999a) also formulate criteria for being the intuitionistic analogue of a classical modal logic. For example, following the definition of Servi, it is not difficult to see that iL is the intuitionistic counterpart of GL. Vakarelov (1981) also shows that above iK there are a continuum of strongly intuitionistic modal logics, i.e. consistent logics that are incompatible with the law of excluded middle. Note that for proper intermediate logics there are none (Rasiowa and Sikorski 1963). Observe that the logic axiomatized by L and Le over iK is only strongly intuitionistic in the weaker sense that it derives $\Box\Box\bot$, see Section 2.5. Modal logics motivated by computer science often turn out to be weaker than iK (Sotirov 1984)(Plotkin and Stirling 1986)(Wijesekera 1990), as do the logics in which the modal operators are viewed as new intuitionistic connectives (Gabbay 1977).

As can be seen from this brief summary of the literature, principles like Le or Vp do not occur, because they neither have a classical counterpart nor do they arise in a natural way from the mentioned interpretations. Thus, looking through the spectacles of provability logic one finds surprising intuitionistic modal logics. Moreover, also on the semantical side certain new possibilities become visible. Besides many other semantics, frame semantics occurs in many of the articles mentioned above. This semantics, defined in Section 3.4.2, consist of a combination of the intuitionistic and the modal frame semantics. That is, frames are sets with two relations: a partial order $\preccurlyeq$ (the intuitionistic relation) and a binary relation R (the modal relation). In the presence of only the modal operator $\Box$ the canonical frames (Section 3.4.6) satisfy $(R;\preccurlyeq) \subseteq R$. Thus it seems harmless to demand this property for frames. However, as we will see in Proposition 4.4.2, some principles can have incompatible frame characterizations with respect to the classes of frames with or without this property. Here again we encounter a deviation from the regular literature on intuitionistic modal logic.

2.7 Arithmetical validity

In this section we prove that the principles and rules given in Section 2.2 and Section 2.5 indeed belong to the preservativity logic of HA. Therefore, the latter belong to the provability logic of HA as well. The main proofs are the one for the Disjunctive Principle and the one for Visser's Scheme. For the principles *Taut*, $P1$, $P2$ and Montagna's Principle, these proofs are rather trivial. For the characteristic axioms and rules of the provability logic of PA, namely the principles

K, 4 and Löb's Principle and the Necessitation Rule, the proofs are analogous to the corresponding proofs for PA. Therefore, we do not include them but refer to the literature instead. The proofs for $4p$ and Löb's Preservativity Principle follow easily from the ones for 4 and Löb's Principle. Finally, we treat the principles that are related to the Disjunction Property and the admissible rules of HA (Section 2.2), the Disjunctive Principle, Visser's Scheme and the Formalized Markov Scheme. For the first two we repeat the proofs by Visser (1994). Then we show that the proof for the last one follows from the fact that Montagna's Principle and Visser's Scheme belong to the preservativity logic of HA.

Note that the fact that a principle belongs to the preservativity logic of HA implies that it is arithmetically valid, i.e. the principle holds for HA. However, it shows more, namely it shows that HA can also prove this fact.

In this section, we write $\Box$ for $\Box_{\mathsf{HA}}$, and similarly for $\rhd$ and $\vdash$. We will use various properties of HA that hold for PA as well, for example the fact that HA proves $(\theta \to \Box\theta)$ for every Σ_1-formula θ. We have not included the proofs of these facts, but will refer to the similar proofs for PA in (Hájek and Pudlák 1991) instead. We write $\Gamma \vdash_m \varphi$ for a derivation, in HA, of φ from Γ, that uses the finitely many axioms of $\mathsf{I\Delta_0} + \mathsf{EXP}$ plus the axioms of HA which Gödelnumber is smaller than m. Similarly for $\Box_m$. The reason that we include $\mathsf{I\Delta_0} + \mathsf{EXP}$ is that this system is strong enough to allow all coding tricks explained in Section 2.1.

2.7.1. Proposition.

(*i*) The principles K, 4 and Löb's Principle belong to the provability logic of HA (and hence to its preservativity logic as well).

(*ii*) Modus Ponens, the Necessitation Rule and the Preservation Rule are rules of the preservativity logic of HA (and hence Modus Ponens and the Necessitation Rule are rules of the provability logic of HA as well).

(*iii*) The principles *Taut*, $P1$, $P2$ and Montagna's Principle belong to the preservativity logic of HA (and hence *Taut* belongs to its provability logic as well).

Proof (*i*) The proofs that K, 4 and Löb's Principle belong to the provability logic of HA are similar to the ones for PA, see for example (Smoryński 1985).

(*ii*) It is trivial that Modus Ponens is a rule of the preservativity logic of HA, because it is a rule of the logic of HA. The proof that HA satisfies the Necessitation Rule, if $\mathsf{HA} \vdash \varphi$ then $\mathsf{HA} \vdash \Box\varphi$, is similar to the one for PA, see (Smoryński 1985). The fact that HA satisfies the Preservation Rule, if $\mathsf{HA} \vdash (\varphi \to \psi)$ then $\mathsf{HA} \vdash \varphi \rhd \psi$, follows almost immediately. Suppose $\mathsf{HA} \vdash (\varphi \to \psi)$. Hence by Necessitation Rule we have $\mathsf{HA} \vdash \Box(\varphi \to \psi)$. It is easy to see that this implies $\mathsf{HA} \vdash \varphi \rhd \psi$. In the next chapter, Section 3.1, we will see that there is an equivalent formulation of preservativity logic for which the Preservation Rule is replaced by the Necessitation Rule.

(*iii*) The statement that *Taut* belongs to the preservativity logic of HA is trivial, since the logic of HA is intuitionistic predicate logic, which contains IPC. The proofs for $P1$ and $P2$ are left to the reader. For Montagna's Principle, consider formulas A, B, C and an arithmetical translation *. We have to show that

$$\mathsf{HA} \vdash A^* \rhd B^* \to (\Box C^* \to A^*) \rhd (\Box C^* \to B^*).$$

Recall that the arithmetical realization of $\Box C$ is a Σ_1-formula (Section 2.1). Therefore, it suffices to show that for all arithmetical formulas φ, ψ we have

$$\text{for all } \Sigma_1\text{-formulas } \theta\text{: } \mathsf{HA} \vdash \varphi \rhd \psi \to (\theta \to \varphi) \rhd (\theta \to \psi). \tag{2.4}$$

In fact, HA even proves: for all Σ_1-formulas θ, $\varphi \rhd \psi$ implies $(\theta \to \varphi) \rhd (\theta \to \psi)$. As we do not need this stronger statement, we prove the weaker (2.4) instead. We use that if a formula is Σ_1, then HA proves this fact, and that HA proves that Σ_1-formulas are closed under conjunction. These properties of HA are proved in a similar way as for PA, see (Hájek and Pudlák 1991).

The proof of (2.4) runs as follows. Let θ be a Σ_1-formula. Reason in HA. Suppose $\varphi \rhd \psi$. We have to prove that for all Σ_1-formulas θ', if $\vdash (\theta' \to (\theta \to \varphi))$ holds, then $\vdash (\theta' \to (\theta \to \psi))$ holds as well. Therefore, suppose $\vdash (\theta' \to (\theta \to \varphi))$, for some Σ_1-formula θ'. Note that $(\theta' \to (\theta \to \varphi))$ is equivalent to $(\theta' \wedge \theta \to \varphi)$. Thus by Necessitation Rule (*ii*) and the axiom K (*i*), we also have that $\vdash (\theta' \to (\theta \to \varphi))$ is equivalent to $\vdash (\theta' \wedge \theta \to \varphi)$. The conjunction of two Σ_1-formulas is a Σ_1-formula, and whence $(\theta' \wedge \theta)$ is a Σ_1-formula. Therefore, by $\varphi \rhd \psi$, $\vdash (\theta' \wedge \theta \to \varphi)$ implies $\vdash (\theta' \wedge \theta \to \psi)$. The latter is again equivalent to $\vdash (\theta' \to (\theta \to \psi))$, which completes the proof. □

2.7.2. Proposition. The principle $4p$ and Löb's Preservativity Principle belong to the preservativity logic of HA.

Proof In Section 4.3 we show that Löb's Preservativity Principle derives the principle $4p$. Therefore, it suffices to show that Löb's Preservativity Principle is a principle of the preservativity logic of HA. We use the well-known fact that, like PA, HA proves Σ_1-completeness i.e. HA proves that for every Σ_1-formula θ we have $\mathsf{HA} \vdash (\theta \to \Box\theta)$. A proof for PA, which is analogous to the one for HA, can be found in (Hájek and Pudlák 1991).

Reason in HA. If for some $\theta \in \Sigma_1$ we have $\vdash (\theta \to (\Box\varphi \to \varphi))$, then we also have $\vdash (\Box\theta \to \Box(\Box\varphi \to \varphi))$ by the Necessitation Rule and the axiom K (Proposition 2.7.1). Since $(\theta \to \Box\theta)$ by Σ_1-completeness, also $\vdash (\theta \to \Box(\Box\varphi \to \varphi))$. Applying Löb's Principle (Proposition 2.7.1) gives $\vdash (\theta \to \Box\varphi)$. Thus by assumption also $\vdash (\theta \to \varphi)$. □

The proofs (Visser 1994) that the Disjunctive Principle and Vissers's Scheme belong to the preservativity logic of HA are related but not similar. This difference is not surprising, as in contrast to the Disjunctive Principle, Visser's Scheme is classically valid, i.e. it belongs to the preservativity logic of PA (see Section 2.3). Before giving those proofs, we will briefly sketch the ideas behind them. They both use a translation on formulas by D. de Jongh. Translations, like for example realizability or the Friedman translation, are a much used tool in meta proofs for constructive theories. In such proofs one often shows that if a formula is derivable then the translation of that formula is also derivable. In our case, we proceed in a similar way. We construct some kind of Σ_1-approximations to the formulas involved, and use the de Jongh translation to show that if the original formula is derivable then these Σ_1-approximations have the desired properties.

In the case of the Disjunctive Principle we have to show, in HA, that if $\varphi \rhd \chi$ and $\psi \rhd \chi$ hold, then also $(\varphi \vee \psi) \rhd \chi$. Thus we have to prove, in HA, that for all Σ_1-formulas θ with $\vdash (\theta \to (\varphi \vee \psi))$, we have $\vdash (\theta \to \chi)$. It suffices to show that for every Σ_1-formula θ with $\vdash (\theta \to (\varphi \vee \psi))$, we can find Σ_1-formulas θ_i, the disjunction of which is implied by θ, and such that $\vdash (\theta_1 \to \varphi)$ and $\vdash (\theta_2 \to \psi)$. Namely, in that case $\varphi \rhd \chi$ implies $\vdash (\theta_1 \to \chi)$ and similarly for ψ.

We will see that for some m, we can take the formulas $\Box_m \varphi$ and $\Box_m \psi$ for θ_i:

(i) for Σ_1-formula θ: $\vdash (\theta \to (\varphi \vee \psi))$ implies $\vdash (\theta \to \Box_m \varphi \vee \Box_m \psi)$

(ii) $\vdash (\Box_m \varphi \to \varphi)$ and $\vdash (\Box_m \psi \to \psi)$.

Only one of these statements has to do with the constructive properties of HA. Namely, (ii) holds for PA as well, while (i) does not. For the latter, this is easy to see. Consider the case $\psi = \neg\varphi$ and $\theta = \top$. Then (i) would show that for all φ, PA derives $\Box_m \varphi \vee \Box_m \neg\varphi$, a fact which is not even true. However, we will see that in the context of HA both properties hold and this will complete the proof.

As mentioned before, Vissers's Scheme is classically valid, and we will see that PA occurs in the proof that Vissers's Scheme belongs to the preservativity logic of HA. Namely, we use the well-known fact that PA is Π_2-conservative over HA, and that HA proves this fact (Friedman 1977). To explain the idea of this proof, consider the following instance of Visser's Scheme:

$$((\varphi_1 \to \psi) \to \varphi_2) \rhd ((\varphi_1 \to \psi)(\varphi_1) \vee (\varphi_1 \to \psi)(\varphi_2)).$$

We have to show, in HA, that for all Σ_1-formulas θ it holds that

$$\vdash \theta \to ((\varphi_1 \to \psi) \to \varphi_2)) \text{ implies } \vdash \theta \to (\varphi_1 \to \psi)(\varphi_1) \vee (\varphi_1 \to \psi)(\varphi_2).$$

We consider only the case that $\theta = \top$. Therefore, suppose

$$\vdash ((\varphi_1 \to \psi) \to \varphi_2)). \tag{2.5}$$

We have to show that

$$\vdash (\varphi_1 \to \psi)(\varphi_1) \vee (\varphi_1 \to \psi)(\varphi_2). \tag{2.6}$$

Note that in the case that the formulas φ_i are not of the form $\Box\varphi'$, the fact that (2.5) implies (2.6), expresses a well-known admissible rule of HA. Therefore, the following proof sketch shows that HA recognizes this admissible rule (compare the part of Section 2.2 on Visser's Scheme).

Since HA is part of PA, the latter derives (2.5) too. By classical reasoning it follows that

$$\mathsf{PA} \vdash (\varphi_1 \vee \varphi_2).$$

If $(\varphi_1 \vee \varphi_2)$ is Π_2 then by the mentioned Π_2-conservativity of PA over HA, we can conclude that HA derives this formula. This already explains the instance of Visser's Scheme for which the formulas φ_i are of the form $\varphi = \Box\varphi'$, and hence Σ_1.

However, the formula $(\varphi_1 \vee \varphi_2)$ is not Π_2 in general. Therefore, we have to find some kind of Σ_1-approximation of φ_i, which means a Σ_1-formula φ_i' such that $(\varphi_i \to \varphi_i')$ and $(\varphi_i' \to (\varphi_1 \to \psi)(\varphi_i))$. Namely, in that case (2.5) implies that $\vdash ((\varphi_1' \to \psi) \to \varphi_2')$. And the same reasoning as above shows that PA derives $(\varphi_1' \vee \varphi_2')$. Since this is a Π_2-formula, by Π_2-conservativity we can conclude that $\mathsf{HA} \vdash (\varphi_1' \vee \varphi_2')$. Using the other property of φ_i' we arrive at the desired conclusion (2.6).

As we will see, these formulas φ_i' actually do not have the property $(\varphi_i \to \varphi_i')$. However, using the de Jongh translation we can show that (2.5) implies that for some ψ', $\vdash ((\varphi_1' \to \psi') \to \varphi_2')$ holds. Then we reason as before and get (2.6) as well.

The properties of Σ_1-formulas in the previous discussion already hints at the special treatment of formulas of the form $\Box C$, which arithmetical translations are Σ_1, in Visser's Scheme.

Before giving the formal proofs of the two principles discussed above, we need some definitions and lemmas. The translation on arithmetical formulas by D. de Jongh, is given by the following inductive definition.

$$\begin{aligned}
&[\![\chi]\!]_m(\varphi) && \equiv_{def} \; \varphi, \text{ for atomic } \varphi \\
&[\![\chi]\!]_m(\cdot) \text{ commutes with } \wedge, \vee, \exists \\
&[\![\chi]\!]_m(\varphi \to \psi) && \equiv_{def} \; ([\![\chi]\!]_m(\varphi) \to [\![\chi]\!]_m(\psi)) \wedge \Box_m(\chi \to (\varphi \to \psi)) \\
&[\![\chi]\!]_m(\forall x\varphi x) && \equiv_{def} \; \forall x[\![\chi]\!]_m(\varphi x) \wedge \Box_m(\chi \to \forall x\varphi x).
\end{aligned}$$

We write $[\![\chi]\!]_m(\Gamma)$ for $\{[\![\chi]\!]_m(\psi) \mid \psi \in \Gamma\}$. Define

$$\begin{aligned}
&(\chi)_m(\varphi) && \equiv_{def} \; \varphi, \text{ for atomic } \varphi \\
&(\chi)_m(\cdot) \text{ commutes with } \wedge, \vee, \exists \\
&(\chi)_m(\varphi \to \psi) && \equiv_{def} \; \Box_m(\chi \to (\varphi \to \psi)) \\
&(\chi)_m(\forall x\varphi x) && \equiv_{def} \; \Box_m(\chi \to \forall x\varphi x)
\end{aligned}$$

2.7.3. Lemma. We have, verifiably in HA, that

$$\llbracket\chi\rrbracket_m(\varphi) \to (\chi)_m(\varphi).$$
$$\llbracket\chi\rrbracket_m(\varphi) \to \Box_m(\chi \to \varphi).$$
$$(\chi)_m(\varphi) \in \Sigma_1.$$
$$\text{for all } \theta \in \Sigma_1\text{: } \llbracket\chi\rrbracket_m(\theta) \leftrightarrow \theta \leftrightarrow (\chi)_m(\theta).$$

Proof Using induction on φ, the proofs of the first and the third statement are straightforward. For the last equation, use the fact that HA derives the formula $\forall x \leq y \Box_m(\varphi x) \to \Box_m \forall x \leq y(\varphi x)$ (a proof for PA can be found in (Hájek and Pudlák 1991)). This implies that $\mathsf{HA} \vdash \llbracket\chi\rrbracket_m(\forall x \leq y \varphi x) \leftrightarrow \forall x \leq y \llbracket\chi\rrbracket_m(\varphi x)$, and the rest of the statement follows easily. The proof of the second statement follows from the fact that, verifiably in HA, $\mathsf{I\Delta_0} + \mathsf{EXP}$ proves Σ_1-completeness: for Σ_1-formulas θ it holds that $(\theta \to \Box\theta)$. An analogous proof for PA can be found in (Hájek and Pudlák 1991). Once this is known, the rest of the proof is easy. □

2.7.4. Lemma. For all formulas A,B in preservativity logic, for all arithmetical realizations *, and for all m, we have

$$\mathsf{HA} \vdash (A^*)_m(B^*) \to (A^*)(B^*).$$

Proof First note that for all natural numbers m, HA proves $\Box_m\varphi \to \varphi$. The proof is completely similar to the one for PA (Hájek and Pudlák 1991). Recall the definition of $(A)(B_1, \ldots, B_n)$ for the case $n = 1$:

$$\begin{array}{lll} (A)(\bot) & \equiv_{def} & \bot \\ (A)(B \wedge B') & \equiv_{def} & (A)(B) \wedge (A)(B') \\ (A)(\Box B) & \equiv_{def} & \Box B \\ (A)(B) & \equiv_{def} & (A \to B), \text{ for } B \text{ not of the form } \bot, (C \wedge C') \text{ or } \Box C. \end{array}$$

For all these cases we have to prove that HA derives $(A^*)_m(B^*) \to (A^*)(B^*)$. Reason in HA. From the definition of $(\chi)_m(\varphi)$ and Lemma 2.7.3 it follows that we have,

$$\begin{array}{lll} (\chi)_m(\bot) & \leftrightarrow & \bot \\ (\chi)_m(\varphi) & \leftrightarrow & \varphi, \text{ if } \varphi \text{ is a } \Sigma_1\text{-formula.} \end{array}$$

We show that $(A^*)_m(B^*) \to (A)(B)^*$ holds with induction to B. In the case that $B = \bot$ it is easy to see that $(A^*)_m(B^*) \to (A)(B)^*$.

If $B = \Box C$, then B^* is a Σ_1-formula. Hence it holds that $(A^*)_m(B^*) \leftrightarrow \Box C^*$, and $\Box C^* = (A^*)(B^*)$. If $B = C \rhd D$, then B^* is of the form $\forall x \varphi x$, because $C^* \rhd D^*$ says 'for all x, if x is the code of a Σ_1-formula θ and $\Box(\theta \to C^*)$ holds, then

$\Box(\theta \to D^*)$ holds as well'. Thus $(A^*)_m(B^*) = \Box_m(A^* \to B^*)$. By the observation above this implies that we have $(A^* \to B^*)$, which is $(A^*)(B^*)$.

If $B = (C \wedge D)$, then $(A^*)_m(B^*) = (A^*)_m(C^*) \wedge (A^*)_m(D^*)$. By the induction hypothesis, $(A^*)_m(C^*) \wedge (A^*)_m(D^*)$ implies $(A^*)(C^*) \wedge (A^*)(D^*)$. By definition, $(A^*)(C^*) \wedge (A^*)(D^*) = (A^*)(C^* \wedge D^*) = (A^*)(B^*)$.

If $B = (C \vee D)$, then $(A^*)_m(B^*) = (A^*)_m(C^*) \vee (A^*)_m(D^*)$. By the induction hypothesis, $(A^*)_m(C^*) \vee (A^*)_m(D^*)$ implies $(A^*)(C^*) \vee (A^*)(D^*)$. It is easy to see that $(A^*)(C^*) \vee (A^*)(D^*)$ implies $(A^*)(C^* \vee D^*)$ (Lemma 3.2.1 (i)).

If $B = (C \to D)$, then $(A^*)_m(B^*) = \Box_m(A^* \to B^*)$. By the observation above this gives $(A^* \to B^*)$, which is $(A^*)(B^*)$. $\Box$

2.7.5. Lemma. (Visser 1994) Let $\varphi = \bigwedge_{i=1}^n(\varphi_i \to \psi_i)$. We have, verifiably in HA,

$$\llbracket\chi\rrbracket_m(\varphi) \leftrightarrow (\llbracket\chi\rrbracket_m(\varphi_i) \to \llbracket\chi\rrbracket_m(\psi_i)) \wedge \Box_m(\chi \to \varphi)$$

$$\Gamma \vdash_m \psi \text{ implies } \llbracket\chi\rrbracket_m(\Gamma) \vdash \llbracket\chi\rrbracket_m(\psi).$$

Proof The proof of the first equation is left to the reader. For the second statement we use induction to the length of the derivation $\Gamma \vdash_{\mathsf{HA},m} \varphi$. We treat the two difficult cases:

Case 1. Γ is empty and $\varphi = \psi 0 \wedge \forall x(\psi x \to \psi(x+1)) \to \forall x \psi x$, i.e. φ is an induction axiom. Since $\vdash \Box_m\varphi$, also $\Box_m(\chi \to \varphi)$. It remains to show that we have $\vdash \llbracket\chi\rrbracket_m(\psi 0 \wedge \forall x(\psi x \to \psi(x+1))) \to \llbracket\chi\rrbracket_m(\forall x \psi x)$, which is equivalent to

$$\begin{aligned}&\llbracket\chi\rrbracket_m(\psi 0) \wedge \forall x(\llbracket\chi\rrbracket_m \psi x \to \llbracket\chi\rrbracket_m \psi(x+1)) \wedge \\ &\quad \wedge \Box_m(\chi \to \forall x(\psi x \to \psi(x+1))) \to \forall x \llbracket\chi\rrbracket_m(\psi x) \wedge \Box_m(\chi \to \forall x \psi x).\end{aligned}$$

As we observed in Lemma 2.7.3, $\llbracket\chi\rrbracket_m(\psi 0)$ implies $\Box_m(\chi \to \psi 0)$. Hence from $\Box_m(\chi \to \varphi)$ it follows that $\Box_m(\chi \to \forall x(\psi x \to \psi(x+1)))$ implies $\Box_m(\chi \to \forall x \psi x)$. By induction we have

$$\llbracket\chi\rrbracket_m(\psi 0) \wedge \forall x(\llbracket\chi\rrbracket_m \psi x \to \llbracket\chi\rrbracket_m \psi(x+1)) \to \forall x \llbracket\chi\rrbracket_m(\psi x).$$

And this concludes Case 1.

Case 2. Suppose $\varphi = (\psi \to \psi')$ and the last step in the proof is $\Gamma, \psi \vdash_m \psi'$ implies $\Gamma \vdash_m (\psi \to \psi')$. By the induction hypothesis, $\Gamma, \psi \vdash_m \psi'$ implies $\llbracket\chi\rrbracket_m(\Gamma), \llbracket\chi\rrbracket_m(\psi) \vdash_{\mathsf{HA}} \llbracket\chi\rrbracket_m(\psi')$. And thus

$$\llbracket\chi\rrbracket_m(\Gamma) \vdash \llbracket\chi\rrbracket_m(\psi) \to \llbracket\chi\rrbracket_m(\psi').$$

Therefore, it remains to shows that

$$\llbracket\chi\rrbracket_m(\Gamma) \vdash \Box_m(\chi \to (\psi \to \psi')).$$

Clearly, we have $\theta, \psi \vdash_m \psi'$ for some conjunction θ of elements of a finite subset of Γ. Thus we have $\llbracket\chi\rrbracket_m(\Gamma) \vdash \Box_m(\chi \to \theta)$ and $\vdash \Box_m(\theta \to (\psi \to \psi'))$. And this leads to the desired conclusion. $\Box$

2.7.6. Theorem. (Visser 1994) The Disjunctive Principle belongs to the preservativity logic of HA.

Proof It is a well-known fact that, like PA, HA proves reflection for its finite fragments, i.e. for every natural number n, HA proves $(\Box_n\varphi \to \varphi)$. Moreover, HA can prove this fact, that is, HA proves that for every x, $\vdash (\Box_x\varphi \to \varphi)$. A proof of this fact for PA, which is similar to the one for HA, can be found in (Hájek and Pudlák 1991). The proof that the Disjunctive Principle belongs to the preservativity logic of HA now runs as follows.

Reason in HA. Suppose that $\varphi \rhd \chi$ and $\psi \rhd \chi$ hold. We have to show that $(\varphi\vee\psi)\rhd\chi$ holds, i.e. that for all Σ_1-formulas θ, $\vdash (\theta \to (\varphi\vee\psi))$ implies $\vdash (\theta \to \chi)$. Therefore, consider a Σ_1-formula θ and suppose $\vdash (\theta \to (\varphi \vee \psi))$. Thus $\theta \vdash_m (\varphi \vee \psi)$, for some m. By Lemma 2.7.5 we have

$$[\![\top]\!]_m(\theta) \vdash [\![\top]\!]_m(\varphi) \vee [\![\top]\!]_m(\psi).$$

By Lemma 2.7.5 this implies

$$\theta \vdash \Box_m\varphi \vee \Box_m\psi. \tag{2.7}$$

Note that $\Box_m\varphi$ and $\Box_m\psi$ are Σ_1-formulas. As observed above, we have that $\vdash (\Box_n\varphi \to \varphi)$ and $\vdash (\Box_n\psi \to \psi)$. Therefore, from $\varphi\rhd\chi$ and $\psi\rhd\chi$, we conclude

$$\vdash (\Box_m\varphi \to \chi) \wedge (\Box_m\psi \to \chi).$$

Together with (2.7) this gives

$$\vdash (\theta \to \chi).$$

This completes our proof. $\square$

2.7.7. Theorem. (Visser 1994) Visser's Scheme belongs to the preservativity logic of HA.

Proof We have to show that for all arithmetical formulae φ_i, ψ_i, for all n, if $\chi = \bigwedge_{i=1}^{n}(\varphi_i \to \psi_i)$, then we have

$$\mathsf{HA} \vdash (\chi \to \varphi_{n+1} \vee \varphi_{n+2}) \rhd (\bigvee_{i=1}^{n+2} (\chi)_m(\varphi_i)). \tag{2.8}$$

Clearly, this implies that for all formulas A_i, B_i in the language of preservativity and for all arithmetical translations *, if $A = \bigwedge_{i=1}^{n}(A_i \to B_i)$, then

$$\mathsf{HA} \vdash (A^* \to A^*_{n+1} \vee A^*_{n+2}) \rhd_{\mathsf{HA}} (\bigvee_{i=1}^{n+2} (A^*)_m(A^*_i)).$$

By Lemma 2.7.4, HA derives $(A^*)_m(A_i^*) \to (A^*)(A_i^*)$. Thus by Preservation Rule (Proposition 2.7.1), HA derives $(A^*)_m(A_i^*) \rhd (A^*)(A_i^*)$. Applying the principle $P1$ (Proposition 2.7.1) now gives

$$\mathsf{HA} \vdash (A^* \to A_{n+1}^* \vee A_{n+2}^*) \rhd (\bigvee_{i=1}^{n+2} (A^*)(A_i^*)).$$

Using the fact that $(\bigvee_{i=1}^{n+2}(A^*)(A_i^*)) = (A^*)(A_1^*, \ldots, A_{n+2}^*)$, this implies Visser's Scheme:

$$\mathsf{HA} \vdash (A^* \to A_{n+1}^* \vee A_{n+2}^*) \rhd (A^*)(A_1^*, \ldots, A_{n+2}^*).$$

Therefore, to show that Visser's Scheme belongs to the preservativity logic of HA it suffices to show that (2.8) holds, i.e. that HA derives that for all $\theta \in \Sigma_1$, $\vdash (\theta \to (\chi \to \varphi_{n+1} \vee \varphi_{n+2}))$ implies $\vdash (\theta \to (\chi)(\varphi_1, \ldots, \varphi_{n+2}))$.
Reason in HA. Let $\theta \in \Sigma_1$ and assume $\vdash (\theta \to (\chi \to \varphi_{n+1} \vee \varphi_{n+2}))$. Hence for some m, we have $\theta \vdash_m (\chi \to \varphi_{n+1} \vee \varphi_{n+2})$. From Lemma 2.7.5 it follows that $[\![\chi]\!]_m(\theta) \vdash [\![\chi]\!]_m(\chi \to \varphi_{n+1} \vee \varphi_{n+2})$. Hence by the same lemma:

$$\theta \vdash \bigwedge_{i=1}^{n} ([\![\chi]\!]_m(\varphi_i) \to [\![\chi]\!]_m(\psi_i)) \wedge \Box_n(\chi \to \chi) \to [\![\chi]\!]_m(\varphi_{n+1} \vee \varphi_{n+2}).$$

Thus clearly,

$$\theta \vdash \bigwedge_{i=1}^{n} ([\![\chi]\!]_m(\varphi_i) \to [\![\chi]\!]_m(\psi_i)) \to [\![\chi]\!]_m(\varphi_{n+1} \vee \varphi_{n+2}).$$

By Lemma 2.7.3 and elementary reasoning this implies that

$$\theta \vdash \bigwedge_{i=1}^{n} ((\chi)_m(\varphi_i) \to [\![\chi]\!]_m(\psi_i)) \to (\chi)_m(\varphi_{n+1}) \vee (\chi)_m(\varphi_{n+2}).$$

Hence

$$\mathsf{PA} \vdash \theta \to (\bigwedge_{i=1}^{n} ((\chi)_m(\varphi_i) \to [\![\chi]\!]_m(\psi_i)) \to (\chi)_m(\varphi_{n+1}) \vee (\chi)_m(\varphi_{n+2})).$$

Using classical logic we can conclude that $\mathsf{PA} \vdash \theta \to \bigvee_{i=1}^{n+2}(\chi)_m(\varphi_i)$. By Lemma 2.7.3, $\theta \to \bigvee_{i=1}^{n+2}(\chi)_m(\varphi_i)$ is a Π_2-formula. By the Π_2-conservativity of PA over HA mentioned above, we have

$$\vdash \theta \to \bigvee_{i=1}^{n+2} (\chi)_m(\varphi_i).$$

This completes the proof of (2.8). $\square$

2.7.8. Corollary. The Formalized Markov Scheme belongs to the provability logic of HA (and hence to its preservativity logic).

Proof In Chapter 3 (Section 3.3) we show that the Formalized Markov Scheme is derivable from Visser's Scheme and Montagna's Principle, using the rules Modus Ponens and Necessitation. □

2.8 Overview of part I

Part I of this thesis is a modal study of the principles of the preservativity logic of HA known so far. In particular, we prove the frame completeness of the conjectured preservativity logic iPH of HA, the main result of this part of the thesis (Chapter 5). As explained in Section 2.1 such a characterization is often the first step for finding embeddings of a provability logic in the corresponding arithmetical theory, i.e. for showing that a system is the provability logic of some theory. We also show in Chapter 5 that the system iPH contains principles of the provability logic of HA that are not captured by iH. This disproves the conjecture that iH is the provability logic of HA.

The proofs in Chapter 5 use the results of Chapter 4, where we study the principles separately. Here we also show that besides the principles Vp_n all principles are independent, as expected. Moreover, there we will see that Visser's Scheme is infinite in an essential way: it is not equivalent to a finite number of Visser's Principles.

As mentioned in the introduction, the characterization of the principles requires many technical tools from modal logic. Moreover, these logics deviate a lot from the logics that are regularly studied in intuitionistic modal logic. Whence some surprising properties and problems become visible, and many proofs are quite different from the ones for the modal logics one usually encounters. Therefore, also from the modal point of view these logics are interesting.

Chapter 7 of part II of this thesis could also be seen in the light of provability logic. This was explained in Section 2.3.1 where we discussed three particular fragments of the preservativity logic of HA.

In Chapter 3 we introduce the tools used in the following chapters of part I. Section 3.4 contains preliminaries. In Section 3.3 we show how the principles of the provability logic of HA, i.e. of the logic iH, are captured by its conjectured preservativity logic iPH.

Chapter 3

Tools and preliminaries

In this chapter we introduce the tools used in the following chapters of part I. In Section 3.1 we discuss some principles that are derivable in preservativity and provability logic. In Section 3.2 we discuss some basic properties of Visser's Scheme and we prove that our formulation of the scheme is equivalent to the one used by Visser (1994). In Section 3.3 we show that all principles of the provability logic we consider are derivable in preservativity logic, and that the converse does not hold. In Section 3.4 we introduce a semantics for preservativity logic, and we define various constructions on the models given by this semantics.

3.1 Basic observations

In this section we discuss some basic principles derivable in preservativity logic. When we say that a principle is arithmetically valid we mean that all the arithmetical realizations of the principle hold. We let * range over arithmetical realizations. Let iP^- be the logic given by the axioms *Taut*, $P1$ $P1$, $P2$ and the rule Modus Ponens and the Preservation Rule, and let iK be the logic given by the axioms *Taut*, and K, and the rule Modus Ponens and the Necessitation Rule (Section 3.4).

3.1.1. Lemma.

(*i*) for any logic iT containing iP^-: $\vdash_{\mathsf{iT}} A$ implies $\vdash_{\mathsf{iT}} \Box A$.

(*ii*) $\vdash_{\mathsf{iP}^-} \Box(A \to B) \to A \rhd B$ and $\vdash_{\mathsf{iP}^-} A \rhd B \to (\Box A \to \Box B)$.

Proof (*i*) Observe that $\vdash_{\mathsf{iT}} (A \to B)$ implies $\vdash_{\mathsf{iT}} \top \to (A \to B)$. Hence by the Preservation Rule $\vdash_{\mathsf{iT}} \top \rhd (A \to B)$, which is equivalent to $\Box(A \to B)$.

(*ii*) The second implication follows immediately from $P1$, using the fact that $\Box A$ is defined as $\top \rhd A$. The following derivation proofs the first implication.

We have

$$
\begin{array}{llll}
\vdash_{\mathsf{iP}^-} & \Box(A \to B) \leftrightarrow \top \rhd (A \to B) & & (1) \\
& A \rhd \top & \text{(Preservation Rule)} & (2) \\
& \Box(A \to B) \to A \rhd (A \to B) & (1)(P1) & (3) \\
& A \rhd A & \text{(Preservation Rule)} & (4) \\
& \Box(A \to B) \to A \rhd (A \wedge (A \to B)) & (3)(4)(P2) & (5) \\
& (A \wedge (A \to B)) \rhd B & \text{(Preservation Rule)} & (6) \\
& \Box(A \to B) \to A \rhd B. & (5)(6)(P1) &
\end{array}
$$

This completes the proof. □

Neither $A \rhd B \leftrightarrow \Box(A \to B)$ nor $A \rhd B \leftrightarrow (\Box A \to \Box B)$ are arithmetically valid. For the first one, we show that if this principle would hold, then so would $\Box\neg\neg\Box\bot$. This means that HA derives $\neg\neg\Box\bot$. By Markov's Rule it follows that then it derives its own inconsistency $\Box\bot$, quod non. The following derivation shows that in the presence of $A \rhd B \leftrightarrow \Box(A \to B)$, also $\Box\neg\neg\Box\bot$ is arithmetically valid.

$$
\begin{array}{ll}
\neg\Box\bot \rhd \Box\neg\Box\bot & (4p) \\
\Box(\neg\Box\bot \to \Box\neg\Box\bot) & \\
\Box(\neg\Box\bot \to \Box\bot) & (L) \\
\Box(\neg\neg\Box\bot). &
\end{array}
$$

A counterexample to the second principle is given by the Rosser sentence; a consistent Σ_1-sentence R such that $(\Box R \to \Box\bot)$. If $R \rhd \bot$ would hold, then by the definition of $\rhd$, we have $\Box(\varphi \to \bot)$ for all Σ_1-sentences φ such that $\Box(\varphi \to R)$ holds. Therefore, we would have $\Box(R \to \bot)$, which contradicts the fact that R is consistent with HA.

The following lemma shows that there is an equivalent formulation of iP^- which, like iK, contains the Necessitation Rule instead of the Preservation Rule. This system is the one that Visser (1994) introduced as a basic system of preservativity.

3.1.2. Lemma.

(*i*) The logic iP^- is equivalent to the logic consisting of the axioms $P1$, $P2$ and $\Box(A \to B) \to A \rhd B$, and the rules Modus Ponens and Necessitation.

(*ii*) iP^- is conservative over iK w.r.t. formulas in the language of provability logic.

Proof (ii) follows from (i) and Lemma 3.1.1. The proof of (i) is straightforward, using the same lemma. $\square$

The following lemma states that $\Box$ distributes over conjunction. This is a well-known property of many modal logics.

3.1.3. Lemma. $\vdash_{\mathsf{iP}^-} (\Box A \wedge \Box B) \leftrightarrow \Box(A \wedge B)$.

Proof Since $(A \wedge B \to A)$ en $(A \wedge B \to B)$ are derivable in IPC, the Preservation Rule gives $\vdash_{\mathsf{iP}^-} (A \wedge B) \rhd A$ and $\vdash_{\mathsf{iP}^-} (A \wedge B) \rhd B$. The implication from right to left now follows by Lemma 3.1.1. For the other direction, observe $(A \to (B \to A \wedge B))$ is derivable in IPC. Thus by Lemma 3.1.1 (i) also $\vdash_{\mathsf{iP}^-} \Box(A \to (B \to A \wedge B))$. Hence by (ii) of the same lemma, $\vdash_{\mathsf{iP}^-} \Box A \to \Box(B \to A \wedge B))$. Applying the same step again gives $\vdash_{\mathsf{iP}^-} \Box A \to (\Box B \to \Box(A \wedge B))$, which implies $\vdash_{\mathsf{iP}^-} (\Box A \wedge \Box B) \to \Box(A \wedge B)$. $\square$

Equivalent formulas preserve the same formulas and are preserved by the same formulas, as the following lemma shows.

3.1.4. Lemma. For any logic iT containing iP^- we have:

if $\vdash_{\mathsf{iT}} A \leftrightarrow B$, then $\vdash_{\mathsf{iT}} C \rhd A \leftrightarrow C \rhd B$ and $\vdash_{\mathsf{iT}} A \rhd C \leftrightarrow B \rhd C$.

Proof It suffices to show that if $\vdash_{\mathsf{iT}} (A \to B)$, then also $\vdash_{\mathsf{iT}} (C \rhd A \to C \rhd B)$ and $\vdash_{\mathsf{iT}} (B \rhd C \to A \rhd C)$. The proof is given by the following derivation.

$\vdash_{\mathsf{iT}}$	$(A \to B)$	by Preservation Rule implies
$\vdash_{\mathsf{iT}}$	$A \rhd B$	by $P1$ implies
$\vdash_{\mathsf{iT}}$	$(C \rhd A \to C \rhd B)$	
$\vdash_{\mathsf{iT}}$	$(B \rhd C \to A \rhd C)$.	

$\square$

In the next lemma we state a property of preservativity logic that we will often use.

3.1.5. Lemma. $\vdash_{\mathsf{iP}} A \rhd B \to (A \vee C) \rhd (B \vee C) \wedge (A \wedge C) \rhd (B \wedge C)$.

Proof Left to the reader. $\square$

We leave it to the reader to verify that the converse of the previous lemma is also valid: the logic given by adding the principle

$$A \rhd B \to (A \vee C) \rhd (B \vee C)$$

to iP^- derives Dp.

The next lemma contains a useful consequence of iP^-.

3.1.6. Lemma. $\vdash_{\mathsf{iP}^-} A \rhd (B \to C) \to (A \wedge B) \rhd C$.

Proof The proof is given by the following derivation.

$$
\begin{array}{llll}
\vdash_{\mathsf{iP}^-} & (A \wedge B) \rhd B \wedge (A \wedge B) \rhd A & \text{(Preservation)} & (1)\\
& A \rhd (B \to C) \to (A \wedge B) \rhd (B \to C) & (1)(P1) & (2)\\
& A \rhd (B \to C) \to (A \wedge B) \rhd (B \wedge (B \to C)) & (1)(2)(P2) & (3)\\
& (B \wedge (B \to C)) \rhd C & \text{(Preservation)} & (4)\\
& A \rhd (B \to C) \to (A \wedge B) \rhd C. & (3)(4)(P1) &
\end{array}
$$

□

The converse of Lemma 3.1.6,

$$(A \wedge B) \rhd C \to A \rhd (B \to C) \tag{3.1}$$

is not arithmetically valid. A counterexample is given by $A = \top$, $B = \neg\Box\bot$ and $C = \Box\bot$. By $4p$ we have $\neg\Box\bot \rhd \Box\neg\Box\bot$. And thus by Lp and $P1$ also $\neg\Box\bot \rhd \Box\bot$. But $\Box(\neg\Box\bot \to \Box\bot)$ does not hold, since this gives $\Box\neg\neg\Box\bot$.
However, Montagna's principle shows that if in (3.1) we restrict C to boxed formulas it becomes derivable in the preservativity logic of HA (and hence is arithmetically valid):

3.1.7. Lemma. $\vdash_{\mathsf{iPH}} (A \wedge \Box C) \rhd B \to A \rhd (\Box C \to B)$.

Proof By Montagna's Principle we have that

$$\vdash_{\mathsf{iPH}} (A \wedge \Box C) \rhd B \to (\Box C \to A \wedge \Box C) \rhd (\Box C \to B).$$

By the Preservation Rule it follows that

$$\vdash_{\mathsf{iPH}} A \rhd (\Box C \to A \wedge \Box C).$$

Combining these two consequences and applying $P1$ gives,

$$\vdash_{\mathsf{iPH}} (A \wedge \Box C) \rhd B \to A \rhd (\Box C \to B).$$

This completes the proof. □

Together with (3.1.6) the last lemma shows that (substituting $\top$ for A)

$$\vdash_{\mathsf{iPH}} \Box C \rhd B \leftrightarrow \Box(\Box C \to B).$$

Observe that $4p$ and Lp can be replaced by equivalent principles in which only $\rhd$ occurs. First note that $\Box A$ implies $B \rhd A$, for all B. Therefore, we can replace $\Box A$

in $4p$ and Lp by $B \rhd A$ and still have arithmetically valid principles which are also derivable from our principles:

$$\vdash_{\mathsf{iPH}} A \rhd (B \rhd A) \qquad \vdash_{\mathsf{iPH}} ((B \rhd A) \to A) \rhd A.$$

Observe that Montagna's Principle derives for all formulas $C = \bigvee_i \bigwedge_j \Box D_{ij}$ the following formula

$$A \rhd B \to (C \to A) \rhd (C \to B). \tag{3.2}$$

The arithmetical validity of this principle is not surprising since the arithmetical realizations of such formulas C are Σ_1. It is a well-known fact that HA proves completeness ($\varphi \to \Box\varphi$), for Σ_1-formulas φ. Hence $\Box(C \to \Box C)$ is in the provability logic of HA for the mentioned formulas C. This follows already from (3.2): if for all A, B, (3.2) is in the preservativity logic of HA then also

$$(A \wedge C) \rhd B \to A \rhd (C \to B). \tag{3.3}$$

Thus in particular, $\Box(C \to \Box C)$ is in the provability logic: by $4p$ we have $C \rhd \Box C$, and thus by (3.3) $\top \rhd (C \to \Box C)$, which is $\Box(C \to \Box C)$.

Noteworthy consequences

The logic iL, given by K and Löb's Principle $\Box(\Box A \to A) \to \Box A$, derives that 'if a theory is consistent then it cannot prove that a formula is unprovable' (a slight generalization of Gödels second incompleteness theorem):

3.1.8. Lemma. $\vdash_{\mathsf{iL}} \Box\neg\Box A \to \Box\bot$.

Proof Observe that $\neg\Box A$ implies $(\Box A \to A)$. By the Necessitation Rule we have $\vdash_{\mathsf{iL}} \Box(\neg\Box A \to A)$. Whence $\vdash_{\mathsf{iL}} \Box\neg\Box A \to \Box A$ by Lemma 3.1.2. In Section 4.3 we show that iL derives the principle 4. Therefore, we have $\vdash_{\mathsf{iL}} \Box\neg\Box A \to \Box\Box A$. Applying Lemma 3.1.3 gives $\vdash_{\mathsf{iL}} \Box\neg\Box A \to \Box(\neg\Box A \wedge \Box A)$. Hence Lemmas 3.1.3 and 3.1.2 leads to $\vdash_{\mathsf{iL}} \Box\neg\Box A \to \Box\bot$. $\Box$

The logic iLLe (the logic axiomatized by L and Le over iK, see Section 3.4) derives that 'if there is a proof of either φ or the unprovability of ψ, then φ is provable' (note that this implies the formula in iL mentioned above):

3.1.9. Lemma. Let $\boxdot$ be short for $(B \wedge \Box B)$. We have

(*i*) $\vdash_{\mathsf{iLLe}} \Box(A \vee \neg\Box B) \to \Box A$.

(*ii*) $\vdash_{\mathsf{iLe}} \Box(A \vee B) \to \Box(A \vee \boxdot B)$.

Proof First we prove (ii):

$$\begin{array}{ll} \vdash_{\mathsf{iLe}} & \Box(A \vee B) \to \Box(A \vee \Box B) \\ & \Box(A \vee B) \to \Box(A \vee B) \\ & \Box(A \vee B) \to \Box((A \vee B) \wedge (A \vee \Box B)) \\ & \Box(A \vee B) \to \Box(A \vee \boxdot B). \end{array}$$

Now the proof of (i) follows from (ii) and the fact that iL derives $(\Box\neg\Box B \to \Box B)$, as was shown in the lemma above. □

There is a consequence of the Formalized Markov Scheme that states that for formulae $(\bigvee \Box A_i \to \bigvee \Box B_i)$ a stronger variant of Löb's Principle is derivable:

3.1.10. Lemma. For $D = (\bigvee \Box A_i \to \bigvee \Box B_i)$ we have

$$\vdash_{\mathsf{iLMa}} \Box(\Box D \to \neg\neg D) \to \Box D.$$

Proof This follows from the following derivation. Let $D = (\bigvee \Box A_i \to \bigvee \Box B_i)$.

$$\begin{array}{lll} \vdash_{\mathsf{iLMa}} \Box(\Box D \to \neg\neg D) & \to & \Box(\Box\neg\neg D \to \neg\neg D) \\ & \to & \Box\neg\neg D \\ & \to & \Box D. \end{array}$$

□

This stronger version of L is not for arbitrary D a principle of HA. For instance, HA derives $\neg\neg(\Box\bot \vee \neg\Box\bot)$, thus also $\Box(\neg\neg(\Box\bot \vee \neg\Box\bot))$ by the Necessitation Rule. Therefore, HA derives $\Box(\Box(\Box\bot \vee \neg\Box\bot) \to \neg\neg(\Box\bot \vee \neg\Box\bot))$. But it does not derive $\Box(\Box\bot \vee \neg\Box\bot)$ as the discussion of iLLe above shows.

3.2 Remarks on Visser's Scheme

In this thesis we use a slightly different formulation of Visser's Scheme then the one used by Visser (1994). The reason for this is that when we use our formulation, the modal characterization of the scheme runs smoother. In this section we prove that the two formulations are equivalent. We also explain that outside the modal context Visser's formulation is to be preferred.

Recall that Visser's Scheme consists of the principles

$$Vp_n \quad (\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_{n+1} \vee A_{n+2}) \rhd (\bigwedge_{i=1}^{n} A_i \to B_i)(A_1, \ldots, A_{n+2}).$$

The notation $(\cdot)(\cdot)$ is given by

$$
\begin{array}{lll}
(A)(B, C_1, \ldots, C_n) & \equiv_{def} & (A)(B) \vee (A)(C_1, \ldots, C_n) \\
(A)(\bot) & \equiv_{def} & \bot \\
(A)(B \wedge B') & \equiv_{def} & (A)(B) \wedge (A)(B') \\
(A)(\Box B) & \equiv_{def} & \Box B \\
(A)(B) & \equiv_{def} & (A \to B) \\
 & & B \text{ not of the form } \bot, (C \wedge C') \text{ or } \Box C.
\end{array}
$$

An equivalent formulation of Visser's Scheme

The notation used in (Visser 1994) is the following

$$
\begin{array}{lll}
\{A\}(B, C_1, \ldots, C_n) & \equiv_{def} & \{A\}(B) \vee \{A\}(C_1, \ldots, C_n) \\
\{A\}(B) & \equiv_{def} & (A)(B), \text{ for } B \text{ no disjunction or conjunction} \\
\multicolumn{3}{l}{\{\cdot\}(\cdot) \text{ commutes with } \wedge \text{ and } \vee.}
\end{array}
$$

Note that the only difference between $(\cdot)(\cdot)$ and $\{\cdot\}(\cdot)$ lies in the treatment of disjunctions: we have $(A)(B \vee C) = (A \to B \vee C)$ and $\{A\}(B \vee C) = \{A\}(B) \vee \{A\}(C)$. If we replace $(\cdot)(\cdot)$ by $\{\cdot\}(\cdot)$ in Visser's Principles, the result is the following principle

$$VR_n \quad (\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_{n+1}) \rhd \{\bigwedge_{i=1}^{n} A_i \to B_i\}(A_1, \ldots, A_{n+1}).$$

Let us call the scheme that consists of all the principles VR_n, Visser's Real Scheme and denote it by VR. Visser (1994) has shown that Visser's Real Scheme belongs to the preservativity logic of HA. In the next proposition we show that Visser's Scheme and Visser's Real Scheme are interderivable. We need (ii) of the following lemma. Part (i) of the lemma will be used in other chapters.

3.2.1. Lemma.

(i) $(A)(B)$ implies $(A \to B)$, and $(A)(B) \vee (A)(C)$ implies $(A)(B \vee C)$.

(ii) For $A = (\bigwedge_{i=1}^{n}(A_i \to B_i))$, for all m, we have

$$\vdash_{\mathsf{iPV}} (A \to A_{n+1} \vee \ldots \vee A_{n+m}) \rhd (A)(A_1, \ldots, A_{n+m}).$$

Proof (i) Left to the reader; for the first statement, use induction on B, for the second statement, use the first one.

(ii) Use induction on m. For $m = 1$, observe that $(A \to A_{n+1})$ is equivalent to $(A \to A_{n+1} \vee \bot)$. We leave the rest of this case to the reader. For $m = 2$ the statement holds by the definition of Visser's Scheme. For $m > 2$, we let $C = A_{n+2} \vee \ldots \vee A_{n+m}$. It is clear that

$$\vdash_{\mathsf{iPV}} (A \to A_{n+1} \vee \ldots \vee A_{n+m}) \rhd (A \to A_{n+1} \vee C).$$

By the definition of Visser's Scheme we have that

$$\vdash_{\mathsf{iPV}} (A \to A_{n+1} \vee C) \rhd (A)(A_1, \ldots, A_{n+1}, C).$$

Note that because C is a disjunction it holds that $(A)(C) = (A \to C)$. By induction hypothesis we have

$$\vdash_{\mathsf{iPV}} (A \to C) \rhd (A)(A_1, \ldots, A_n, A_{n+2}, \ldots, A_{n+m}).$$

We leave it to the reader to check that, using the Disjunctive Principle and $P1$, all this leads to the desired result,

$$\vdash_{\mathsf{iPV}} (A \to A_{n+1} \vee \ldots \vee A_{n+m}) \rhd (A)(A_1, \ldots, A_{n+m}).$$

□

3.2.2. Proposition. Visser's Scheme derives Visser's Real Scheme and vice versa.

Proof We leave the proof that Visser's Real Scheme derives Visser's Scheme to the reader (use the fact that $\{A\}(B)$ implies $(A)(B)$). For the other part, consider a formula $(A \to A_{n+1})$, where $A = \bigwedge_{i=1}^{n}(A_i \to B_i)$. We have to show that

$$\vdash_{\mathsf{iPV}} (A \to A_{n+1}) \rhd \{A\}(A_1, \ldots, A_{n+1}). \tag{3.4}$$

It is easy to see that every A_i is equivalent to a formula of the form

$$A'_i = \bigvee_{j=1}^{k_i} \bigwedge_{h=1}^{m_{ij}} (C_{ijh} \wedge \Box D_{ijh}),$$

where every C_{ijh} is a propositional variable, an implication or a preservation that is not a boxed formula, and such that for every E, $\{E\}(A_i)$ is equivalent to $\{E\}(A'_i)$. Namely, A'_i can be obtained by replacing, in A_i, occurrences $(B \vee C) \wedge D$ by $(B \wedge D) \vee (C \wedge D)$.

Observe that A is equivalent to A', where

$$A' = \bigwedge_{i=1}^{n} (\bigwedge_{j=1}^{k_i} (\bigwedge_{h} (C_{ijh} \wedge \Box D_{ijh}) \to B_i)).$$

By Lemma 3.2.1 and the definition of $(\cdot)(\cdot)$ we have

$$\vdash_{\mathsf{iPV}} (A' \to A'_{n+1}) \rhd \bigvee_{i=1}^{n+1} \bigwedge_j (\bigwedge_h ((A' \to C_{ijh}) \wedge \Box D_{ijh})).$$

Since clearly, $\{A'\}(A'_i) = \bigwedge_j (\bigwedge_h ((A' \to C_{ijh}) \wedge \Box D_{ijh}))$, this implies that

$$\vdash_{\mathsf{iPV}} (A' \to A'_{n+1}) \rhd \{A'\}(A'_1, \ldots, A'_{n+1}).$$

As we just observed, $\{A'\}(A'_i)$ is equivalent to $\{A'\}(A_i)$. Moreover, A is equivalent to A'. Thus we can conclude (3.4), and we are done. $\Box$

The logic given by Visser's Scheme

In contrast to the other principles of preservativity logic, Visser's Scheme consists of infinitely many principles; in Section 4.6.1 we will prove that it cannot be reduced to one principle. However, also in another respect Visser's Scheme deviates from the other principles of iPH. Namely, for all of these principles it is trivial to see that the set of all (substitution) instances of the principle is closed under substitution, and hence the logic given by the principle is closed under substitution. For Visser's Scheme the latter holds but the former does not. Consider for example the following two instances of Visser's Scheme:

$$\begin{aligned} &((p_1 \to q) \to p_2 \vee p_3) \rhd (((p_1 \to q) \to p_1) \vee ((p_1 \to q) \to p_2) \vee \\ &\quad \vee ((p_1 \to q) \to p_3))) && (3.5) \\ &((\Box p_1 \to q) \to \Box p_2 \vee \Box p_3) \rhd (\Box p_1 \vee \Box p_2 \vee \Box p_3). && (3.6) \end{aligned}$$

If we substitute $\Box p_i$ for p_i in (3.5) we arrive at the formula

$$\begin{aligned} &((\Box p_1 \to q) \to \Box p_2 \vee \Box p_3) \rhd (((\Box p_1 \to q) \to \Box p_1) \vee && (3.7) \\ &\quad \vee ((\Box p_1 \to q) \to \Box p_2) \vee ((\Box p_1 \to q) \to \Box p_3))). \end{aligned}$$

This formula is not an instance of Visser's Scheme, as (3.6) shows. However, this formula is derivable in the system iPV: it is easy to see that it follows from (3.6), using the fact that $(\Box p_1 \vee \Box p_2 \vee \Box p_3)$ implies the formula $((\Box p_1 \to q) \to \Box p_1) \vee (\Box p_1 \to q) \to \Box p_2) \vee ((\Box p_1 \to q) \to \Box p_3))$ by propositional logic. Similar reasoning shows that the logic iPV is closed under substitution. However, in contrast to the other principles of iPH, this example shows that the collection of all instances of Visser's Scheme is not closed under substitution.

Visser's Scheme versus Visser's Real Scheme

Although Visser's Real Scheme and Visser's Scheme are interderivable, it is not difficult to see that $\{A\}(B)$ derives $(A)(B)$, while in general the converse does not hold. In this sense Visser's Real Scheme is more efficient than Visser's Scheme. Let us also illustrate this with one example.

Let $A_1 = (p_1 \vee p_2)$ and $A_2 = ((p_3 \vee p_4) \vee (p_5 \wedge p_6))$, and consider the formula $A = ((A_1 \to q) \to A_2)$. It is clear that

$$\{A\}(A_1, A_2) = (\bigvee_{i=1}^{4}(A \to p_i) \vee ((A \to p_5) \wedge (A \to p_6))).$$

Thus by Proposition 3.2.2 it follows that

$$\vdash_{\mathsf{iPV}} A \rhd (\bigvee_{i=1}^{4}(A \to p_i) \vee ((A \to p_5) \wedge (A \to p_6))).$$

However, while the derived formula is an instance (hence just one application) of Visser's Real Scheme, this derivation in iPV uses many application of Visser's Scheme. Namely, the application of Visser's Scheme to A is

$$A \rhd ((A \to p_1 \vee p_2) \vee (A \to p_3 \vee p_4) \vee ((A \to p_5) \wedge (A \to p_6))).$$

Is is clear that $(\bigvee_{i=1}^{4}(A' \to p_i) \vee ((A' \to p_5) \wedge (A' \to p_6)))$ derives the formula $((A \to p_1 \vee p_2) \vee (A \to p_3 \vee p_4) \vee ((A \to p_5) \wedge (A \to p_6)))$, but not vice versa.

Note that if $A \rhd B$ is an instance of one of the schemes, then B derives A, while in general the converse does not hold. Thus A can be a stronger formula than A (see the previous examples). For now, let us call a formula *simple* when either it is a propositional variable, a preservation or it is an implication for which either the antecedent is not a conjunct of implications or the consequent is a propositional variable, an implication or a preservation. Note that for simple formulas, the application of Visser's Real Scheme or Visser's Scheme does not lead to stronger formulas. We do not prove this fact, but the previous discussion indicates that if $A \rhd B$ is an instance of Visser's Real Scheme, then every subformula of B that is not in the scope of an implication is simple. Therefore, the application of Visser's Real Scheme does no longer lead to stronger formulas. This does not hold for Visser's Scheme, as the example above shows. Thus in this sense Visser's Real Scheme is more efficient than Visser's Scheme. However, as mentioned before, for the modal study of the logic given by the scheme, we prefer to work with Vissers Scheme instead of Visser's Real Scheme.

3.3 Preservativity versus provability

In this section we explain that the logic iH is contained in the system iPH, i.e. the principles of the provability logic of HA discussed in Section 2.5 are derivable in iPH. Then we show that the converse does not hold: iPH derives principles in the language of provability which are not captured by the system iH. These two facts show that iH is properly contained in the $\mathcal{L}_{\Box}$-part of iPH. It would be interesting to know if one can obtain a decent axiomatization of the $\mathcal{L}_\Box$-part of iPH. Although we did not find such an axiomatization yet, we conjecture that it exists.

First of all, the Necessitation Rule $(A/\Box A)$ is an admissible rule for iPH: If $\vdash A$ then $\vdash (\top \to A)$. Hence by the Preservation Rule $\vdash \top \rhd A$, which is $\vdash \Box A$. Lemma 3.1.1 shows that the axioms K, 4 and L belong to iPH. Leivant's Principle can be derived as follows.

$$
\begin{array}{lll}
\vdash_{\mathsf{iPH}} & A \rhd A \wedge B \rhd \Box B & \\
& A \rhd (A \vee \Box B) \wedge B \rhd (A \vee \Box B) & \text{(Lemma 3.1.4)}(P1) \\
& (A \vee B) \rhd (A \vee \Box B) & (Dp) \\
& \Box(A \vee B) \to \Box(A \vee \Box B). & \text{(Lemma 3.1.1)}
\end{array}
$$

Finally, we have to see that the Formalized Markov Scheme belongs to iPH. It is easy to see that $(\neg\neg \bigvee_i \Box B_i) \rhd (\bigvee_i \Box B_i)$ is derivable from Visser's Scheme:

$$
\begin{array}{ll}
\vdash_{\mathsf{iPH}} & (\neg\neg \bigvee_{i=1}^n \Box B_i) \leftrightarrow \neg(\bigwedge_{i=1}^n \neg \Box B_i) \\
& \neg(\bigwedge_{i=1}^n \neg \Box B_i) \rhd (\bigwedge_{i=1}^n \neg \Box B_i)(\bot, \Box B_1, \ldots, \Box B_n) \\
& (\bigwedge_{i=1}^n \neg \Box B_i)(\bot, \Box B_1, \ldots, \Box B_n) = \bigvee_{i=1}^n \Box B_i.
\end{array}
$$

By Montagna's Principle we then have $(\Box A \to \neg\neg \bigvee_i \Box B_i) \rhd (\Box A \to \bigvee_i \Box B_i)$. Since $\neg\neg(\Box A \to \bigvee_i \Box B_i)$ implies, $(\Box A \to \neg\neg \bigvee_i \Box B_i)$, this leads to the Formalized Markov Scheme $\neg\neg(\Box A \to \bigvee_i \Box B_i) \rhd (\Box A \to \bigvee_i \Box B_i)$.

We show that iPH is not conservative over iH. Note that for all axioms of iH of the form $(\Box A \to \Box B)$, iPH derives $A \rhd B$. For example, $(\Box A \to A) \rhd A$ and $(\neg\neg \Box B) \rhd \Box B$ belong to iPH. Using the Disjunctive Principle it follows that iPH derives $((\Box A \to A) \vee \neg\neg \Box B) \rhd (A \vee \Box B)$ as well. Therefore, by Lemma 3.1.1 also $\Box((\Box A \to A) \vee \neg\neg \Box B) \to \Box(A \vee \Box B)$ is derivable in iPH. In Section 5.4 we show that this formula does not belong to iH.

The observation above has some interesting consequences. For example it shows that

$$\vdash_{\mathsf{iPH}} ((\Box A \to A) \vee (\Box B \to B)) \rhd (A \vee B).$$

And hence $\Box((\Box A \to A) \vee (\Box B \to B)) \to \Box(A \vee B)$ holds for HA, a principle which does not hold for PA.

3.4 Preliminaries

In this section we introduce a semantics for the preservativity and provability operators, and we define the canonical model and the construction method. These are all fairly standard definitions except for the way in which the operator $\rhd$ is interpreted in models. This semantics for $\rhd$ is an idea from Visser. We also define the 'new' notion of an extendible property. In the proofs that this or that logic is canonical we need extensions of given sets of formulas. These extensions are all special instances of a 'general' principle of extension, which gave rise to the definition of an extendible property. First we introduce all these notions for preservativity logic. Most definitions are similar for provability logic. The ones that do differ are discussed in Section 3.4.6.

3.4.1 Definitions

The language $\mathcal{L}_\rhd$ of preservativity logic is that of propositional logic extended with one binary modal operator, $\rhd$. We assume $\bot$ (falsum) and $\top$ (true) to be present as primitive symbols in our propositional language. Recall that $\Box A$ is defined as $\top\rhd A$. A formula of the form $A\rhd B$ is called a *preservation* and a formula of the form $\Box A$ is called a *boxed formula*. We adhere to some reading conventions and omit parentheses when possible. The negation binds stronger than $\rhd$ which binds stronger than $\wedge$ and $\vee$, which in turn bind stronger than $\to$. We use a 'sequent-calculus' abbreviation: $\Gamma\rhd\Delta$ is short for $\bigwedge\Gamma\rhd\bigvee\Delta$.

A *logic* is a theory closed under substitution. We call the logic in $\mathcal{L}_\rhd$ which has as axioms all tautologies of intuitionistic propositional logic IPC and the principles $P1$, $P2$ (and Dp) and as rules Modus Ponens and the Preservation Rule (Section 2.2) the *arithmetical* (*semantical*) *base preservativity logic* and denote it with iP^- (iP). Following the notation of (Chagrov and Zakharyaschev 1997) we define $\mathsf{iP(A{\oplus}B)}$ to be the preservativity logic consisting of the axioms of iP plus A and B, and the Preservation Rule and Modus Ponens. When X denotes the infinite set of principles $A_1, A_2, \ldots$, we also write iPX for $\mathsf{iP(A_1{\oplus}\ A_2 \oplus \ldots)}$. When Xp is one of the principles of the preservativity logic given above we write iPX for iPXp. We write $\vdash_{\mathsf{iT}} A$ when A is derivable in iT. We write $\Gamma \vdash_{\mathsf{iT}} A$ when there is a derivation of A in iT from Γ without use of the Preservation Rule, in other words, when A is derivable by Modus Ponens from theorems of iT and formulae in Γ.

The name 'semantical base preservativity logic' for iP arises from the fact that it is sound and complete with respect to the frame semantics defined in Section 3.4.2. Thus, semantically seen, it is a base preservativity logic. On the other hand, the only axioms of iP for which it is trivial to see that HA derives all their arithmetical realizations are *Taut*, $P1$ and $P2$, and this accounts for the name 'arithmetical base preservativity logic' for iP^-.

3.4.2 Semantics

A possible semantics for preservativity logic can be produced via frames: we just add one extra clause for the interpretation of $\rhd$. The frames we use occur already in the literature (Section 2.6). The semantics for $\rhd$ came from Visser.

First some notation. When R and S are two binary relations, $(R;S)$ is the relation defined via $w(R;S)u \equiv \exists v(wRvSu)$.

A *frame* is a triple $\mathcal{F} = (W, \preccurlyeq, R)$, where W is a nonempty set (the set of *nodes*), $\preccurlyeq$ is a partial ordering on W (the *intuitionistic relation*) and R a binary relation on W (the *modal relation*) such that $(\preccurlyeq;R) \subseteq R$.

A *model* is a quadruple $\mathcal{M} = (W, \preccurlyeq, R, V)$, where $(W, \preccurlyeq, R)$ is a frame and V a *valuation relation* on pairs consisting of nodes and propositional variables. We demand that V is persistent, i.e.

(*persistence*) if $w \preccurlyeq v$ and wVp, then vVp.

We inductively define what it means for a formula A to be *forced (or valid) at a node* w of a model $\mathcal{M}$ ($\mathcal{M}, w \Vdash A$):

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash p & \equiv_{def} & wVp \\
\mathcal{M}, w \Vdash A \wedge B & \equiv_{def} & \mathcal{M}, w \Vdash A \text{ and } \mathcal{M}, w \Vdash B \\
\mathcal{M}, w \Vdash A \vee B & \equiv_{def} & \mathcal{M}, w \Vdash A \text{ or } \mathcal{M}, w \Vdash B \\
\mathcal{M}, w \Vdash A \to B & \equiv_{def} & \forall v \succcurlyeq w \ (\mathcal{M}, v \Vdash A \text{ implies } \mathcal{M}, v \Vdash B) \\
\mathcal{M}, w \Vdash A \rhd B & \equiv_{def} & \forall v \ (\text{if } wRv \text{ and } \mathcal{M}, v \Vdash A \text{ then } \mathcal{M}, v \Vdash B) \\
\mathcal{M}, w \Vdash \Box A & \equiv_{def} & \forall v \ \ (\text{if } wRv \text{ then } \mathcal{M}, v \Vdash A).
\end{array}
$$

Note that the definition of forcing for $\Box A$ agrees with the fact that $\Box A$ is defined as $\top \rhd A$, and that $\Box A$ gets the standard interpretation on frames. When $\mathcal{M}$ is clear from the context we write $w \Vdash A$ instead of $\mathcal{M}, w \Vdash A$. The formula A is *valid* or *forced* in $\mathcal{M}$, notation $\mathcal{M} \models A$, if A is forced in all nodes in $\mathcal{M}$. The formula A is *valid* in a frame $\mathcal{F}$, notation $\mathcal{F} \models A$, if A is valid in all models with underlying frame $\mathcal{F}$.

Note that $w \Vdash A$ and $w \preccurlyeq v$ implies $v \Vdash A$, and that $w \Vdash \Box A$ and wRv implies $v \Vdash A$.

A node v in a frame is called a *successor* of w if wRv, in which case w is called a *predecessor* of v. We use an abbreviation for the relation $(R;\preccurlyeq)$:

$$\tilde{R} \equiv_{def} (R;\preccurlyeq).$$

For a relation R we define $wR = \{v \mid wRv\}$. For a set U , we write $u \preccurlyeq U$ if for all $x \in U$, $u \preccurlyeq x$. We write '$x \preccurlyeq y_1, \ldots, y_n$' for '$x \preccurlyeq y_1 \wedge x \preccurlyeq y_2 \wedge \ldots \wedge x \preccurlyeq y_n$'. Similarly for other relations. A node v in a frame is *above* w if $w \preccurlyeq v$. In this case w is called *below* v. A node x is called an (*intuitionistic*) *top node* if there is no element above it except the node itself. $Top(\mathcal{F})$ is the set of all top nodes in a frame $\mathcal{F}$. We write Top instead of $Top(\mathcal{F})$ if no confusion is possible. When w is a node, $T(w)$ is the set of all top nodes above w.

3.4.1. Remark. The condition $(\preccurlyeq;R) \subseteq R$, included to guarantee persistence for formulas $A \rhd B$, may be weakened to

$$(\preccurlyeq;R) \subseteq (R;\preccurlyeq) \qquad (w \preccurlyeq w'Rv' \Rightarrow \exists u(wRu \preccurlyeq v')).$$

However we prefer to work with the simple condition where possible. For more discussion on this topic, see (Simpson 1993).

A property P on frames *corresponds* to a set T of formulas if for all frames $\mathcal{F}$: $\mathcal{F} \models T$ iff F has property P. Note that in this case we have

if $\vdash_{\mathsf{iT}} A$ then A is valid on all frames with property P.

When a frame $\mathcal{F}$ has a property P we say that $\mathcal{F}$ *is a P-frame*. We call $\mathcal{F}$ a $P_1 \dots P_n$-frame when it has the properties $P_1 \dots P_n$. If $\mathcal{C}$ is a class of frames, a logic iT is called *complete with respect to* $\mathcal{C}$ if

for all A: $\vdash_{\mathsf{iT}} A$ iff A valid on all frames in $\mathcal{C}$.

The logic iT is called *complete* if $\mathcal{C}$ is the class of frames to which iT corresponds.

3.4.3 Canonicity

Canonical models are defined in a similar manner as in classical modal logic. To define the canonical (X)-model for a logic we have to introduce the notion of an *X-saturated set*. A set of formulas X is called *adequate* if it is closed under subformulas and contains $\top$ and $\bot$. A set of formulas Γ is called *X-saturated* with respect to a logic T if it is a consistent subset of X such that

- $\Gamma \vdash_T A$ implies $A \in \Gamma$, for all $A \in X$,
- $\Gamma \vdash_T A \vee B$ implies $A \in \Gamma$ or $B \in \Gamma$, for all $A \vee B \in X$.

If X is the set of all formulas, an X-saturated set is just called *saturated*. It can be easily seen that for any (finite) adequate set X and for any A for which $\nvdash A$, there is an (finite) X-saturated set Γ such that $\Gamma \nvdash A$. Note also that any $\Delta \subseteq X$ for which $\Delta \nvdash A$, can be extended to an X-saturated Γ such that $\Gamma \nvdash A$.

For any logic T, for any adequate set X, the *T-canonical X-model* is the model $(W, \preccurlyeq, R, V)$ defined as follows:

$$\begin{array}{lll} \multicolumn{3}{l}{W \text{ consists of the } X\text{-saturated sets (with respect to } \vdash_T)} \\ w \preccurlyeq v & \equiv_{def} & w \subseteq v \\ wRv & \equiv_{def} & \text{if } A_1, \dots, A_n, B \in X, w \vdash_T A_1, \dots, A_n \rhd B \text{ and} \\ & & A_1, \dots, A_n \in v, \text{ then } B \in v \\ w \Vdash p & \equiv_{def} & p \in w, \text{ for propositional variables } p \in X. \end{array}$$

Recall that $A_1, \ldots, A_n \rhd B$ is short for $(\bigwedge A_i) \rhd B$ (Section 3.4.1). Note that in the definition of R we take formulas $A \rhd B$ into account which do not belong to X.

To see that this indeed defines a model, see the completeness proof for iP. This proof shows another fact we will often use, namely that for any canonical X-model:

$$\text{for all nodes } w, \text{ for all } A \in X: \; w \Vdash A \text{ iff } A \in w.$$

When X is the set of all formulas, we call the canonical X-model the *canonical model of T*. We call a logic iT *canonical* if the canonical model has the frame property to which the logic corresponds. Note that canonical logics are always complete, namely with respect to the class of frames to which they correspond.

Note that in the iT-canonical frame in general $(R;\preccurlyeq) \subseteq R$ does not hold. On the other hand, if we restrict our language to $\Box$ and the connectives, the canonical models do satisfy $(R;\preccurlyeq) \subseteq R$, see Section 3.4.6. That $(R;\preccurlyeq) \subseteq R$ is too strong a requirement in the context of preservativity logic follows from the fact that $A \rhd B \to \Box(A \to B)$ is valid on such frames. This principle is not in the preservativity logic of HA as was explained in Section 3.1.

3.4.4 Extendible properties

In this section we introduce a general construction to make certain extensions of sets of formulas. In many proofs to come we will extend certain sets of formulas to saturated sets with certain properties. It turns out that the way these extensions are made follow the same pattern. Therefore, we choose to define a general notion of extension which covers this.

Let iT be a preservativity logic and X an adequate set (Section 3.4.3). A property $*(\cdot)$ on sets of formulas such that we have both

$$\begin{array}{ll} \text{for all } A \in X: & \text{if } *(x) \text{ and } x \vdash_{\mathsf{iT}} A, \text{ then } *(x \cup \{A\}) \\ \text{for all } (A \vee B) \in X: & \text{if } *(x \cup \{A \vee B\}), \text{ then} \\ & *(x \cup \{A\}) \text{ or } *(x \cup \{B\}), \end{array}$$

is called an iT-*extendible property* (*w.r.t.* X). If in addition it holds that

$$\text{for all } A \in X: \quad \text{if } *(x) \text{ and } y \vdash_{\mathsf{iT}} x \rhd A, \text{ then } *(x \cup \{A\})$$

then it is called an iT-*extendible y-successor property*. For a property $*$ such that $*(\Gamma)$ holds, the $*$-*extension of* Γ is the union $x = \bigcup x_i$ of sets x_i which are constructed as follows. Given an enumeration $B_0, B_1, \ldots$ of all formulas in X, in which every formula occurs infinitely often, we define

$$\begin{array}{lll} x_0 & = & \Gamma \\ x_{i+1} & = & \begin{cases} x_i & \text{if} \quad \text{not } *(x_i \cup \{B_i\}) \\ x_i \cup \{B_i\} & \text{if} \quad *(x_i \cup \{B_i\}), B_i \text{ no disjunction} \\ x_i \cup \{B_i, E\} & \text{if} \quad *(x_i \cup \{B_i\}), B_i = C \vee D, \\ & \qquad E = C \text{ if } *(x_i \cup \{B_i, C\}), \\ & \qquad E = D \text{ otherwise.} \end{cases} \end{array}$$

Observe that $x \supset \Gamma$ is X-saturated. Thus, x is a node in the canonical X-model, and if Γ is a node in the canonical X-model as well, then $\Gamma \preccurlyeq x$. If in addition $*$ is an iT-extendible y-successor property, then also yRx holds in the iT-canonical X-model.

3.4.2. Remark. Note that for an iT-extendible w-successor property, the first requirement is redundant, because it follows from the third one. Namely, if $x \vdash A$ holds we have $\vdash_{\mathsf{iT}} (x \to A)$, and hence by Preservation Rule $\vdash_{\mathsf{iT}} x \rhd A$. Thus clearly $w \vdash_{\mathsf{iT}} x \rhd A$.

In the completeness proofs in the next chapters we often use extendible properties in the following way. Given a set Δ with a certain property, we want to extend it to a *saturated* set with this property, i.e. to a node in the canonical model with this property. There are two particular properties which often occur in this setting. The following lemma shows that these properties are extendible w-successor properties.

3.4.3. Lemma. For any logic iT containing iP, for any formula C and for all nodes w, v in the iT-canonical model, the following two properties are extendible w-successor properties:

$*(x)$ $\quad w \nvdash_{\mathsf{iT}} x \rhd C$.

$\star(x)$ $\quad$ for all D: $w \vdash_{\mathsf{iT}} x \rhd D$ implies $D \in v$.

Proof We write $\vdash$ for $\vdash_{\mathsf{iT}}$. First we consider the property $*(\cdot)$. We have to show that

$$\begin{array}{ll} \text{for all } A \in X: & \text{if } w \nvdash x \rhd C \text{ and } x \vdash A, \text{ then } w \nvdash x, A \rhd C \\ \text{for all } (A \vee B) \in X: & \text{if } w \nvdash x, (A \vee B) \rhd C, \text{ then } w \nvdash x, A \rhd C \text{ or} \\ & w \nvdash x, B \rhd C \\ \text{for all } A \in X: & \text{if } w \nvdash x \rhd C \text{ and } w \vdash x \rhd A, \text{ then } w \nvdash x, A \rhd C. \end{array}$$

Recall that we write $x, A \rhd C$ for $(\bigwedge x \wedge A) \rhd C$. By Remark 3.4.2 we know that if the third requirement holds, so does the first. Therefore, it suffices to show that the last two requirements holds.

For the second requirement, assume $w \vdash x, A \rhd C$ and $w \vdash x, B \rhd C$. To show that $*(\cdot)$ satisfies the second requirement we have to prove that $w \vdash x, (A \vee B) \rhd C$. This follows immediately from Dp.

For the third requirement assume $w \vdash x \rhd A$ and $w \vdash x, A \rhd C$. We show that $w \vdash x \rhd C$, and this will show that $*(\cdot)$ satisfies the third requirement. By the Preservation Rule we have $\vdash x \rhd \bigwedge x$, which is short for $\vdash \bigwedge x \rhd \bigwedge x$. Therefore, we certainly have $w \vdash x \rhd \bigwedge x$. Thus by $P2$ we have $w \vdash x \rhd (\bigwedge x \wedge A)$. Together with $w \vdash x, A \rhd C$ and $P1$ this leads to $w \vdash x \rhd C$.

Consider the property $\star$. To show that $\star$ is an extendible w-successor property we have to prove that

$$
\begin{array}{ll}
\text{for all } A \in X\text{:} & \text{if } \star(x) \text{ and } x \vdash A, \text{ then} \\
& (\text{for all } D\text{: } w \vdash x, A \rhd D \text{ implies } D \in v) \\
\text{for all } (A \vee B) \in X\text{:} & \text{if } \star(x \cup \{A \vee B\}), \text{ then} \\
& (\text{for all } D\text{: } w \vdash x, A \rhd D \text{ implies } D \in v) \text{ or} \\
& (\text{for all } D\text{: } w \vdash x, B \rhd D \text{ implies } D \in v) \\
\text{for all } A \in X\text{:} & \text{if } \star(x) \text{ and } w \vdash x \rhd A, \text{ then} \\
& (\text{for all } D\text{: } w \vdash x, A \rhd D \text{ implies } D \in v).
\end{array}
$$

By Remark 3.4.2, it suffices to show that the last two requirements holds.

For the second requirement, assume that neither $\star(x \cup \{A\})$ nor $\star(x \cup \{B\})$ holds. We prove that $\star(x \cup \{A \vee B\})$ does not hold. By assumption there are formulas C and D such that $C \notin v$ and $D \notin v$, and both $w \vdash x, A \rhd C$ and $w \vdash x, B \rhd D$. Clearly, both C and D imply $(C \vee D)$. Hence by Preservation Rule we have $\vdash C \rhd (C \vee D)$ and $\vdash D \rhd (C \vee D)$. Applying $P1$ gives $w \vdash x, A \rhd (C \vee D)$ and $w \vdash x, B \rhd (C \vee D)$. Thus by Dp we have $w \vdash x, (A \vee B) \rhd (C \vee D)$. If $\star(x \cup \{A \vee B\})$ would hold, this would imply that $(C \vee D) \in v$. Since v is a node in the canonical model it is a saturated set. Therefore, this would imply that $C \in v$ or $D \in v$, which contradicts our assumption.

We show that the third requirement holds. Assume that $\star(x)$ and $w \vdash x \rhd A$ hold, and that we have $w \vdash x, A \rhd D$, for some D. We have to show that $D \in v$. The same reasoning as above for $*(\cdot)$, shows that we have $w \vdash x \rhd (\bigwedge x \wedge A)$. Therefore, $w \vdash x, A \rhd D$ implies $w \vdash x \rhd D$ by $P1$. The fact that $\star(x)$ holds, gives $D \in v$. $\square$

3.4.5 The Construction Method

We define a method, *the construction method*, to obtain from a given model a new one. This method is similar to the construction method in classical model logic. The construction method is often used to obtain a completeness result with respect to some class of finite frames. Let $\mathcal{M} = (W, \preccurlyeq, R, V)$ be some canonical model, let X be an adequate set for which $A \rhd B \in X$ implies $\Box B \in X$. The method allows us to construct for any $w \in W$ a model $\mathcal{M}' = (W', \preccurlyeq', R', V')$ the domain of which consists of (copies of) nodes in W, which intuitively is the minimal set of nodes required to have w forcing the same formulae in X in the models $\mathcal{M}$ and $\mathcal{M}'$. We will restrict ourselves to a construction method for models that besides iP also satisfy Lp and Mp.

The construction proceeds as follows. We choose step by step, starting with w, a subset of W which will be the domain W' of our new model $\mathcal{M}'$. Note that the elements of W are sets of formulas. First, define

$$w^X_{\rhd} = \{A \rhd B \in X \mid A \rhd B \in w\}$$

$$w^X_{\not\rhd} = \{A \rhd B \in X \mid A \rhd B \notin w\}.$$

Similarly for $\to$. We omit the superscript X when possible. Let $*$ denote the concatenation function on strings:

$$\langle x_1, \ldots, x_n \rangle * \langle y_1, \ldots, y_m \rangle = \langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle.$$

Put $\alpha_{\langle\rangle} = w$. Suppose $v = \alpha_\sigma$ is defined. We choose elements $\alpha_{\sigma * \langle A \to B \rangle}$ and $\alpha_{\sigma * \langle A \rhd B \rangle}$ in W, for all elements $(A \to B) \in \sigma_{\not\to}$, $A \rhd B \in \sigma_{\not\rhd}$.

The node $\alpha_{\sigma * \langle A \to B \rangle}$ is an element $u \in W$ such that $v \preccurlyeq u$, $A \in u$ and $B \notin u$. Note that such elements can always be found. The node $\alpha_{\sigma * \langle A \rhd B \rangle}$ is an element $u \in W$ such that vRu, $A \in u$, $B \notin u$ and $\Box B \in v$. Observe that u contains more boxed formulas than v, for in the presence of Lp, and hence of $4p$ and 4, vRu and $\Box C \in v$ implies that $\Box C \in u$. To prove that such a node u exists it suffices to show that in any canonical model for a logic containing iPLM, if $A \rhd B \notin v$ there exists a v-successor extension of $\{A, \Box B\}$ omitting $\{B\}$. Thus we have to see that $v \nvdash A, \Box B \rhd B$. Suppose not. Then we have, using Lp and Mp:

$$\begin{array}{rcl} v & \vdash & A, \Box B \rhd B \\ & & (\Box B \to A \wedge \Box B) \rhd (\Box B \to B) \\ & & A \rhd (\Box B \to B) \\ & & A \rhd B. \end{array}$$

Define $W' = \{\sigma \mid \sigma \text{ is defined }\}$, and V via

$$\sigma \Vdash p \equiv_{def} \alpha_\sigma \Vdash p, \text{ for p} \in X.$$

We define the intuitionistic and the modal relation such that

$$\text{for all } A \in X, \text{ for all } \sigma \in W' : \quad \alpha_\sigma \Vdash A \text{ iff } \sigma \Vdash A.$$

As the choice of the relations will differ from case to case we do not give any specific examples here besides the obvious one;

$$\begin{array}{lll} \sigma \preccurlyeq' \tau & \equiv_{def} & \alpha_\sigma \preccurlyeq \alpha_\tau \\ \sigma R' \tau & \equiv_{def} & \alpha_\sigma R \alpha_\tau. \end{array}$$

It is not difficult to see that this choice gives a model with the desired property, be it not always on a frame with the desired properties.

3.4.4. Remark. It is easy to see that W' is finite if X is. First note that by construction, a node (saturated set) $\sigma * \langle B \rhd C \rangle$ contains more boxed formulas (formulas of the form $\Box C$) that belong to X than σ. A node $\sigma * \langle B \to C \rangle$ contains more implications that belong to X than σ. Moreover, for a node $\tau = \sigma * \langle B \to C \rangle$ we have that $\alpha_\sigma \preccurlyeq \alpha_\tau$ holds in the canonical model, i.e. $\alpha_\sigma \subseteq \alpha_\tau$. Clearly, all the implications that have to be treated, i.e. all implications for which we possibly have to add a new node in the construction, belong to X. And similarly for boxed formulas and preservations. Therefore, in going from σ to $\sigma * \langle B \rhd C \rangle$ or $\sigma * \langle B \to C \rangle$ either the number of boxed formulas that have to be treated decreases, or it stays the same and the number of implications that have to be treated decreases. Finally, if there are no more boxed formulas to be treated this means that for all $\Box B \in X$, it holds that $\Box B \in \alpha_\sigma$. Hence for all $B \rhd C \in X$, we have $\Box C \in \alpha_\sigma$ and thus $B \rhd C \in \alpha_\sigma$. Therefore, if there are no more boxed formulas to be treated there are no formulas of the form $B \rhd C$ to be treated either. Since the preservations and implications that belong to X are the only formulas that have to be treated in the construction method, this shows that the method is finite if X is.

3.4.6 The language of provability logic

The language of provability logic $\mathcal{L}_\Box$ is that of propositional logic extended with one modal operator $\Box$. We write

$$\boxdot A \equiv_{def} A \wedge \Box A.$$

The definition of $w_{\not\Box}$ is similar to $w_{\rhd}$.

For any principles A and B, $\mathsf{iK(A{\oplus}B)}$ is the logic in $\mathcal{L}_\Box$ consisting of all formulas provable in intuitionistic propositional logic IPC and the axioms K plus A and B, and the rules Modus Ponens and Necessitation ($C/\Box C$). As in classical provability logic, we write iT for iKT, for any set of principles T. We write $\vdash_{\mathsf{iT}} A$ when A is derivable in iT. We write $\Gamma \vdash_{\mathsf{iT}} A$ when there is a derivation of A in iT from Γ without use of Necessitation, in other words, when A is derivable by Modus Ponens from theorems of iT and formulae in Γ.

A (non)boxed formula is a formula (not) of the form $\Box A$.

The definition of a frame, a model and the notion of correspondence are inherited from preservativity logic, by reading $\top \rhd A$ for $\Box A$. As observed before, in that way $\Box A$ gets the standard interpretation on frames.

Canonicity in provability logic

It is convenient to change the definition of a canonical model (Section 3.4.3) slightly in the context of provability logic. For any logic T in $\mathcal{L}_\Box$ and for any adequate set X, the *T-canonical X-model* is the model $(W, \preccurlyeq, R, V)$ defined as follows:

W consists of the X-saturated sets (with respect to $\vdash_T$)

$w \preccurlyeq v \quad \equiv_{def} \quad w \subseteq v$

$wRv \quad \equiv_{def} \quad$ if $\Box A \in w$ then $A \in v$

$w \Vdash p \quad \equiv_{def} \quad p \in w$, for propositional variables $p \in X$.

Given this definition, the definition of canonicity is similar to the one in preservativity logic. The difference between this definition on canonical model and the one in the context of preservativity logic lies in the definition of R, which for the latter would read wRv iff for all $A \in X$, if $w \vdash \Box A$ then $A \in v$.

Brilliant frames

Recall that in a frame we always have $(\preccurlyeq;R) \subseteq R$. A frame is called *brilliant* if in addition it holds that

(*brilliant*) $\qquad \tilde{R} \subseteq R$

where $\tilde{R}$ is defined as $(R;\preccurlyeq)$ (Section 3.4.2). Note that in $\mathcal{L}_\Box$, canonical models have brilliant frames. In $\mathcal{L}_\rhd$ they do not have this property. For example, $A \rhd B \to \Box(A \to B)$ is valid on these frames, a principle which is not arithmetically valid (see Section 3.1). However, we will see that if we restrict ourselves to $\mathcal{L}_\Box$ all provability principles considered are complete with respect to some class of brilliant frames, even though they are sometimes also complete with respect to some nice class of non-brilliant frames.

Extendible properties in provability logic

The definition of an extendible property (Section 3.4.4) does not change in the context of $\mathcal{L}_\Box$.

3.4.5. Remark. Let $*$ be a extendible property w.r.t. an adequate set X. Note that if x is the $*$-extension of a set which contains $\{A \mid \Box A \in y\}$, then x is a node in the canonical X-model and in this model yRx holds.

Chapter 4

The principles

In Chapter 2 we introduced and discussed the meaning of the principles of the preservativity logic of HA known so far. In this and the next chapter we consider these principles from a modal point of view. In this chapter we study them separately and in the next chapter together. Here we describe to which frame properties the principles correspond and prove that all principles but Löb's Preservativity Principle are canonical. Since every canonical logic is complete (Section 3.4.3), this implies that besides Löb's Preservativity Principle, all these principles are complete with respect to a certain class of frames. Except for Löb's Preservativity Principle and Visser's Scheme, we show that all these principles have the finite model property as well, i.e. they are complete with respect to a certain class of finite frames. For the study of classical modal logics via frame characterizations and the like, we refer the reader to (van Benthem 1983)(van Benthem 1984)(Chagrov and Zakharyaschev 1997)(Blackburn, de Rijke and Venema 2001).

In Section 4.4 we show that iLe is conservative over iP4 with respect to formulas in $\mathcal{L}_\Box$. Thus in the absence of other principles, the Disjunctive Principle does not capture more of the Disjunction Property than Leivant's Principle (compare the discussion on the Disjunctive Principle in Section 2.3). However, in the next chapter we will see that this no longer holds in the presence of principles like the Formalized Markov Scheme. Namely, in Section 5.4, we show that iPH derives $\Box((\Box A \to A) \vee \neg\neg\Box B) \to \Box(A \vee \Box B)$, while the logic iH does not derives this principle, although it contains Leivant's Principle and the Formalized Markov Scheme.

We will see in Section 4.7 that besides the principles Vp_n, none of the preservativity principles derive one another, and that the same holds for all provability principles. In Section 4.6.1 we show that Vp_m does not derive Vp_n for $n > m$. However, sometimes two principles interfere in a different way. For example, Montagna's Principle and Visser's Scheme are both canonical, i.e. their canonical models have respectively the Ma- and the Vp^∞-property to which these principles correspond. But the canonical model for the logic iPMV given by both these principles has a stronger frame property than just these two properties, as will be shown in Corollary 4.6.3.

The results that will be used in Chapter 5 in the completeness proof for the logic iPH given by all principles together, are the correspondence for the principle Lp (Lemma 4.3.1), the canonicity of the logics $\mathsf{iP4}$ (Proposition 4.2.1) and iPM (Proposition 4.2.1), and the mentioned completeness proof for the logic iPV (Corollary 4.6.3). In Chapter 5 we also give a completeness proof for the logic iH given by the first known principles of the provability logic of HA. There we use the following results from this chapter: the completeness proof for the logic iMa (Proposition 4.6.7), and the completeness proof with respect to finite frames for the logic iLLe (Proposition 4.4.1).

In Section 4.1 we show that the base logics iP and iK are complete with respect to their given frame semantics. The completeness proofs for iP, $\mathsf{iP4}$ and iPL are similar to the ones in classical modal logic. The proofs for iK, $\mathsf{iK4}$ and iL occur already in the literature (Božić and Dǒsen 1984)(Kirov 1984)(Ursini 1979b). We treat a preservativity principle and its corresponding counterpart, like Lp and L, in one Section. The only exception is Leivant's Principle. Although it is derivable from $4p$ we treat it in a separate section because in this way all 'standard' proofs, for $\mathsf{iP}, \mathsf{iK}, 4p, Lp, 4, L$, precede the more interesting and non-standard proofs for Le, Mp, Vp and Ma.

We recall the known principles of the preservativity logic of HA that were discussed in Section 2.2.

$$\Box A \equiv_{def} \top \rhd A$$

$P1$	$A \rhd B \wedge B \rhd C \to A \rhd C$	
$P2$	$C \rhd A \wedge C \rhd B \to C \rhd (A \wedge B)$	
Dp	$A \rhd C \wedge B \rhd C \to (A \vee B) \rhd C$	(Disjunctive Principle)
$4p$	$A \rhd \Box A$	
Lp	$(\Box A \to A) \rhd A$	(Löb's Preservativity Principle)
Mp	$A \rhd B \to (\Box C \to A) \rhd (\Box C \to B)$	(Montagna's Principle)
Vp_n	$(\bigwedge_{i=1}^n (A_i \to B_i) \to A_{n+1} \vee A_{n+2}) \rhd (\bigwedge_{i=1}^n A_i \to B_i)(A_1, \ldots, A_{n+2})$	
Vp	$Vp_1, Vp_2, Vp_3, \ldots$	(Visser's Scheme)

The fragment of the provability logic of HA treated in Section 2.5 consists of the following principles.

K	$\Box(A \to B) \to (\Box A \to \Box B)$	
4	$\Box A \to \Box\Box A$	
L	$\Box(\Box A \to A) \to \Box A$	(Löb's Principle)
Le	$\Box(A \vee B) \to \Box(A \vee \Box B)$	(Leivant's Principle)
Ma	$\Box\neg\neg(\Box A \to \bigvee \Box B_i) \to \Box(\Box A \to \bigvee \Box B_i)$	(Formalized Markov Scheme)

Preservativity logic has the rules Modus Ponens and the

Preservation Rule if $\vdash (A \to B)$, then $\vdash A \rhd B$.

In the case of provability logic the Preservation Rule is replaced by

Neccesitation $A/\Box A$.

The logic iP is given by the axioms *Taut*, *Dp*, $P1$ and $P2$. The logic iK is given by the axioms *Taut* and K.

4.1 The base of preservativity logic

In this section we show that the frames defined in Section 3.4.2 are exactly the frames we need for the semantical base preservativity logic iP and for the base logic iK of provability logic.

4.1.1. Proposition. $\vdash_{\mathsf{iP}} A$ iff A is valid on all finite frames.

Proof We treat the direction from right to left. Suppose $\mathsf{iP} \nvdash A$. We have to show that there is a model for iP which does not force A. Let X be a finite adequate set containing A. We prove that the canonical X-model is such a model. Observe that the canonical X-model is indeed a model, i.e. $(\preccurlyeq;R) \subseteq R$, and that every model satisfies the axioms of iP. It is easy to see that there is an X-saturated set (hence a node in this model) which does not contain A. Therefore, to see that A is not valid on this model it suffices to show that

$$\forall B \in X \forall w : \ B \in w \text{ iff } w \Vdash B.$$

This can be easily shown by formula induction. We only treat implication and preservation for the direction from right to left. Suppose $B = (C \to D)$ and $B \notin w$. If $w \cup \{C\}$ would derive D, then also $w \vdash (C \to D)$. Thus $w \cup \{C\} \nvdash D$. This implies that $w \cup \{C\}$ is consistent. Let v be an X-saturated extension of

$w \cup \{C\}$ which does not derive D. Then $w \preccurlyeq v$, $v \Vdash C$ and $v \nVdash D$ hold, hence $w \nVdash (C \to D)$.

Now suppose $B = C \rhd D \notin w$. It suffices to construct an X-saturated set v such that wRv and $C \in v$ while $D \notin v$. Consider the property

$*(x) \qquad w \nvdash x \rhd D.$

By Lemma 3.4.3, $*(\cdot)$ is an iP-extendible w-successor property. Note that $*(C)$ holds. Any $*$-extension of $\{C\}$ can be taken for v. The fact that v does not contain D follows from the definition of a $*$-extension. $\square$

4.1.1 The base of provability logic

The defined semantics is correct for the base logic iK of provability logic:

4.1.2. Proposition. In $\mathcal{L}_\square$: $\vdash_{\mathsf{iK}} A$ iff A is valid on all finite brilliant frames.

Proof This proof is similar to the completeness proof for iP above. The only difference is that one has to observe that the canonical X-model is brilliant in this case, see Section 3.4.6. $\square$

4.2 The principle $4p$

The logic $\mathsf{iP4}$ is axiomatized over iP by

$4p \quad A \rhd \square A.$

We show that $\mathsf{iP4}$ is complete with respect to the class of gathering frames. We call a model or a frame *gathering* if it satisfies

$(gathering) \qquad wRvRu \to v \preccurlyeq u.$

We show that $\mathsf{iK4}$ is complete with respect to a different class of frames (Section 4.2.1). Although the Leivant Principle is derivable in $\mathsf{iP4}$ we treat it in a separate section. The reason for this given at the beginning of this chapter.

4.2.1. Proposition.

- (i) The principle $4p$ corresponds to gatheringness.
- (ii) The logic $\mathsf{iP4}$ is canonical.
- (iii) $\vdash_{\mathsf{iP4}} A$ iff A valid on all finite gathering frames.

Proof The three statements are easy to prove. We leave (i), (ii) and the direction from left to right of (iii) to the reader. For the the direction from right to left of the last statement it suffices to observe that for any finite adequate set which contains $\square B$ for any nonboxed $B \in X$, the $\mathsf{iP4}$-canonical X-model is gathering. $\square$

4.2.1 The principle 4

The logic $\mathsf{iK4}$ is axiomatized over iK by

4 $\qquad \Box A \to \Box\Box A.$

We show that $\mathsf{iK4}$ is complete with respect to finite transitive frames; this is similar to the situation in the classical case. We will see however that it corresponds to a weaker property than transitivity $(R;R \subseteq R)$, namely

$(\textit{semi-transitivity}) \qquad (R;R) \subseteq \tilde{R}.$

As is explained in Section 3.4.6 one cannot have both gatheringness and brilliancy. Here we see that for 4 we have brilliancy and for $4p$ we have gatheringness. Note that in the same way as in the classical case it can be shown that $\vdash_{\mathsf{iL}} 4$ (see Section 4.3).

4.2.2. Proposition. In $\mathcal{L}_\Box$:

(i) 4 corresponds to semi-transitivity.

(ii) $\mathsf{iK4}$ is canonical.

(iii) $\vdash_{\mathsf{iK4}} A$ iff A is valid on all finite transitive brilliant frames.

Proof The first two statements and the direction from left to right of the third statement are left to the reader. We treat the direction from right to left of (iii). Assume $\nvdash_{\mathsf{iK4}} A$. Let X be a finite adequate set which contains A and let Γ be an X-saturated set such that $\Gamma \nvdash_{\mathsf{iK4}} A$. Consider the model $(W, \preccurlyeq, R, V)$ such that $W, \preccurlyeq$ and V are like in the iK-canonical X-model and R is defined via

$$wRv \equiv_{def} \forall \Box B \in w \ (B, \Box B \in v).$$

It is clear that this frame is transitive. Therefore, to see that this model refutes A, we only have to show that ($w \Vdash B$ iff $B \in w$) holds for all $B \in X$. The only step which is different from the completeness proof for iK, is the direction from left to right for the case $B = \Box C$. Suppose $\Box C \notin w$. It is not difficult to see that the following property is an $\mathsf{iK4}$-extendible property w.r.t. X (compare Lemma 3.4.3),

$*(x) \qquad x \nvdash C.$

Observe that $*(\{D, \Box D \mid \Box D \in w\})$ holds. It is easy to see that any $*$-extension of the set $\{D, \Box D \mid \Box D \in w\}$ is an X-saturated set v such that $C \notin v$, and wRv. This proves $w \nVdash \Box C$. $\Box$

4.3 Löb's Preservativity Principle

The logic iPL is axiomatized over iP by Löb's Preservativity Principle

$Lp \quad (\Box A \to A) \rhd A.$

We show that Lp corresponds to the gathering conversely well-founded frames. We call a frame *conversely well-founded* if the modal relation on the frame is conversely well-founded. We do not know if iPL is also complete with respect to these frames. If we restrict ourselves to the language $\mathcal{L}_\Box$, then Löb's Principle is complete with respect to the gathering conversely well-founded frames, which is shown in Section 4.3.1. However, the 'trick' used in this completeness proof for iL breaks down for iPL in the absence of the principle Mp. The completeness proof for iL is similar to the one in classical logic. We have included it for completeness' sake.

Classically we have, in $\mathcal{L}_\Box$, that $\mathsf{iL} \vdash 4$. Here we also have

$$\vdash_{\mathsf{iPL}} L \text{ and } \vdash_{\mathsf{iPL}} 4p \text{ and } \vdash_{\mathsf{iP(4p \oplus L)}} Lp.$$

The first deduction is trivial. The second one has a similar proof as the above mentioned analogue in $\mathcal{L}_\Box$:

$$\begin{array}{ll} \vdash_{\mathsf{iPL}} & A \to (\Box(\Box A \wedge A) \to \Box A \wedge A)) \\ & A \rhd (\Box(\Box A \wedge A) \to \Box A \wedge A)) \\ & A \rhd (\Box A \wedge A) \\ & A \rhd \Box A \end{array}$$

The third derivation runs as follows.

$$\begin{array}{ll} \vdash_{\mathsf{iL}} & \Box(\Box(\Box A \to A) \to \Box A) \\ \vdash_{\mathsf{iL}} & \Box(\Box A \to A) \rhd \Box A \\ \vdash_{\mathsf{iP4}} & (\Box A \to A) \rhd \Box(\Box A \to A) \wedge (\Box A \to A) \\ \vdash_{\mathsf{iP(4p \oplus L)}} & (\Box A \to A) \rhd (\Box A \wedge (\Box A \to A)) \\ \vdash_{\mathsf{iP(4p \oplus L)}} & (\Box A \to A) \rhd A \end{array}$$

4.3.1. Lemma. The principle Lp corresponds to gatheringness plus converse well-foundedness of the modal relation.

Proof Left to the reader. □

4.3.1 Löb's Principle

The logic iL is axiomatized over iK by Löb's Principle

$L \qquad \Box(\Box A \to A) \to \Box A.$

In a manner similar to the classical case, we show that iL is complete with respect to the finite transitive conversely well-founded brilliant frames. We call these frames *L-frames.* We saw that although the principle 4 is complete with respect to transitive frames it corresponds to the weaker property of being semi-transitive. A similar difference occurs in the case of L. This is not surprising, since iL derives the principle 4 (the proof of this fact is similar to the one that Lp derives $4p$ above).

4.3.2. Proposition. In $\mathcal{L}_\Box$:

(i) L corresponds to semi-transitivity plus converse well-foundedness.

(ii) $\vdash_{\mathsf{iL}} A$ iff A is valid on all finite transitive conversely well-founded brilliant frames.

Proof (i) We only treat the direction from left to right. First assume that $\mathcal{F}$ is not semi-transitive; choose w, v, u such that

$$wRvRu \wedge \forall v'(wRv' \to v' \not\preccurlyeq u). \tag{4.1}$$

Consider the model on $\mathcal{F}$ given by the valuation $x \Vdash p \equiv_{def} (x \not\preccurlyeq v \wedge x \not\preccurlyeq u)$. We have to see that $w \Vdash \Box(\Box p \to p)$. Therefore, take x, y with wRx, $x \preccurlyeq y$ and $y \Vdash \Box p$. Because $u \nVdash p$, $v \nVdash \Box p$ and thus $y \not\preccurlyeq v$. Also $y \not\preccurlyeq u$ by (4.1), hence $y \Vdash p$. But clearly $w \nVdash \Box p$ since $v \nVdash p$.

For the second part, assume $\mathcal{F}$ is a semi-transitive but non conversely well-founded frame. Let $w_0 R w_1 R w_2 R \ldots$ be a chain in $\mathcal{F}$. Define a valuation on $\mathcal{F}$ via

$$x \Vdash p \equiv_{def} \forall i (x \not\preccurlyeq w_i).$$

In this model on $\mathcal{F}$, if $w_0 R x$ and $x \Vdash \Box p$ then $x \not\preccurlyeq w_i$, for all i, as $x \preccurlyeq w_i$ implies xRw_{i+1}. Hence $w \Vdash \Box(\Box p \to p)$. But not $w \Vdash \Box p$.

(ii) The direction from right to left. Let A be such that $\nvdash_{\mathsf{iL}} A$. Let X be a finite adequate set containing A, such that there is an X-saturated Γ for which $\Gamma \nvdash_{\mathsf{iL}} A$. We build a model $(W, \preccurlyeq, R, V)$, which does not make A valid. W is the set of X-saturated sets. $\preccurlyeq, V$ are defined in the same way as for the iL-canonical model. But define

$$wRv \equiv_{def} \forall \Box B \in w(\Box B, B \in v) \wedge \exists \Box D \in v(\Box D \notin w).$$

This makes W into a finite, transitive, conversely well-founded brilliant frame. We show that $w \Vdash B$ iff $B \in w$, for $B \in X$. We only treat the left to right direction for the case $B = \Box C$. Assume $\Box C \notin w$. Let $\Delta = \{D, \Box D \mid \Box D \in w\} \cup \{\Box C\}$. From $\Delta \vdash C$ it would follow that $w \vdash \Box(\Box C \to C)$, and hence $w \vdash \Box C$, which is false. Therefore, $\Delta \nvdash C$. Let v be an X-saturated extension of Δ which does not derive C, then wRv, and therefore $w \nVdash \Box C$. $\Box$

4.4 Leivant's Principle

The logic iLe is axiomatized over iK by Leivant's Principle

$Le \quad \Box(A \vee B) \to \Box(A \vee \Box B).$

Although the Leivant Principle is derivable in $\mathsf{iP4}$ (Section 3.3) we have not treated it in the section on the principle $4p$. The reason for this is given at the beginning of this chapter. As was explained in Sections 2.2 and 2.5, Leivant's Principle and the Disjunctive Principle are related to the Disjunction Property. In Section 4.4.1 we show that from the viewpoint of provability logic the Disjunctive Principle, in combination with the principle $4p$, does not capture more than Leivant's Principle, i.e. we show that the former is conservative over the latter.

In this section we show that iLe is complete with respect to finite transitive frames which have the Le-property

$(Le\text{-}property) \quad wRv \to \exists x(wRx \preccurlyeq v \wedge \forall u(vRu \to x \preccurlyeq u)).$

This completeness proof will be the first non-standard proof so far. One cannot use classical analogies, since in the context of natural classical modal logics Leivant's principle does not occur (Section 2.2). We will see that Le corresponds to the property

$(Le^{\infty}\text{-}property) \quad wRvRu \to \exists x(wRx \wedge x \preccurlyeq v \wedge x \preccurlyeq u).$

However, on finite frames Le corresponds to the Le-property. The proof of this fact will explain how this difference occurs when no infinite frames are allowed. Finally we show that iLe is also complete with respect to finite gathering frames. This implies that

for all $A \in \mathcal{L}_\Box$: $\vdash_{\mathsf{iP4}} A$ iff $\vdash_{\mathsf{iLe}} A$.

Note that one loses the brilliancy in this case. One cannot have both gatheringness and brilliancy: in these frames $\Box\neg\neg\Box\bot$ is valid, which clearly is not arithmetically valid. A node $x \preccurlyeq v$ for which $\forall u(vRu \to wRx \preccurlyeq u)$ holds is called a *leivant-node for the pair* (w, v).

We remind the reader of the following consequence of iLe that we will often use (it is proved in Section 3.1):

$$\vdash_{\mathsf{iLe}} \Box(A \vee B) \to \Box(A \vee \boxdot B). \tag{4.2}$$

Clearly $\mathsf{iLe} \vdash 4$. Observe that this is reflected in the corresponding frame properties; an Le^{∞}-frame is semi-transitive.

In Chapter 5 on the completeness of iH we will need the fact that iLLe is complete with respect to the finite transitive conversely well-founded brilliant Le-frames. (Recall that iLLe is iKLLe. The principle L is treated in the previous section.) Since this proof is similar to the completeness proof for iLe we treat it at this place.

4.4.1. Proposition. In $\mathcal{L}_\Box$:

(i) Le corresponds to the Le^∞-property.

(ii) On finite frames Le corresponds to the Le-property.

(iii) iLe is canonical.

(iv) $\vdash_{\mathsf{iLe}} A$ iff A is valid on all finite brilliant Le-frames.

(v) $\vdash_{\mathsf{iLLe}} A$ iff A is valid on all finite transitive conversely well-founded brilliant Le-frames.

Proof We only treat the direction from left to right of (i) and (ii) and the direction from right to left of (v). The proof of (iv) is similar to the one of (v). The rest of the statements are easy.

(i) Assume that a frame $\mathcal{F}$ does not have the Le^∞-property. Take $wRvRu$ such that

$$\forall x \neg(wRx \wedge x \preccurlyeq v \wedge x \preccurlyeq u). \tag{4.3}$$

Now define a model on $\mathcal{F}$ via

$$\begin{array}{lll} y \Vdash p & \equiv_{def} & \exists x \preccurlyeq y(wRx \wedge x \preccurlyeq u) \\ y \Vdash q & \equiv_{def} & \exists x \preccurlyeq y(wRx \wedge x \not\preccurlyeq u). \end{array}$$

With this valuation clearly $w \Vdash \Box(p \vee q)$. It is also easy to see that $u \nVdash q$, hence $v \nVdash \Box q$. Moreover, $v \nVdash p$. For, if $v \Vdash p$, then there is a node $x \preccurlyeq v$ such that $wRx \wedge x \preccurlyeq u$. This contradicts (4.3). Hence $v \nVdash p$, thus $v \nVdash p \vee \Box q$. Therefore, $w \nVdash \Box(p \vee \Box q)$.

(ii) The direction from left to right. It suffices to show that a finite Le^∞ frame is an Le-frame. Let $\mathcal{F}$ be a finite Le^∞-frame. Consider wRv. We show that there is a node x such that $(wRx \preccurlyeq v \wedge \forall u(vRu \to x \preccurlyeq u))$. Let n be the number of successors of v. If $n = 0$, then we can take $v = x$, and we are done. Therefore, assume $n \geq 1$. We show that there is an enumeration $u_1, \ldots, u_n$ of the successors of v such that there is a sequence $x_1 \succcurlyeq x_2 \succcurlyeq \ldots \succcurlyeq x_n$ of nodes, for which

$$wRx_i \preccurlyeq v \wedge x_i \preccurlyeq u_i.$$

Clearly, x_n has the desired properties, i.e. we can take $x = x_n$, since $\forall i(x_n \preccurlyeq u_i)$, and $wRx_n \preccurlyeq v$ by construction.

We construct these sequences as follows. Let u_1 be a successor of v. Since $wRvRu_1$, by the Le^∞-property, there is a node x_1 such that $wRx_1 \preccurlyeq v \wedge x_1 \preccurlyeq u_1$. Assume that for $j \preccurlyeq i$, x_j and u_j are defined. Let u_{i+1} be a successor of v distinct from $u_1, .., u_i$. Since $wRx_i \preccurlyeq vRu_{i+1}$ holds, we also have wRx_iRu_{i+1}. Hence by the Le^∞-property there is a node x_{i+1} such that

$$wRx_{i+1} \preccurlyeq x_i \wedge x_{i+1} \preccurlyeq u_{i+1}.$$

Since $x_{i+1} \preccurlyeq x_i \preccurlyeq v$, we also have $x_{i+1} \preccurlyeq v$. Thus x_{i+1} has the desired properties.

(v) The direction from right to left. An adequate set X is called *Le-adequate* if it is the single closure under $\Box$ of an adequate set which is of the form $\{\bigvee Z \mid Z \subseteq X_0\}$ for some set X_0 which does not contain formulas of the form $A \vee B$. X_0 is called the *base* of X. If $\nvdash_{\mathsf{iLLe}} A$, there is a finite *Le*-adequate set X which contains A, and such that there is an X-saturated set Γ which does not derive A. For the base of X just take the set X_0 of all subformulas of A (and their negations) minus the disjunctive ones. Consider the model $(W, \preccurlyeq, R, V)$, where W, $\preccurlyeq$ and V are defined as on the iLe-canonical X-model, and R is defined via

$$\begin{aligned} wRv \quad \equiv_{def} \quad & \forall Z \subseteq X_0 \text{ (if } \Box(\bigvee Z) \in w \text{ then } \exists Z_i \in Z(Z_i, \Box Z_i \in v)) \wedge \\ & \exists \Box B \in v \ (\Box B \notin w). \end{aligned}$$

First, we show that

$$\text{for all } B \in X\text{: } w \Vdash B \text{ iff } B \in w. \tag{4.4}$$

And second we show that the given frame is an *Le*-frame. This will complete the proof, because is easy to see that it is a transitive conversely well-founded brilliant frame. Just consider singleton sets Z in the definition of R.

The proof of (4.4). We need some notation. Let σ range over all functions on $\{Z \mid Z \subseteq X_0 \wedge Z \neq \emptyset\}$ for which $\sigma Z \in Z$. For any set x, x_σ denotes the set $\{\sigma Z, \Box \sigma Z \mid \Box(\bigvee Z) \in x\}$. Note that if w and v are X-saturated, then

$$wRv \text{ iff } \exists \sigma \exists \Box B \notin w \ (w_\sigma \cup \{\Box B\} \subseteq v). \tag{4.5}$$

For the proof of (4.4) the only nontrivial step is the direction from left to right in the case that $B = \Box C$. Assume $\Box C \notin w$. It is clear that the property

$$*(x) \qquad x \nvdash C$$

is an iLLe-extendible property w.r.t. X. We show that if for all σ we have $w_\sigma \cup \Box C \vdash C$, then $w \vdash \Box C$. Then we can conclude that there is a σ such that $*(w_\sigma, \Box C)$. Clearly, any $*$-extension of $w_\sigma, \Box C$ is an X-saturated set v such that $C \notin v$ and wRv. This would show that $w \nVdash \Box C$.

Arguing by contradiction suppose that for all σ we have $w_\sigma, \Box C \vdash C$. Let $Z_1, \ldots, Z_n$ be all the subsets Z of X_0 for which $\Box(\bigvee Z) \in w$.

$$\begin{array}{rcl} \forall \sigma (w_\sigma & \vdash & \Box C \to C) \\ \forall \sigma (\boxdot \sigma Z_1, \ldots, \boxdot \sigma Z_n & \vdash & \Box C \to C) \\ \forall \sigma \forall B \in Z_1 (\boxdot B, \boxdot \sigma Z_2, \ldots, \boxdot \sigma Z_n & \vdash & \Box C \to C) \\ \forall \sigma (\bigvee_{B \in Z_1} \boxdot B, \boxdot \sigma Z_2, \ldots, \boxdot \sigma Z_n & \vdash & \Box C \to C) \\ & \vdots & \\ \bigvee_{B \in Z_1} \boxdot B, \ldots, \bigvee_{B \in Z_n} \boxdot B & \vdash & \Box C \to C \\ \Box(\bigvee_{B \in Z_1} \boxdot B), \ldots, \Box(\bigvee_{B \in Z_n} \boxdot B) & \vdash & \Box(\Box C \to C). \end{array}$$

As $\vdash_{\mathsf{iLe}} \Box(\bigvee Z) \to \Box(\bigvee_{B \in Z} \boxdot B)$, this implies that $w \vdash \Box(\Box C \to C)$. Hence by L also $w \vdash \Box C$, a contradiction. This concludes the proof of (4.4).

To verify the Le-property, let wRv, $vR = \{u_1, \ldots, u_n\}$ and $u = v \cap u_1 \cap \ldots \cap u_n$. We have to find a node x such that wRx and $x \subseteq u$. Let

$$E = \bigvee\{D \in X_0 \mid u \nvdash \boxdot D\}.$$

In order to find x, we will construct w_σ and B such that

$$w_\sigma, \Box B \nvdash E, \quad w_\sigma \subseteq v, \quad \Box B \in v \cap w_{\not\Box}. \tag{4.6}$$

Assuming (4.6) to be satisfied, we will show that there is a y such that

$$w_\sigma, \Box B \subseteq y \subseteq u,\ \forall Z \subseteq X_0(\text{if } y \vdash \bigvee Z \text{ then } \exists Z_i \in Z(Z_i \in y)). \tag{4.7}$$

Then $x = \{D \in X \mid y \vdash D\}$ is X-saturated, $x \subseteq u$ and by (4.5) also wRx. So it meets our conditions. Therefore, it remains to prove (4.6) and (4.7).

Assuming (4.6) we show (4.7) as follows. Let $Y_1, Y_2, \ldots$ be an enumeration of all the subsets of X_0 with infinite repetition. We construct sets x_n, such that the required $y = \bigcup_n x_n$.

$$\begin{array}{rcl} x_1 & = & w_\sigma \cup \{\Box B\} \\ x_{i+1} & = & \begin{cases} x_i & \text{if} \quad x_i \nvdash \bigvee Y_i \\ x_i \cup \{\bigvee Y_i, D\} & \text{if} \quad x_i \vdash \bigvee Y_i, \text{ and } D \in Y_i \text{ is such} \\ & \phantom{\text{if}} \quad \text{that } x_i \cup \{\bigvee Y_i, D\} \nvdash E. \end{cases} \end{array}$$

Now all x_i are subsets of u. The set x_1 is a subset of u since $\Box B \in v$. Also $w_\sigma \subseteq u$; formulae in w_σ come in pairs $D, \Box D$ with $D \in X_0$. Suppose $D, \Box D \in w_\sigma$ and either of $D, \Box D \notin u$. Then $u \nvdash \boxdot D$, so $D \vdash E$, and thus $w_\sigma \vdash E$, which is not the case. So $D, \Box D \in u$. Assume we have already shown $x_i \subseteq u$. If $x_i = x_{i+1}$ it is trivial; so let $x_{i+1} = x_i \cup \{\bigvee Y_i, D\}$, and assume, arguing by contradiction, that $x_{i+1} \nsubseteq u$. Since $x_i \vdash \bigvee Y_i$, so $u \vdash \bigvee Y_i$, thus $\bigvee Y_i \in u$, this implies that $D \notin u$. Hence $u \nvdash D$; but then $D \vdash E$, and $x_{i+1} \vdash E$, a contradiction.

Now we turn to the proof of (4.6). We have two cases.

Case $\Box E \notin v$. If $\Box E \notin v$, then for all $\Box B \in v$, all $w_\sigma \subseteq v$ we have $w_\sigma, \Box B \nvdash E$; for otherwise we have $\Box E \in v$, since $v \vdash \Box(\bigwedge w_\sigma \wedge \Box B)$. Since wRv by (4.5), there is a $w_\sigma \subseteq v$ and a $\Box B \in v \cap w_{\not\Box}$, hence we are done.

Case $\Box E \in v$. We show that there is a $w_\sigma \subseteq v$ such that $w_\sigma, \Box E \nvdash E$. This would prove (4.6) with $\Box E$ for the $\Box B$, since it is easy to see that $\Box E \notin w$. Arguing by contradiction, let us assume that for all $w_\sigma \subseteq v$ we have $w_\sigma, \Box E \vdash E$; then we can derive the incorrect statement $w \vdash \Box E$, as follows. Again, let $Z_1, \ldots Z_n$ be all subsets $Z \subseteq X_0$ for which $\Box(\bigvee Z) \in w$. First note that

$$Z_i = \bigvee_{D \in Z_i \cap u} D \vee \bigvee_{D \in Z_i, D \notin u} D.$$

Hence also

$$Z_i \vdash E \vee \bigvee_{D \in Z_i \cap u} D.$$

And thus

$$w \vdash \Box(E \vee \bigvee_{D \in Z_i \cap u} \boxdot D).$$

Now we reason as follows.

$$\begin{array}{l}
\forall\sigma(\text{ if } w_\sigma \subseteq v, \text{ then } w_\sigma, \Box E \vdash E) \\
\forall\sigma(\text{ if } w_\sigma \subseteq v, \text{ then } \boxdot\sigma Z_1, \ldots, \boxdot\sigma Z_n, \Box E \vdash E) \\
\forall\sigma(\text{ if } w_\sigma \subseteq v, \text{ then } \bigvee_{B \in Z_1 \cap u} \boxdot B, \boxdot\sigma Z_2, \ldots, \boxdot\sigma Z_n, \Box E \vdash E) \\
\qquad\qquad\qquad\qquad\qquad\vdots \\
E \vee \bigvee_{D \in Z_1 \cap u} \boxdot D, \ldots, E \vee \bigvee_{D \in Z_n \cap u} \boxdot D, \Box E \vdash E \\
\Box(E \vee \bigvee_{D \in Z_1 \cap u} \boxdot D), \ldots, \Box(E \vee \bigvee_{D \in Z_n \cap u} \boxdot D) \vdash \Box(\Box E \to E) \\
w \vdash \Box(\Box E \to E) \\
w \vdash \Box E.
\end{array}$$

This completes the proof of (v). □

4.4.1 Conservativity

As promised, we show that iLe is also complete with respect to finite gathering frames. This implies that iP4 is conservative over iLe with respect to formulas in $\mathcal{L}_\Box$. (A theory T is called *conservative over* T' *with respect to formulas in* $\mathcal{L}$ if T' derives every formula in $\mathcal{L}$ that T derives.) As was explained in Section 4.4, we cannot have both gatheringness and brilliancy.

As explained before, the fact that iLe is conservative over iP4 with respect to formulas in $\mathcal{L}_\Box$, shows that in the absence of other principles, the Disjunctive Principle does not capture more of the Disjunction Property than Leivant's Principle. In the next chapter we will see that this no longer holds in the presence of principles like the Formalized Markov Scheme: in Section 5.4, we show that iPH derives $\Box((\Box A \to A) \vee \neg\neg\Box B) \to \Box(A \vee \Box B)$, while the logic iH does not derives this principle, although it contains Leivant's Principle and the Formalized Markov Scheme.

4.4.2. Proposition. $\vdash_{\mathsf{iLe}} A$ iff A is valid on all finite gathering frames.

Proof Suppose $\nvdash_{\mathsf{iLe}} A$. Let $\mathcal{M} = (W, \preccurlyeq, R, V)$ be a finite brilliant *Le*-model which does not validate A in some node b. We define a new relation $R' \subseteq R$ on W such that the the model $\mathcal{M}' = (W, \preccurlyeq, R', V)$ has a gathering frame and validates the same formulas as $\mathcal{M}$.

Intuitively we 'erase' those modal relationships R between elements which violate the gatheringness of the frame, i.e. between nodes w, v such that there is a vRu with $v \not\preccurlyeq u$. That is, we define

$$wR'v \equiv_{def} wRv \text{ and } \forall u(vRu \to v \preccurlyeq u).$$

To prove that $\mathcal{M}, w \Vdash B$ iff $\mathcal{M}', w \Vdash B$, is straightforward once one knows

$$wRv \to w(R';\preccurlyeq)v.$$

We will show this last fact. Let $S(x)$ be short for $\forall u \in W(xRu \to x \preccurlyeq u)$. Now, assume wRv. We show that there is a node v' with $wR'v' \preccurlyeq v$, that is, with $wRv' \preccurlyeq v$ and $S(v')$. The idea is as follows. By the Leivant property there is a node x_1 below v and all its successors (in $\mathcal{M}$), and such that wRx_1. If $x_1 = v$, we have $wR'v$ and are done. If $x_1 \neq v$ we consider a node x_2 below x_1 and all its successors, and such that wRx_1, which again exists by the Leivant property. If $x_2 = x_1$, we can take $v' = x_1$. If $x_2 \neq x_1$ we consider a node x_3 which is below x_2 and all its successors , and such that wRx_3, etc.

More formally, we construct a sequence of elements $v = x_1 \succcurlyeq x_2 \succcurlyeq \ldots$ in W such that for all i, it holds that wRx_i. And such that if $S(x_i)$ does not hold, then $(x_i \neq x_{i-1})$. As $\mathcal{M}$ is finite this implies we can find an element x_i with the desired properties. We show how to construct x_{n+1} from x_n. If $S(x_n)$ holds, put $x_{n+1} = x_n$. If $S(x_n)$ does not hold, x_{n+1} is a node which is below x_n and its successors, and moreover such that wRx_{n+1}. □

4.4.3. Corollary. The logic iP4 is conservative over iLe with respect to formulas in $\mathcal{L}_\Box$.

4.5 Montagna's Principle

The logics iPM is axiomatized over iP by Montagna's Principle

$Mp \quad A \rhd B \to (\Box C \to A) \rhd (\Box C \to B).$

We show that Mp corresponds to the Mp-property defined as

$(Mp\text{-}property) \quad wRv \preccurlyeq u \to \exists x(wRx \wedge v \preccurlyeq x \preccurlyeq u \wedge x\tilde{R} \subseteq u\tilde{R}).$

Then we prove that iPM is canonical.

If a principle corresponds to a frame property in which expressions like $x\tilde{R} \subseteq y\tilde{R}$ occur, like Montagna's Principle, then for a proof of its canonicity we need to know what $x\tilde{R} \subseteq y\tilde{R}$ means on the canonical model, i.e. in terms of saturated sets. This is the content of the following lemma. Note the difference with the language $\mathcal{L}_\Box$, in which case $w_\Box \subseteq v_\Box$ iff $vR \subseteq wR$. The proof is similar to the proof of the following lemma.

4.5.1. Lemma. In any canonical model: $v\tilde{R} \subseteq w\tilde{R}$ iff $w_{\Box} \subseteq v_{\Box}$.

Proof First the direction from left to right. Suppose $\Box A \in w$ while $\Box A \notin v$. By Lemma 3.4.3, the property

$*(x) \qquad v \nvdash x \rhd A,$

is an extendible v-successor property. Note that $*(\{\top\})$ holds, and let u be any $*$-extension of $\{\top\}$. Clearly, vRu, and $A \notin u$ hence $w(R;\preccurlyeq)u$ cannot hold.

For the other direction, assume $w_{\Box} \subseteq v_{\Box}$ and vRu. We have to construct a node u' such that $wRu' \subseteq u$. By Lemma 3.4.3, the property

$*(x) \qquad$ for all A: $w \vdash x \rhd A$ implies $A \in u$,

is an extendible w-successor property. Clearly, $*(\{\top\})$ holds. Therefore, any $*$-extension of $\{\top\}$ will do for u'. $\Box$

4.5.2. Proposition.

(i) The principle *Mp* corresponds to the the *Mp*-property.

(ii) The logic iPM is canonical.

Proof We prove part (ii) of the proposition and leave (i) to the reader. Consider $wRv \preccurlyeq u$ in the iPM-canonical model. Define the property

*(x) for all A: $w \vdash x \rhd A$ implies $A \in u$.

It is easy to see that $*(\cdot)$ is an iPM-extendible w-successor property. Thus if $*(v \cup u_{\Box})$ holds, then any $*$-extension of $v \cup u_{\Box}$ is a node x such that wRx (Section 3.4.4) and $v \preccurlyeq x \preccurlyeq u$ and $x\dot{R} \subseteq u\dot{R}$ hold (Lemma 4.5.1). Thus it remains to show that $*(v \cup u_{\Box})$ holds. This follows from the fact that for all finite subsets $\Gamma \subseteq v$ and $\Delta \subseteq u_{\Box}$, and for all B we have that $w \vdash \Gamma, \Delta \rhd B$ implies $B \in u$. Therefore, suppose that for some such Γ, Δ, B it does hold that $w \vdash \Gamma, \Delta \rhd B$. Replace Δ by the equivalent $\Box A$ where $A = (\bigwedge\{C \mid \Box C \in \Delta\})$. Then

$$\begin{array}{rl} w \vdash & \Gamma, \Box A \rhd B \\ & (\Box A \to \bigwedge \Gamma \wedge \Box A) \rhd (\Box A \to B) \\ & \Gamma \rhd (\Box A \to B). \end{array}$$

This implies that $(\Box A \to B) \in v$, whence that $B \in u$. $\Box$

4.6 Visser's Scheme

The logic iPV is axiomatized over iP by Visser's Scheme which consists of the principles $Vp_1, Vp_2, \ldots$, where

$$Vp_n \quad (\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_{n+1} \vee A_{n+2}) \rhd (\bigwedge_{i=1}^{n} A_i \to B_i)(A_1, \ldots, A_{n+2}).$$

Recall that $(A)(B_1, \ldots, B_n)$ is defined as

$$\begin{array}{lll} (A)(B, C_1, \ldots, C_n) & \equiv_{def} & (A)(B) \vee (A)(C_1, \ldots, C_n) \\ (A)(\bot) & \equiv_{def} & \bot \\ (A)(B \wedge B') & \equiv_{def} & (A)(B) \wedge (A)(B') \\ (A)(\Box B) & \equiv_{def} & \Box B \\ (A)(B) & \equiv_{def} & (A \to B) \\ & & B \text{ not of the form } \bot, (C \wedge C') \text{ or } \Box C. \end{array}$$

In Section 2.3 we discussed the meaning of Visser's Scheme and its relation with the admissible rules of HA. In Section 3.2 we discussed the workings of the scheme. In this section we show that Visser's Scheme Vp corresponds to a certain frame property Vp^∞, and that the iPV-canonical frame has a stronger frame property. This proves that the logic iPV is complete. We also prove that in combination with Mp the logic is complete with respect to a class of frames which have a more elegant property, which will be called the Vp-property. In Section 4.6.1 we show that $\mathsf{iPV_n}$ does not derive $Vp_{(n+1)}$, a result which does not play a role in the completeness proof of iPH. Of course, this result shows that Visser's Scheme is an essentially infinite collection of principles. In Section 4.6.2 we treat the Formalized Markov Scheme, which is derivable in iPMV.

For the frame characterization of Visser's Scheme we need the notion of a tight predecessor. We will first give the intuition behind it. Let $\bar{v}, \bar{u}$ range over finite sets of nodes, and write e.g. $x \preccurlyeq \bar{v}$ for 'for all $v \in \bar{v}(x \preccurlyeq v)$'. Consider two main instances of Visser's Scheme (see also Section 2.2):

$$(\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_{n+1} \vee A_{n+2}) \rhd (\bigvee_{j=1}^{n+2}(\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_j)) \tag{4.8}$$

$$(\bigvee_{i=1}^{n} \neg\neg\Box A_i) \rhd (\bigvee_{i=1}^{n} \Box A). \tag{4.9}$$

The second one is treated in Section 3.3. The first one arises if we restrict Visser's Scheme to pure propositional variables, the second one if we restrict it to boxed formulas and $\bot$. These two principles are related to two parts of the frame characterization of Visser's Scheme. It is easy to see that (4.9) is valid on frames which satisfy

$$wRvR\bar{u} \to \exists x\ (v \preccurlyeq x \wedge x\tilde{R}\bar{u} \wedge \neg\exists y(x \prec y)). \tag{4.10}$$

Formula (4.8) holds on frames which satisfy

$$wRv \preccurlyeq \bar{v} \to \exists x(v \preccurlyeq x \preccurlyeq \bar{v} \wedge \forall y \succ x \exists z \in \bar{v}(z \preccurlyeq y)). \tag{4.11}$$

We show this for $n = 3$. If for nodes wRv in such a frame we have $v \Vdash ((p_1 \to q) \to p_2 \vee p_3)$, and not $v \Vdash (p_1 \to q) \to p_j$ then there are nodes $u_1, u_2, u_3 \succcurlyeq v$ that force $(p_1 \to q)$ and such that u_i does not force p_i. Let $\bar{v} = \{u_1, u_2, u_3\}$ and let x be the node such that $v \preccurlyeq x \preccurlyeq \bar{v}$ and such that for all $y \succ x$, it holds that $u_i \preccurlyeq y$ for some i. Observe that x forces $(p_1 \to q)$ but that it does not force $(p_2 \vee p_3)$, contradicting the assumption that v forces $((p_1 \to q) \to p_2 \vee p_3)$. For arbitrary n the reasoning is the same.

The combination of the two frame properties above leads to the frame property with respect to which Vp is complete. However, Vp does correspond to a weaker property, which will be called the Vp^∞-property. This is best illustrated by the discussion on formula (4.8) above. Namely, one can weaken (4.11) by requiring that all nodes y above x are either below all nodes in $\bar{v}$ or above at least one node in $\bar{v}$:

$$wRv \preccurlyeq \bar{v} \to \exists x(v \preccurlyeq x \preccurlyeq \bar{v} \wedge \forall y \succcurlyeq x(y \preccurlyeq \bar{v} \vee \exists z \in \bar{v}(z \preccurlyeq y))).$$

The same reasoning as above shows that Vp is still valid on frames with this property.

Tight predecessors (in modal logic)

We say that a node x in K is a *semi-tight predecessor of $\bar{v}$ holding $\bar{u}$*, if

$$x \preccurlyeq \bar{v} \wedge x\tilde{R}\bar{u} \wedge \forall y \succ x(\exists z \in \bar{v}(z \preccurlyeq y) \vee (y \preccurlyeq \bar{v} \wedge y\tilde{R}\bar{u})).$$

It is called a *tight predecessor of $\bar{v}$ holding $\bar{u}$* if in addition there holds the stronger

$$\forall y \succ x \exists z \in \bar{v}(z \preccurlyeq y).$$

If in addition we have

$$v\tilde{R} \subseteq x\tilde{R} \wedge \forall y \succ x \exists z \in \bar{v}(z \preccurlyeq y),$$

then x is called *a tight predecessor of $\bar{v}$ for v*.

We call a frame (model) a *Vp^∞-frame (model)* if it has the Vp^∞-property:

(Vp^∞-*property*) for all finite sets of nodes $\bar{v}, \bar{u}$: $wRv \wedge v \preccurlyeq \bar{v} \wedge vR\bar{u} \to$
$\exists x \succcurlyeq v(x$ is a semi-tight predecessor of $\bar{v}$ holding $\bar{u})$.

An inspection of the Vp^∞-property will convince the reader that there are hardly any finite models that have this property.

Observe that if one reads tight for semi-tight in the Vp^∞-property, it expresses (4.10) if $\bar{v}$ is empty, and (4.11) if $\bar{u}$ is empty. To show that iPV is complete with respect to Vp^∞-frames we need the following lemma (compare Lemma 4.5.1 and the discussion just before it).

4.6.1. Lemma. In any canonical model: $w\tilde{R}v$ iff for all $\Box A \in w$, $A \in v$.

Proof Only the direction from right to left. Let $*(\cdot)$ be the property

$*(y)$ for all A: $w \vdash y \rhd A$ implies $A \in v$.

By Lemma 3.4.3, $*$ is an extendible w-successor property. It is easy to see that $*(u_0)$ holds, where $u_0 = \{A \mid \Box A \in w\}$. Let u be the $*$-extension of u_0. Clearly, $wRu \preccurlyeq v$ holds. □

4.6.2. Proposition.

(*i*) Visser's Scheme corresponds to the Vp^∞-property.

(*ii*) The logic iPV is canonical.

(*iii*) The canonical model iPV satisfies the following property which is stronger than the Vp^∞-property:

$$\text{for all finite sets of nodes } \bar{v}, \bar{u}\text{: } wRv \wedge v \preccurlyeq \bar{v} \wedge vR\bar{u} \rightarrow$$
$$\exists x \succcurlyeq v(x \text{ is a tight predecessor of } \bar{v} \text{ holding } \bar{u}).$$

Proof We often use lemma 3.2.1 (*i*) without mentioning. (*i*) First we show that Vp holds on a Vp^∞-frame. Suppose wRv and $v \nVdash (A)(D_1, \ldots, D_{n+2})$ hold, for some $A = \bigwedge_{i=1}^n (D_i \rightarrow E_i)$, on some Vp^∞-frame. We show that $v \nVdash (A \rightarrow D_{n+1} \vee D_{n+2})$. Assume $D_i = B_i \wedge \Box C_i$, where B_i is not of the form $\Box C$. From the assumption it follows that $v \nVdash (A \rightarrow B_i) \wedge \Box C_i$, whence either $v \nVdash \Box C_i$ or $v \nVdash (A \rightarrow B_i)$. Therefore, there are finite sets of nodes $\bar{v}$ and $\bar{u}$ such that for all i we have that *either* there is a node $x \in \bar{u}$ with vRx and $x \nVdash C_i$ *or* there is a node $x \in \bar{v}$ with $v \preccurlyeq x$, $x \Vdash A$ and $x \nVdash B_i$. Let $\bar{v}$ and $\bar{u}$ be a smallest pair of sets with these properties. Let $u \succcurlyeq v$ be a semi-tight predecessor of $\bar{v}$ holding $\bar{u}$. We show that $u \Vdash A$ and $u \nVdash (D_{n+1} \vee D_{n+2})$. This will prove that $v \nVdash (A \rightarrow D_{n+1} \vee D_{n+2})$.

To see that that $u \nVdash (D_{n+1} \vee D_{n+2})$, note that for $i = n+1, n+2$ we have that either there is node $x \in \bar{u}$ with $x \nVdash C_i$ *or* there is a node $x \in \bar{v}$ with $x \Vdash A$ and $x \nVdash B_i$. In the first case we have that $u\tilde{R}x$, and hence $u \nVdash \Box C_i$. In the second case we have that $u \preccurlyeq x$ and thus $u \nVdash (A \rightarrow B_i)$. Hence in both cases we can conclude $u \nVdash D_i$. To see that $u \Vdash A$, consider $y \succcurlyeq u$. Then either $y \preccurlyeq \bar{v}$ and $y\tilde{R}\bar{u}$, or $z \preccurlyeq y$ for some $z \in \bar{v}$. In the last case y forces A because all nodes in $\bar{v}$ force A. In the first case, it suffices to show that for all $i \leq n$, we have that $y \nVdash B_i \wedge \Box C_i$. Note that for all $i \leq n$ either there is node $x \in \bar{u}$ with $x \nVdash C_i$ *or* there is a node $x \in \bar{v}$ with $x \Vdash A$ and $x \nVdash B_i$. In the first case we have that $y\tilde{R}x$ holds, and whence $y \nVdash \Box C_i$. In the second case we have $y \preccurlyeq x$, and therefore $y \nVdash B_i$. Hence in both cases we can conclude $y \nVdash D_i$.

The other part of (i) follows from part (i) of Lemma 4.6.4; a frame which does not have the Vp^∞-property does not have the Vp_n-property, for some n.

(ii) This follows from (iii).

(iii) Consider nodes wRv, $v \preccurlyeq v_1, \ldots, v_m$ and $vRu_1, \ldots, u_n$, in the iPV-canonical model. Let $\hat{v}, \hat{u}$ denote $v_1 \cap \ldots \cap v_m$ and $u_1 \cap \ldots \cap u_n$ respectively. First note that in general $\hat{v}$ and $\hat{u}$ are not saturated. Therefore, they are not necessarily nodes in the canonical model. Let

$$\Delta = \{(E \wedge \Box E' \to F) \mid F \in \hat{v} \wedge (E \notin \hat{v} \vee E' \notin \hat{u})\}.$$

(Thus in particular the implications $(E \to F)$ and $(\Box E \to F)$, for which $F \in \hat{v}$ and respectively $E \notin \hat{v}$ and $E \notin \hat{u}$, are in Δ.) Note that $\Delta \subseteq \hat{v}$. Let $*(\cdot)$ be the property

$*(x) \quad x \vdash A_1 \vee \ldots \vee A_m \vee \Box B_1 \vee \ldots \vee \Box B_n$ implies $\exists i \ (A_i \in \hat{v}$ or $B_i \in \hat{u})$.

Clearly, $*(\cdot)$ is an extendible property (Section 3.4.4). We show that $*(v \cup \Delta)$ holds. Let $C = A_1 \vee \ldots \vee A_m \vee \Box B_1 \vee \ldots \vee \Box B_n$ and suppose $v \cup \Delta \vdash C$. This implies that there is a conjunct $D = \bigwedge_{i=1}^{k}(E_i \to F_i)$ of implications in Δ, such that $v \vdash (D \to C)$. Thus $(D \to C) \in v$, because v is saturated. Since

$$(D \to C) \rhd (D)(E_1, \ldots E_k, A_1, \ldots, A_m, \Box B_1, \ldots, \Box B_n),$$

also $(D)(E_1, \ldots E_k, A_1, \ldots, A_m, \Box B_1, \ldots, \Box B_n) \in v$. From the construction of Δ it follows that v does not contain any of $(D \to E) \wedge \Box E'$, for $E_i = E \wedge \Box E'$. Therefore v contains either $(D \to A_i)$ or $\Box B_i$ for some i. This proofs that $*(v \cup \Delta)$ holds. Let u be the $*$-extension of $v \cup \Delta$. As described in Section 3.4.4, u is saturated. We show that u is a semi-tight predecessor of $\bar{v}$ holding $\bar{u}$. Clearly, $v \preccurlyeq u \preccurlyeq v_1, \ldots, v_n$ holds, and by Lemma 4.6.1we have $u\tilde{R}u_i$, for all i.

It remains to show that

$$\forall y \succ u \exists i (z`v_i \preccurlyeq y).$$

Arguing by contradiction, suppose $u \prec u'$ for some saturated set u' and assume that no v_i is contained in u'. For all $i \leq m$, we choose a formula $A_i \in v_i$ outside u'. Then the formula $(A_1 \vee \ldots \vee A_m)$ is in $\hat{v}$ but not in u'. From the construction of u, and the fact that u' is a superset of u, it follows that there is a formula $(E \wedge \Box E') \in u'$ such that either $E \notin \hat{v}$ or $E' \notin \hat{u}$. Now $(E \wedge \Box E' \to A_1 \vee \ldots \vee A_m)$ is an element of Δ, thus also of u. Hence $(A_1 \vee \ldots \vee A_n)$ should be in u', a contradiction. This proves that iPV is canonical. $\Box$

We saw that Vp is complete with respect to a stronger frame property than the property to which it corresponds. On frames for which for every two nodes $x \preccurlyeq y$ there is a node $x \preccurlyeq z \preccurlyeq y$ such that there is no node $z \prec z' \prec y$, the two frame properties coincide. Since the canonical model has such a frame, (iii) follows in fact from (ii).

In the presence of Montagna's principle

Happily, in the presence of Montagna's Principle, Visser's Scheme has a more compact characterization. It is given by the following property

$$(Vp\text{-}property) \quad wRv \preccurlyeq v_1, \ldots, v_m \to$$
$$\exists x(v \preccurlyeq x \preccurlyeq v_1, \ldots, v_m \wedge v\tilde{R} \subseteq x\tilde{R} \wedge \forall y \succ x \exists i(v_i \preccurlyeq y)).$$

Recall that in this case x is called *a tight predecessor of* $v_1, \ldots, v_n$ *for* v. The following corollary of the previous lemma plus the canonicity of Montagna's Principle (Proposition 4.5.2) shows that the logic iPMV is complete with respect to $Mp\,Vp$-frames.

4.6.3. Corollary. Any canonical model of a logic containing the principles Mp and Vp has the Vp-property.

Proof The proof is analogous to the proof of the canonicity of iPV above. Consider $wRv \preccurlyeq v_1, \ldots, v_m$. The set Δ will now be

$$\Delta = \{(E \wedge \Box E' \to F) \mid F \in \hat{v} \wedge (E \notin \hat{v} \vee \Box E' \notin v)\}.$$

In the same way as in the proof of Proposition 4.6.2, define a property

$$*(x) \quad x \vdash A_1 \vee \ldots \vee A_m \vee \Box B_1 \vee \ldots \vee \Box B_n \text{ implies } \exists i(A_i \in \hat{v} \text{ or } \Box B_i \in v).$$

and construct a node u via this property. This leads to a node u such that $v \preccurlyeq u \preccurlyeq v_1, \ldots, v_m$ and $u_\Box \subseteq v_\Box$. Applying Lemma 4.5.1 gives $v\tilde{R} \subseteq u\tilde{R}$. Following the proof of Proposition 4.6.2 it is easy to see that u has the desired properties. □

4.6.1 The independence of Visser's Principles

In this section we show that $\mathsf{iPV_n}$ does not derive $Vp_{(n+1)}$. The proof is rather unpleasant but we think that the result needs to be established. It implies that Visser's Scheme is infinite in an essential way. The result does not play a role in the next chapter on the completeness of iPH.

This proof is based on the fact that $\mathsf{iPV_n}$ is complete with respect to frames which satisfy the Vp_n-property:

$$(Vp_n\text{-}property) \quad \text{for all finite sets of nodes } \bar{v}_+ = \bar{v} \cup \bar{v}_-, \bar{u}_+ = \bar{u} \cup \bar{u}_-$$
$$\text{such that } |\bar{v}| + |\bar{u}| \leq n,\ |\bar{v}_-| + |\bar{u}_-| \leq 2\text{: } wRv \wedge v \preccurlyeq \bar{v}_+ \wedge vR\bar{u}_+ \to$$
$$\exists x \succcurlyeq v(x \preccurlyeq \bar{v}_+ \wedge x\tilde{R}\bar{u}_+ \wedge \forall y \succ x(\exists z \in \bar{v}_+(z \preccurlyeq y) \vee (y \preccurlyeq \bar{v} \wedge y\tilde{R}\bar{u}))).$$

Note that in the formula above, if $|\bar{v}_-|+|\bar{u}_-|=0$, then x is just a tight predecessor of $\bar{v}$ holding $\bar{u}$. Thus a frame which satisfies all Vp_n-properties has the Vp^∞-property (Section 4.6). This is what we expect, since Visser's Scheme, which corresponds to the Vp^∞-property, consists of all principle Vp_n.

Before we treat the completeness proof for Visser's Principles we clarify the difference between the Vp-property and the not very elegant Vp_n-property. To make the discussion more transparent, we forget about the special treatment of boxed formulas in these schemes. Therefore, consider the following principle which is a special instance of Vp_1:

$$((A_1 \to B) \to A_2 \vee A_3) \rhd \bigvee_{i=1}^{3}((A_1 \to B) \to A_i).$$

If we look for the minimal requirement on frames for which they validate this principle, we arrive at the following property

$$\begin{aligned} &wRv \preccurlyeq v_1, v_2, v_3 \to \\ &\quad \exists x(v \preccurlyeq x \preccurlyeq v_1, v_2, v_3 \wedge \forall y \succ x((v_2 \preccurlyeq y \vee v_3 \preccurlyeq y) \vee y \preccurlyeq v_1)). \end{aligned}$$

(v_1 is $\bar{v}$ and v_2, v_3 is $\bar{v}_-$ in the definition of the Vp_n-property above.) We do not prove that the principle corresponds to this property, but it is instructive to see why the principle is valid on these frames. Suppose v forces $((A_1 \to B) \to A_2 \vee A_3)$ but not $(A_1 \to B) \to A_i$. Select nodes $v_i \succcurlyeq v$ such that $v_i \Vdash (A_1 \to B)$ but $v_i \nVdash A_i$. To derive a contradiction, we use the existence of a node x such that $v \preccurlyeq x \preccurlyeq v_1, v_2, v_3$ and for all $y \succ x$, we have $((v_2 \preccurlyeq y \vee v_3 \preccurlyeq y) \vee y \preccurlyeq v_1)$. Namely, since $x \preccurlyeq v_2, v_3$, it follows that $x \nVdash (A_2 \vee A_3)$. We show that $x \Vdash (A_1 \to B)$, and then we have a contradiction with $v \preccurlyeq x$ because $v \Vdash ((A_1 \to B) \to A_2 \vee A_3)$. Therefore, consider $y \succcurlyeq x$ and assume $y \Vdash A_1$. From $y \succcurlyeq x$ it follows that $y = x$ or $(y \preccurlyeq v_1 \vee v_2 \preccurlyeq y \vee v_3 \preccurlyeq y)$. Thus $(v_2 \preccurlyeq y \vee v_3 \preccurlyeq y)$. But then $y \Vdash (A_1 \to B)$, and whence $y \Vdash B$.

The example above showed that the sets $\bar{v}_-$ and $\bar{u}_-$ correspond to the formulas A_{n+1} and A_{n+2} in the principle Vp_n. In the example it could be that $v_1 = v_2$, in which case $\bar{v}_- = v_3$. This explains the requirement $|\bar{v}_-| + |\bar{u}_-| \leq 2$ in the Vp_n-property.

4.6.4. Lemma.

(*i*) The principle Vp_n corresponds to the Vp_n-property.

(*ii*) The logic $\mathsf{iPV_n}$ is canonical.

Proof (*i*) We only show that any frame which does not have the Vp_n-property has a valuation which refutes Vp_n. Consider such a frame $\mathcal{F}$. We leave it to the reader to check that if, in the Vp_n-property above, we change the words 'for all finite sets of nodes $\bar{v}_+ = \bar{v} \cup \bar{v}_-, \bar{u}_+ = \bar{u} \cup \bar{u}_-$ such that $|\bar{v}| + |\bar{u}| \leq n$, $|\bar{v}_-| + |\bar{u}_-| \leq 2$' to 'for all finite sets of nodes $\bar{v}_+ = \bar{v} \cup \bar{v}_-, \bar{u}_+ = \bar{u} \cup \bar{u}_-$ such that

$$\forall x \in \bar{v}_+ \forall y \in \bar{u}_- \neg(x\tilde{R}y) \text{ and } \forall x \in \bar{v} \forall y \in \bar{u} \neg(x\tilde{R}y) \text{ and} \qquad (4.12)$$
$$\forall x, y \in \bar{v}_+ (x \neq y \rightarrow (x \not\preccurlyeq y \wedge y \not\preccurlyeq x)).$$

and $|\bar{v}| + |\bar{u}| \leq n$, $|\bar{v}_-| + |\bar{u}_-| \leq 2$' , we still have an equivalent property. Therefore, we can conclude that in $\mathcal{F}$ there are finite sets of nodes $\bar{v}_+ = \bar{v} \cup \bar{v}_-, \bar{u}_+ = \bar{u} \cup \bar{u}_-$ such that (4.12) holds and $|\bar{v}| + |\bar{u}| \leq n$, $|\bar{v}_-| + |\bar{u}_-| \leq 2$, for which

$$wRv \preccurlyeq \bar{v}_+ \wedge vR\bar{u}_+ \wedge \forall u \succcurlyeq v(u \not\preccurlyeq \bar{v}_+ \vee \neg(u\tilde{R}\bar{u}_+) \vee \qquad (4.13)$$
$$\exists u' \succ u(\forall x \in \bar{v}_+(x \not\preccurlyeq u') \wedge (u' \not\preccurlyeq \bar{v} \vee \neg(u'\tilde{R}\bar{u})))).$$

Observe that if both $\bar{v}$ and $\bar{u}$ are empty, (4.13) cannot hold. For if so, then there exists a node $u' \succ v$ such that $u' \not\preccurlyeq \bar{v}$ or $\neg(u'\tilde{R}\bar{u})$ holds, quod non. We have to consider three cases: (a) $\bar{v}_+ = \{v'\}$, $\bar{u}_+$ is empty, (b) $\bar{u}_+ = \{u'\}$ and $\bar{v}_+$ is empty, (c) both $\bar{v}_+$ and $\bar{u}_+$ contain at least one node or one of them contains at least two nodes.

(a) In this case (4.13) cannot hold (take $u = v'$).

(b) If $\bar{u}$ is empty, (4.13) cannot hold (take $u = v$). If $\bar{u} = \bar{u}_+$, by (4.13), for all $x \succcurlyeq v$, if $x\tilde{R}u'$ there exists $x' \succcurlyeq x$ such that $x'\tilde{R}u'$ does not hold. Define the valuation

$$x \Vdash p \equiv_{def} x \not\preccurlyeq u'.$$

Clearly, in this model $v \nVdash \Box p$ holds. We show that $v \Vdash \neg\neg\Box p$ holds, and this proves that Vp_1 does not hold on the frame. To see that $v \Vdash \neg\neg\Box p$, consider $x \succcurlyeq y$. We have to show that $x \nVdash \neg\Box p$. By assumption there exists a node $x' \succcurlyeq x$ such that $x'\tilde{R}u'$ does not hold. Hence $x' \Vdash \Box p$, and thus $x \nVdash \neg\Box p$.

(c) To define a valuation which is going to refute Vp_n on $\mathcal{F}$, we want that either $\bar{u}_-$ contains at least one element or that $\bar{v}_-$ contains two elements. First we show how we can amend $\bar{v}_-$ and $\bar{u}_-$ in such a way that this holds, while keeping (4.13) and (4.12) valid. If $\bar{u}_-$ and $\bar{v}_-$ are empty, we take $x_1, x_2 \in \bar{v}$, $y_1, y_2 \in \bar{u}$, and redefine $\bar{v}_- = x_1$, $\bar{u}_- = y_1$ or $\bar{u}_- = y_1, y_2$ or $\bar{v}_- = x_1, x_2$ (if $x_1 \neq x_2$). Let us see that, depending on $\bar{v}$ and $\bar{u}$, nodes can be chosen such that this can be done and such that (4.13) and (4.12) hold for the new $\bar{v}, \bar{u}$. If $\bar{u}_-$ is empty and $\bar{v}_- = x$ there are two possibilities. If $\bar{v}$ contains a node $y \neq x$ we redefine $\bar{v}_- = x, y$. If not, $\bar{v} = \bar{v}_- = x$. By assumption $\bar{u}$ contains at least one element y. If for all $y \in \bar{u}$, $x\tilde{R}y$, then (4.13) cannot hold (take $u = x$). Take $y \in \bar{u}$ for which $x\tilde{R}y$ does not hold and redefine $\bar{u}_- = y$. This shows that from now on we can assume that $\bar{u}_-$ contains at least one element or that $\bar{v}_-$ contains two elements.

We only treat the case that both $\bar{v}_-$ and $\bar{u}_-$ contain one element, the other cases are similar. Let $\bar{v} = v_1, \ldots, v_k$, $\bar{u} = u_1, \ldots, u_m$ and $\bar{v}_- = v_{k+1}$, $\bar{u}_- = u_{m+1}$. Define a model on the given frame via the following valuation:

$$\begin{array}{lll} x \Vdash p_i & \equiv_{def} & x \not\preccurlyeq v_i \\ x \Vdash q_i & \equiv_{def} & v_i \preccurlyeq x, \text{ for some } i \leq k+1 \\ x \Vdash r_i & \equiv_{def} & x \not\preccurlyeq u_i. \end{array}$$

Let

$$A = \bigwedge_{i=1}^{k}(p_i \to q_i) \wedge \bigwedge_{i=1}^{m} \neg\Box r_i.$$

We show that $v \Vdash A \to p_{k+1} \vee \Box r_{m+1}$, $v \nVdash (A)(p_1, \ldots, p_{k+1}, \Box r_1, \ldots, \Box r_{m+1})$. To see that the second statement holds it suffices to see that $v_i \Vdash A$, $v_i \nVdash p_i$ and $u_i \nVdash r_i$, which we leave to the reader. We prove that $v \Vdash A \to p_{k+1} \vee \Box r_{m+1}$. Consider a node $u \succcurlyeq v$ such that $u \Vdash A$. Hence for all $i \leq m$, $u' \succcurlyeq u$, $u'\tilde{R}\bar{u}$. Furthermore, for all $i \leq m$, $u' \succcurlyeq u$, if $u' \not\preccurlyeq v_i$ then $u' \Vdash q_i$. In particular,

$$\forall u' \succcurlyeq u((u' \preccurlyeq \bar{v} \wedge u'\tilde{R}\bar{v}) \vee \exists x \in \bar{v}_+(x \preccurlyeq u')).$$

By (4.12), $\exists x \in \bar{v}_+(x \preccurlyeq u)$ implies that not $u\tilde{R}u_{m+1}$. Therefore, we can conclude $u \preccurlyeq \bar{v}$ or not $u\tilde{R}u_{m+1}$. All together this leads, by (4.13), to $u \not\preccurlyeq v_{k+1}$ or not $u\tilde{R}u_{m+1}$. Hence $u \Vdash p_{k+1} \vee \Box r_{m+1}$; what we wanted to show.

(*ii*) Assume that in the $\mathsf{iPV_n}$-canonical model we have $wRv \wedge v \preccurlyeq v_1, \ldots, v_{k+i}$ and $vRu_1, \ldots, u_{m+j}$, for some k, m, i, j such that $k + m \leq n$ $i + j = 2$. The proof that there is a node u which is a tight predecessor of $v_1, \ldots, v_k$ holding $u_1, \ldots, u_m$ and such that $u \preccurlyeq v_{k+1}, \ldots, v_{k+i}$ and $uRu_{m+1}, \ldots, u_{m+j}$, is similar to the proof of Proposition 4.6.2. The only difference occurs in the definition of the set Δ, which we define in this case as follows. Let $\hat{v} = v_1 \cap \ldots \cap v_k$ and $v^* = v_1 \cap \ldots \cap v_{k+i}$ and $\hat{u} = u_1 \cap \ldots u_m$ and $u^* = u_1 \cap \ldots u_{m+j}$, and let

$$\Delta = \{(E \to F) \mid F \in v^* \wedge E \notin \hat{v}\} \cup \{(\Box E \to F) \mid F \in v^* \wedge E \notin \hat{u}\}.$$

Let $*(\cdot)$ be the property

$*(x)$ $x \vdash A_1 \vee \ldots \vee A_q \vee \Box B_1 \vee \ldots \vee \Box B_r$ implies $\exists h$ ($A_h \in v^*$ or $B_h \in u^*$).

In a similar way as in Lemma 3.4.3 one can show that $*(\cdot)$ is an extendible property. We show that $*(v \cup \Delta)$ holds. Let $C = A_1 \vee \ldots \vee A_q \vee \Box B_1 \vee \ldots \vee \Box B_r$ and suppose $v \cup \Delta \vdash C$. This implies that there are conjuncts $D_1 = \bigwedge_{h=1}^{l_1}(E_h \to F_h)$ and $D_2 = \bigwedge_{h=1}^{l_2}(\Box E'_h \to F'_h)$, such that $F_h, F'_h \in v^*$ and $E_h \notin \hat{v}$ and $E'_h \notin \hat{u}$. Let F be the conjunction of all F_h, F'_h, and let $H_p = (\bigvee_{E_h \notin v_p} E_h)$ and $H'_p = (\bigvee_{E'_h \notin u_p} E'_h)$. Clearly, $F \in v^*$, $H_p \notin v_p$ and $H'_p \notin u_p$. Define a new conjunct of formulas in Δ:

$$D = \bigwedge_{p=1}^{k}(H_p \to F) \wedge \bigwedge_{p=1}^{m}(\Box H'_p \to F).$$

We have $v \vdash (D \to C)$ and thus $(D \to C) \in v$. Since

$$(D \to C) \rhd (D)(H_1, \ldots, H_k, \Box H'_1, \ldots, \Box H'_m, A_1, \ldots, A_q, \Box B_1, \ldots, \Box B_r)$$

and $k + m \leq n$, also

$$(D)(H_1, \ldots, H_k, \Box H'_1, \ldots, \Box H'_m, A_1, \ldots, A_q, \Box B_1, \ldots, \Box B_r) \in v.$$

From the construction of Δ it follows that v does not contain any of $(D \to H_h)$ or $\Box H'_h$. Therefore v contains either $(D \to A_h)$ or $\Box B_h$ for some h. This proves that $*(v \cup \Delta)$ holds. Let u be the $*$-extension of $v \cup \Delta$. The proof that u has the desired properties is similar to the corresponding part in the proof of Proposition 4.6.2 and is therefore left to the reader. $\Box$

4.6.5. Corollary. For all $0 < m < n$, $\nvdash_{\mathsf{iPV_m}} Vp_n$.

4.6.2 The Formalized Markov Scheme

The logic iMa is axiomatized over iK by the Formalized Markov Scheme

$$Ma \quad \Box\neg\neg(\Box A \to \bigvee \Box B_i) \to \Box(\Box A \to \bigvee \Box B_i).$$

Recall that the Formalized Markov Scheme is the partial formalization of Markov's Rule (Section 2.5). In Section 2.2 the relation between this rule and Visser's Scheme was explained. In Section 3.3 we saw that Ma is derivable in iPMV.

In this section we show that iMa is complete with respect to frames with the property

$$(Ma\text{-}property) \quad wRv \to \exists x \in Top\ (w\tilde{R}x \wedge v\tilde{R} = x\tilde{R}).$$

Recall (Section 3.4.2) that a node in Top, a top node, is a node x such that there is no node y with $x \prec y$. Note the similarity between the frame property for the Formalized Markov Scheme and property (4.10) discussed in Section 4.6 on Visser's Scheme.

A top node x for which $w\tilde{R}x$ and $v\tilde{R} = x\tilde{R}$ hold will be called a *markov-node for the pair* (w, v). On brilliant frames the Ma-property reads

$$wRv \to \exists x \in Top\ (wRx \wedge vR = xR).$$

Note that the logic iMa cannot be complete with respect to gathering frames which satisfy this stronger property. Since in such frames every top node which is a successor, satisfies $\Box\bot$, the formula $\Box\neg\neg\Box\bot$ holds on such frames.

Before we give the completeness proof for iMa we need a lemma.

4.6.6. Lemma. For the logic iMa we have that if $\Delta = \{D \mid \Gamma \vdash \Box D\}$, for some set Γ, and the set of formulas $\Delta, \Box A_1, .., \Box A_n, \neg\Box B_1, .., \neg\Box B_m$ is inconsistent, then Δ derives $(\bigwedge \Box A_i \to \bigvee \Box B_i)$.

Proof The first derivation shows that from the inconsistency of the formulas $\Delta, \Box A_1, .., \Box A_n, \neg\Box B_1, .., \neg\Box B_m$, it follows that $\Delta \vdash_{\mathsf{iMa}} \neg\neg(\Box \bigwedge A_i \to \bigvee \Box B_i)$.

$$\begin{array}{rcl}
\Delta, \Box A_1, .., \Box A_n, \neg\Box B_1, .., \neg\Box B_m & \vdash_{\mathsf{iMa}} & \bot \\
\Delta, \Box \bigwedge A_i, \neg\Box B_1 \ldots, \neg\Box B_m & \vdash_{\mathsf{iMa}} & \bot \\
\Delta & \vdash_{\mathsf{iMa}} & \Box \bigwedge A_i \to \neg\neg \bigvee \Box B_i \\
\Delta & \vdash_{\mathsf{iMa}} & \neg\neg(\Box \bigwedge A_i \to \bigvee \Box B_i).
\end{array}$$

The following derivation shows that $\Delta \vdash_{\mathsf{iMa}} \neg\neg(\Box \bigwedge A_i \to \bigvee \Box B_i)$ implies that $\Delta \vdash_{\mathsf{iMa}} (\bigwedge \Box A_i \to \bigvee \Box B_i)$.

$$\begin{array}{rll} \Delta & \vdash_{\mathsf{iMa}} & \neg\neg(\Box \bigwedge A_i \to \bigvee \Box B_i) \\ \Gamma & \vdash_{\mathsf{iMa}} & \Box\neg\neg(\Box \bigwedge A_i \to \bigvee \Box B_i) \\ \Gamma & \vdash_{\mathsf{iMa}} & \Box(\Box \bigwedge A_i \to \bigvee \Box B_i) \\ \Delta & \vdash_{\mathsf{iMa}} & \Box \bigwedge A_i \to \bigvee \Box B_i \\ \Delta & \vdash_{\mathsf{iMa}} & \bigwedge \Box A_i \to \bigvee \Box B_i. \end{array}$$

Note that the second step of the last derivation is the only place where the Formalized Markov Scheme is used. The special form of Δ is used in the first and the third step of the last derivation. $\Box$

4.6.7. Proposition. In $\mathcal{L}_\Box$:

(i) On finite frames Ma corresponds to the Ma-property.

(ii) The iMa-canonical model has the Ma-property.

Proof (i) Only the direction from left to right. Let $\mathcal{F}$ be a finite frame which does not have the Ma-property; there are $w, v, u_1, .., u_n$ with wRv and $v\tilde{R} = \{u_1, \ldots, u_n\}$ and

$$\forall x \in Top(w\tilde{R}x \wedge x\tilde{R} \subseteq v\tilde{R} \to v\tilde{R} \not\subseteq x\tilde{R})).$$

Let $T_i = \{y \mid w\tilde{R}y \wedge y\tilde{R} \subseteq v\tilde{R} \wedge \neg y\tilde{R}u_i\}$. Define a valuation on $\mathcal{F}$ via

$$x \Vdash p \quad \equiv_{def} \quad v\tilde{R}x$$

$$x \Vdash q_i \quad \equiv_{def} \quad \exists y \in T_i \ (y\tilde{R}x).$$

We show that with this valuation, $w \Vdash \Box\neg\neg(\Box p \to \bigvee \Box q_i)$ and $w \nVdash (\Box p \to \bigvee \Box q_i)$. The last part is obvious, since $w\tilde{R}v$ and $v \Vdash \Box p$. That $v \nVdash \Box q_i$ for all i, follows from the fact that $u_i \nVdash q_i$, for all i. Therefore, consider any top node y above some successor z of w. It suffices to show that $y \Vdash \Box p \to \bigvee \Box q_i$, because this would imply $z \Vdash \neg\neg(\Box p \to \bigvee \Box q_i)$. Note that $w\tilde{R}y$. If $y \Vdash \Box p$, then $y\tilde{R} \subseteq v\tilde{R}$. Hence, by assumption, $\neg y\tilde{R}u_i$, for some i. Therefore, $y \in T_i$, and thus $y \Vdash \Box q_i$.

(ii) Let $(W, \preccurlyeq, R, V)$ be the iMa-canonical model. Let w, v be two nodes such that wRv. We show that there is a *top* node x such that wRx and $vR = xR$. Let $\Delta = \{D \mid \Box D \in w\}$. It suffices to show that the set $\Delta, v_\Box, \{\neg\Box E \mid \Box E \notin v\}$ is consistent, as any maximal consistent extension of this set will have the desired properties. If it is not consistent, there are $\Box E_1, \Box E_2, \ldots, \Box E_n \notin v$ and $\Box B_1, \ldots, \Box B_m \in v$ such that $\Delta, \Box B_1, \ldots, \Box B_m, \neg\Box E_1 \ldots, \neg\Box E_n$ is inconsistent. But then lemma 4.6.6 implies that $\Delta \vdash \bigwedge \Box B_i \to \bigvee \Box E_i$. Hence $(\Box E_1 \vee \ldots \vee \Box E_n)$ is in v, and that cannot be. $\Box$

4.7 Independence

In this section we explain why all principles discussed above are independent. We call two principles A and B independent if they do not derive one another. In particular, if A is a principle of the preservativity logic then we say that B is independent from A if $\mathsf{iPA} \nvdash B$, i.e. if B is not derivable from A in the system given by $P1$, $P2$ and Dp and the rules Modus Ponens and the Preservation Rule. If A is a principle of the provability logic, we say that B is independent from A if $\mathsf{iA} \nvdash B$, i.e. if B is not derivable from A in the system given by K and the rules Modus Ponens and the Necessitation Rule.

To show that B is independent from A it suffices to show that there is a model for A on which B is not valid. If A is complete with respect to some class of frames, this model can be obtained by giving a valuation on such a frame such that B is not valid under this valuation. Using the results in this chapter it is easy to prove the following proposition.

4.7.1. Proposition.

- (i) The following principles are independent: Löb's Preservativity Principle, Montagna's Principle and Visser's Scheme. Löb's Preservativity Principle derives the principle $4p$.
- (ii) For all $n > m$, the m-th Visser's Principle Vp_m does not derive the n-th Visser's Principle Vp_n, and Vp_n derives Vp_m.
- (iii) The following principles are independent: Löb's Principle, Leivant's Principle and the Formalized Markov Scheme. Löb's Principle as well as Leivant's Principle derive the principle 4.
- (iv) Löb's Preservativity Principle derives Löb's Principle. The principle $4p$ derives the principle 4 and Leivant's Principle. Montagna's Principle and Visser's Scheme derive the Formalized Markov Scheme.

Proof (i) As explained above this follows easily from the completeness results in the previous sections. That Löb's Preservativity Principle derives the principle $4p$ is shown in Section 4.3.

(ii) This is Corollary 4.6.5.

(iii) As explained above this follows easily from the completeness results in the previous sections. That Löb's Principle derives the principle 4 is explained in Section 4.3.1.

(iv) These statements are proved in Section 3.3. □

Chapter 5

Completeness

In this chapter we show that the logic iPH which we conjecture to be the preservativity logic of HA is complete with respect to the gathering conversely well-founded *Mp Vp*-frames (Section 5.1). In Section 5.2 we use this result to prove that some rules are admissible for iPH. These two sections are the heart of Part I. In Section 5.3 we return to the logic iH in the language $\mathcal{L}_{\Box}$. There we present a completeness proof for iH with respect to the class of finite transitive irreflexive brilliant *Le Ma*-frames. In Section 3.3 we showed that iH is contained in iPH. In Section 5.4 we use the completeness proof for iH to show that iPH is not conservative over iH with respect to $\mathcal{L}_{\Box}$. Thereby we disprove the conjecture that iH is the provability logic of HA.

5.1 Modal completeness of preservativity logic

First we sketch the completeness proof for iPH.

Proof sketch

For formulas A that are not derivable in iPH we have to show that there is a model that refutes A and which has a gathering conversely well-founded *Mp Vp*-frame. To construct such a model we use the construction method (Section 3.4.5) with respect to a certain finite adequate set X. As expected, the resulting model will in general be infinite, since most of the frames which validate Visser's Scheme are not finite (Section 4.6). We use four subconstructions $\beta, \delta, \zeta, \xi$. Each of them expands a frame by adding nodes from the canonical model to it in an adequate way. We will explain how they select these nodes. To ensure that the new nodes x have certain properties we require that α_x has the corresponding properties in the canonical model. For example, if we demand $x \preccurlyeq \sigma$, then we choose α_x in such a way that $\alpha_x \preccurlyeq \alpha_\sigma$ holds in the canonical model. If $x\tilde{R} \subseteq \sigma\tilde{R}$ is the desired property, we demand that $(\alpha_\sigma)_{\Box} \subseteq (\alpha_x)_{\Box}$. Note that by Lemma 4.5.1 this is equivalent with $\alpha_x\tilde{R} \subseteq \alpha_\sigma\tilde{R}$.

The construction β choses nodes $\sigma * \langle B \to C\rangle$ and $\sigma * \langle B \rhd C\rangle$ as is usual in the construction method. In combination with δ, the construction ζ ensures that the final frame has the *Mp*-property (Section 4.5): for nodes $\sigma R\tau \preccurlyeq \tau'$ it constructs a node $a = \sigma * \langle m, \tau, \tau'\rangle$ such that in the final model σRa and $\tau \preccurlyeq a \preccurlyeq \tau'$ and $a\tilde{R} \subseteq \tau'\tilde{R}$ hold.

In combination with δ, the construction ξ ensures that the final frame has the *Vp*-property (Section 4.6): for nodes $\sigma R\tau \preccurlyeq \tau_1, \ldots, \tau_n$ it constructs a node $a = \tau * \langle v, \tau_1, \ldots, \tau_n\rangle$ such that in the final model $v \preccurlyeq a$ and a is a tight predecessor of $\tau_1, \ldots, \tau_n$ for τ.

The construction δ is an addition to both ζ and ξ. If we want $\pi\tilde{R} \subseteq \pi'\tilde{R}$ to hold in the final model and we add a node $\pi\tilde{R}\pi''$, then δ constructs a node $a = \pi' * \langle m, \pi''\rangle$ such that $\pi' Ra \preccurlyeq \pi''$. Therefore, $\pi'\tilde{R}\pi''$ will hold. The discussion above shows that we have to ensure that $\pi\tilde{R} \subseteq \pi'\tilde{R}$ holds in the following cases: $\pi = \sigma * \langle m, \tau, \pi'\rangle$ or $\pi = \sigma * \langle m, \pi'\rangle$ or $\pi' = \pi * \langle v, \tau_1, \ldots, \tau_n\rangle$.

The following tricks are used in the construction in order to guarantee that no unnecessary nodes are selected. The reader interested in the construction but not in the complications which arise from this attempt for efficiency can skip these details.

If we want to guarantee that $\pi\tilde{R} \subseteq \pi'\tilde{R}$ we do not have to add a node $\pi' * \langle m, \pi''\rangle$ for all the nodes π'' with $\pi\tilde{R}\pi''$. For example, it could be that $\pi'\tilde{R}\pi$ already holds. Therefore, we will define a property $*(\pi, \pi', \pi'')$ that holds exactly when we want $\pi\tilde{R} \subseteq \pi'\tilde{R}$ to hold, and $\pi\tilde{R}\pi''$ holds but not $\pi'\tilde{R}\pi''$. For a similar reason, we define properties $\star(\sigma, \tau, \tau')$ and $\circ(\tau, \tau_1, \ldots, \tau_n)$ which holds exactly when we have to add nodes $\sigma * \langle m, \tau, \tau'\rangle$ or $\tau * \langle v, \tau_1, \ldots, \tau_n\rangle$ respectively. To recognize if $*$ or $\star$ hold we use a function γ. If for example $\sigma R\tau \preccurlyeq \tau'$ holds but $\tau = \sigma * \langle m, \tau'\rangle$, then we do not have to add a node $\sigma * \langle m, \tau, \tau'\rangle$ since τ itself will have the desired properties. The same holds for instance in the case that $\tau = \sigma * \langle m, \pi, \pi'\rangle$ and $\pi' = \sigma' * \langle m, \tau'\rangle$. We let the function γ cover all these cases by defining $\gamma(\sigma)$ inductively as: if $\sigma = \sigma'' * \langle m, \tau, \sigma'\rangle$ or $\sigma = \sigma'' * \langle m, \sigma'\rangle$, then $\gamma(\sigma) = \gamma(\sigma')$, and $\gamma(\sigma) = \sigma$ otherwise.

We use one device more to lower the number of nodes we have to construct. We let R and $\preccurlyeq''$ be one-step relations: intuitively, we have $\sigma R\tau$ if there is no $\sigma R\tau' R\tau$, and similarly for $\preccurlyeq''$. We let $\preccurlyeq$ be the transitive closure of $\preccurlyeq''$ and define R^* and $\preccurlyeq^*$ as the minimal extensions of R and $\preccurlyeq$ for which R^* is gathering and $\preccurlyeq^*$ is a partial order and $(\preccurlyeq^*;R^*) \subseteq R^*$ holds (Lemma 5.1.3). For example, when $\sigma R\sigma'(\preccurlyeq'';R)\tau$ we put $\sigma R^*\sigma' R^*\tau$ and $\sigma' \preccurlyeq^* \tau$. The relations R^* and $\preccurlyeq^*$ will be the modal and intuitionistic relation in our final model. The use of R and $\preccurlyeq''$ is best illustrated by an example. Suppose we have to define a node $\sigma * \langle m, \tau, \tau'\rangle$, and $\sigma R^*\tau$ holds and $\sigma R\tau$ does not hold. It follows from the definition of R^* that there is a node $\sigma' R\tau$. We only construct the node $\sigma' * \langle m, \tau, \tau'\rangle$ and observe that also $\sigma R^*\sigma' * \langle m, \tau, \tau'\rangle$. Hence $\sigma' * \langle m, \tau, \tau'\rangle$ has the desired properties of $\sigma * \langle m, \tau, \tau'\rangle$ in the final model. Therefore, the latter node does not have to be constructed.

Finally, in construction δ we select the nodes $a = \pi' * \langle m, \pi'' \rangle$ in such a way that $(\alpha_{\pi''})_\Box = (\alpha_a)_\Box$. This allows us to ensure that $\pi''\tilde{R} = a\tilde{R}$ in the final model. And that guarantees that for the situation $\pi' Ra \preccurlyeq \pi''$, which arises from the definition of a, we do not again have to add a node $\pi' * \langle m, a, \pi'' \rangle$, since the node a has the same properties. Lemma 5.1.1 shows that we can choose α_a as desired. This completes the informal discussion of the completeness proof for iPH.

5.1.1. Lemma. In any canonical model of a logic containing Mp it holds that

$$\text{if } w_\Box \subseteq v_\Box \wedge vRu \text{ then } \exists u'(wRu' \preccurlyeq u \wedge u'_\Box = u_\Box).$$

(By Lemma 4.5.1 this is equivalent with the property that if $w_\Box \subseteq v_\Box$ and vRu, then there exists a node u' such that $wRu' \preccurlyeq u$ and $u'\tilde{R} = u\tilde{R}$.)

Proof Let $*(\cdot)$ be the property

*(x) for all A: $w \vdash x \rhd A$ implies $A \in u$.

In Lemma 3.4.3 we have shown that $*$ is an extendible w-successor property (in the lemma it is denoted with $\star$) and that $*(u_\Box)$ holds. Let u' be the $*$-extension of $u_\Box$. Clearly, u' has the desired properties. □

5.1.2. Remark. In any gathering model,

$$\text{if } w'Rw \preccurlyeq v_1Rv_2 \preccurlyeq v_3Rv_4 \preccurlyeq \ldots v_n \text{ then, for all } i,\ w \preccurlyeq v_i.$$

This can be easily seen, using the gatheringness and the fact that $(\preccurlyeq;R) \subseteq R$.

The relations R^* and $\preccurlyeq^*$

Let R and $\preccurlyeq$ respectively be a binary relation and a partial order on a finite set W. We define relations $\preccurlyeq^*$ and R^* which are the minimal extensions of $\preccurlyeq$ and R such that R^* is gathering and $(\preccurlyeq^*;R^*) \subseteq R^*$ holds. The idea behind these extensions is given by Remark 5.1.2. We define $\preccurlyeq^*$ and R^* via

$$\begin{aligned} wR^*v \quad &\equiv_{def} \ \exists x(w \preccurlyeq xRv) \vee \exists x_1 \ldots x_n yy'(w \preccurlyeq y \preccurlyeq x_1 \wedge \\ &\qquad \wedge y'Ry \wedge w \preccurlyeq x_1Rx_2 \preccurlyeq x_3 \ldots x_nRv) \\ w \preccurlyeq^* v \quad &\equiv_{def} \ w \preccurlyeq v \vee \exists xyzz'(w \preccurlyeq z \preccurlyeq x \wedge z'Rz \wedge w \preccurlyeq xR^*y \preccurlyeq v). \end{aligned}$$

We write $\tilde{R}^*$ for $(R^*;\preccurlyeq^*)$. The first disjunct in the definition of R^* arises from the fact that we want to have $(\preccurlyeq^*;R^*) \subseteq R^*$. The second disjunct arises from the fact that we want R^* to be gathering. Namely, by Remark 5.1.2, $y'Ry$ and $y \preccurlyeq x_1Rx_2 \preccurlyeq x_3 \ldots x_n$ implies $y \preccurlyeq^* x_n$, since we construct $\preccurlyeq^*$ and R^* in such a way that R^* is gathering. Thus we have $w \preccurlyeq y \preccurlyeq^* x_nRv$, hence $w(\preccurlyeq^*;R)v$. As we want to have $(\preccurlyeq^*;R^*) \subseteq R^*$, we have to demand wR^*v. Similar explanations apply to the definition of $\preccurlyeq^*$.

5.1.3. Lemma. Let R and $\preccurlyeq$ respectively be a binary relation and a partial order on a finite set W. If both

$$wRv \preccurlyeq^* u \to \exists x(wR^*x \wedge v \preccurlyeq^* x \preccurlyeq^* u \wedge x(\preccurlyeq;R) \subseteq u\tilde{R}^*)$$

$$wRv \preccurlyeq^* u_1, \ldots, u_n \to \exists x(v \preccurlyeq^* x \preccurlyeq^* u_1, \ldots, u_n \wedge$$
$$v(\preccurlyeq;R) \subseteq x\tilde{R}^* \wedge \forall y \succ^* x(u_i \preccurlyeq^* y, \text{ for some } i))$$

then $(W, R^*, \preccurlyeq^*)$ is a gathering $Mp\,Vp$-frame.

Proof Although one have to check many cases, it is not difficult to see that $(W, R^*, \preccurlyeq^*)$ is indeed a frame, ($\preccurlyeq^*$ is a partial order and $(\preccurlyeq^*;R^*) \subseteq R^*$ holds), and that it is gathering. We show that

$$wR^*v \preccurlyeq^* u \to \exists x(wR^*x \wedge v \preccurlyeq^* x \preccurlyeq^* u \wedge x\tilde{R}^* \subseteq u\tilde{R}^*)$$

$$wR^*v \preccurlyeq^* u_1, \ldots, u_n \to \exists x(v \preccurlyeq^* x \preccurlyeq^* u_1, \ldots, u_n \wedge$$
$$x\tilde{R}^* \subseteq u\tilde{R}^* \wedge \forall y \succ^* x(u_i \preccurlyeq^* y, \text{ for some } i))$$

hold, that is, that $(W, R^*, \preccurlyeq^*)$ is an $Mp\,Vp$-frame. The following two Claims suffice.
Claim 1 If wR^*v then there exists a node w' such that $w'Rv$ and for all $w'R^*v'$, also wR^*v'.
Proof of Claim 1 Suppose wR^*v. This implies that there are $x_1, y_1, \ldots x_n, y_n$, such that

$$w \preccurlyeq x_1Ry_1 \preccurlyeq x_2R \ldots \preccurlyeq x_nRy_n = v,$$

and *either* $n = 1$, so $v = y_1$, *or* $\exists zz'(w \preccurlyeq z \preccurlyeq x_1 \wedge z'Rz)$. In both cases, $w' = x_n$ has the desired properties. This proves Claim 1.
Claim 2 If $x(\preccurlyeq;R) \subseteq u\tilde{R}^*$, then $x\tilde{R}^* \subseteq u\tilde{R}^*$.
Proof of Claim 2 Suppose xR^*a. This implies there are $x \preccurlyeq a_1Ra_2 \preccurlyeq \ldots Ra$ such that *either* $a_2 = a$ *or* $\exists bb'(x \preccurlyeq b \preccurlyeq a_1 \wedge b'Rb)$. By assumption $u\tilde{R}^*a_2$, say $uR^*u' \preccurlyeq a_2$. In the first case, we clearly have $u\tilde{R}^*a$. In the second case, since uR^*u' there is $u''Ru'$, from which it follows that $u' \preccurlyeq^* a$. Hence also $u\tilde{R}^*a$. This proves Claim 2. □

5.1.4. Theorem. $\vdash_{\mathsf{iPH}} A$ iff A is valid on all gathering $Mp\,Vp$-frames for which the modal relation is conversely well-founded.

Proof[1] We only treat the direction from right to left. Suppose $\nvdash_{\mathsf{iPH}} A$. We construct a gathering $Mp\,Vp$-model for which the modal relation is conversely well-founded by the construction method (see Subsection 3.4.5). Let X be a finite adequate set, containing A, such that $B \rhd C \in X$ implies $\Box C \in X$. Consider the

[1] The port proof.

iPH-canonical model and let R' and $\preccurlyeq'$ be the relations on this model. Let $\alpha_{\langle\rangle}$ be a node at which A is not valid. With $W^*, R^*, \preccurlyeq^*$ we denote respectively the domain and the relations of the model M^* we are going to construct.

Using the construction method, we construct binary relations R and $\preccurlyeq''$ along with a set W^*. We denote the reflexive transitive closure of $\preccurlyeq''$ by $\preccurlyeq$. Then we define R^* and $\preccurlyeq^*$ as explained above, and show that in $(W^*, \preccurlyeq^*, R^*)$,

$$\sigma R\tau \preccurlyeq^* \tau' \to \exists x(\sigma R^* x \wedge \tau \preccurlyeq^* x \preccurlyeq^* \tau' \wedge x(\preccurlyeq;R) \subseteq \tau' \tilde{R}^*) \tag{5.1}$$

$$\begin{aligned}\sigma R\tau \preccurlyeq^* \tau_1, \ldots, \tau_n \to \exists x(&\tau \preccurlyeq^* x \preccurlyeq^* \tau_1, \ldots, \tau_n \wedge \\ &\tau(\preccurlyeq;R) \subseteq x\tilde{R}^* \wedge \forall y \succ^* x(\tau_i \preccurlyeq^* y, \text{ for some } i))\end{aligned} \tag{5.2}$$

and apply Lemma 5.1.3 to conclude that $(W^*, R^*, \preccurlyeq^*)$ is a gathering $Mp\,Vp$-frame. Finally, we show that R^* is conversely well-founded.

During the construction we guarantee that

$$\text{if } \sigma R\tau \text{ respectively } \sigma \preccurlyeq \tau, \text{ then } \alpha_\sigma R' \alpha_\tau \text{ respectively } \alpha_\sigma \preccurlyeq' \alpha_\tau. \tag{5.3}$$

We will often use (5.3) without mentioning it.

First some notations and conventions. We write $\sigma * A$ for $\sigma * \langle A \rangle$ and $\sigma_\Box$ for $(\alpha_\sigma)_\Box$. For a sequence σ, we define $\gamma(\sigma)$ inductively via: if $\sigma = \sigma'' * \langle m, \tau, \sigma' \rangle$ or $\sigma = \sigma'' * \langle m, \sigma' \rangle$, then $\gamma(\sigma) = \gamma(\sigma')$, and $\gamma(\sigma) = \sigma$ otherwise.

For $B \rhd C \in X \backslash \alpha_\sigma$, the node $\alpha_{\sigma * \langle B \rhd C \rangle}$ is a node such that $\alpha_\sigma R' \alpha_{\sigma * \langle B \rhd C \rangle}$, and $B \in \alpha_{\sigma * \langle B \rhd C \rangle}$ while $C \notin \alpha_{\sigma * \langle B \rhd C \rangle}$. For $(B \to C) \in X \backslash \alpha_\sigma$ the node $\alpha_{\sigma * \langle B \to C \rangle}$ is a node for which $\alpha_\sigma \preccurlyeq' \alpha_{\sigma * \langle B \to C \rangle}$, and $B \in \alpha_{\sigma * \langle B \to C \rangle}$ while $C \notin \alpha_{\sigma * \langle B \to C \rangle}$ (Section 3.4.5).

For $\sigma R\tau \preccurlyeq^* \tau'$, the node α_a, where $a = \sigma * \langle m, \tau, \tau' \rangle$, is a node with the following properties: $\alpha_\sigma R' \alpha_a$, $(\alpha_a)_\Box = (\alpha_{\tau'})_\Box$ and $\alpha_\tau \preccurlyeq' \alpha_a \preccurlyeq' \alpha_\tau$. Note that the existence of α_a is guaranteed by Proposition 4.5.2, using (5.3). Observe that, by Lemma 4.5.1, $(\alpha_a)_\Box = (\alpha_{\tau'})_\Box$ implies $\alpha_a \tilde{R}' \subseteq \alpha_{\tau'} \tilde{R}'$ $(\tilde{R}' = (R';\preccurlyeq'))$.

For $\sigma R\tau \preccurlyeq^* \tau_1, \ldots, \tau_n$, the node α_a, where $a = \tau * \langle v, \tau_1, \ldots, \tau_n \rangle$, is a node such that $\alpha_\tau \preccurlyeq' \alpha_a$ and α_a is a tight predecessor of $\alpha_{\tau_1}, \ldots, \alpha_{\tau_n}$ for α_τ, that is: $\alpha_a \preccurlyeq' \alpha_{\tau_1}, \ldots, \alpha_{\tau_n}$, $\alpha_\tau \tilde{R}' \subseteq \alpha_a \tilde{R}'$ and for all $x \succ' \alpha_a$, $\alpha_{\tau_i} \preccurlyeq' x$ for some i. Note that such a node exists by Corollary 4.6.3.

If $\sigma'(\preccurlyeq;R)\pi$, and σ is either $\gamma(\sigma')$ or $\sigma' * \langle v, \tau_1, \ldots, \tau_n \rangle$, the node $\alpha_{\sigma * \langle m, \pi \rangle}$ is a node such that $\alpha_\sigma R' \alpha_{\sigma * \langle m, \pi \rangle} \preccurlyeq' \alpha_\pi$ and $(\alpha_{\sigma * \langle m, \pi \rangle})_\Box = \pi_\Box$. Note that such nodes exist by Lemma 5.1.1, using the fact that $\sigma'_\Box = \sigma_\Box$.

We define properties $i(\cdot), p(\cdot)$ and $*(\cdot), \circ(\cdot), \star(\cdot)$ on respectively pairs of nodes and formulas, and sequences of nodes:

$$\begin{array}{ll} i(\sigma, B \to C) & (B \to C) \in X \backslash \sigma \wedge \neg \exists \sigma'(\sigma \preccurlyeq \sigma' \wedge B \in \sigma' \wedge C \notin \sigma') \\ p(\sigma, B \rhd C) & B \rhd C \in X \backslash \sigma \wedge \neg \exists \sigma'(\sigma R^* \sigma' \wedge B \in \sigma' \wedge C \notin \sigma') \end{array}$$

$$\begin{array}{ll}
\star(\sigma,\tau,\tau') & \sigma R\tau \preccurlyeq^* \tau' \wedge \gamma(\tau)\neq\gamma(\tau') \\
\circ(\tau,\tau_1,\ldots,\tau_n) & \exists\sigma(\sigma R\tau)\wedge \tau \preccurlyeq^* \tau_1,\ldots,\tau_n \\
*(\sigma,\sigma',\tau) & \sigma(\preccurlyeq;R)\tau \wedge \neg(\sigma'(R^*;\preccurlyeq^*)\tau)\wedge \\
 & (\sigma'=\gamma(\sigma)\vee \sigma'=\sigma*\langle\tau_1,\ldots,\tau_n\rangle,\ \text{for some } \tau_i).
\end{array}$$

Note that these properties can change during the construction. For example, if Y, Y' are two distinct sets of constructed nodes containing σ, σ', τ, $*(\sigma,\sigma',\tau)$ can hold in Y but not in Y'.

The construction of $(W^*, R, \preccurlyeq)$ uses four subconstructions, $\beta, \delta, \zeta, \xi$, which we will apply in a certain order. Every subconstruction consists of making an extension of the frame constructed so far by constructing some new nodes. The result, $\beta(Y)$, of the application of β to a frame $Y = (W_Y, \preccurlyeq_Y, R_Y)$ results in a frame $(W_{\beta(Y)}, \preccurlyeq_{\beta(Y)}, R_{\beta(Y)})$. Similarly for δ, ζ and ξ. When we say that for some nodes σ, τ, τ' in Y, $\star(\sigma,\tau,\tau')$ does (not) hold in Y, we read $\preccurlyeq_Y$ for $\preccurlyeq$, and similarly for the other relations. Similarly for the other properties. Again, $\preccurlyeq_Y$ is the transitive closure of $\preccurlyeq''_Y$, thus to define $\preccurlyeq_Y$ it suffices to define $\preccurlyeq''_Y$. The definitions of $\beta, \delta, \zeta, \xi$ run as follows.

$$\begin{array}{rcl}
W_{\beta(Y)} & = & W_y \cup \{\sigma * \langle B \to C\rangle \mid i(\sigma, B\to C) \text{ holds in } Y\} \cup \\
 & & \{\sigma * \langle B \rhd C\rangle \mid p(\sigma, B\to C) \text{ holds in } Y\} \\
\preccurlyeq''_{\beta(Y)} & = & \preccurlyeq''_Y \cup \{(\sigma, \sigma*\langle B\to C\rangle) \mid \sigma*\langle B\to C\rangle \notin Y\} \\
R_{\beta(Y)} & = & R_Y \cup \{(\sigma, \sigma*\langle B\rhd C\rangle) \mid \sigma*\langle B\rhd C\rangle \notin Y\}
\end{array}$$

$$\begin{array}{rcl}
W_{\zeta(Y)} & = & W_y \cup \{\sigma*\langle m,\tau,\tau'\rangle \mid \star(\sigma,\tau,\tau') \text{ holds in } Y\} \\
\preccurlyeq''_{\zeta(Y)} & = & \preccurlyeq''_Y \cup \\
 & & \{(\tau, \sigma*\langle m,\tau,\tau'\rangle), (\sigma*\langle m,\tau,\tau'\rangle, \tau') \mid \sigma*\langle m,\tau,\tau'\rangle \notin Y\} \\
R_{\zeta(Y)} & = & R_Y \cup \{(\sigma, \sigma*\langle m,\tau,\tau'\rangle) \mid \sigma*\langle m,\tau,\tau'\rangle \notin Y
\end{array}$$

$$\begin{array}{rcl}
W_{\xi(Y)} & = & W_y \cup \{\tau*\langle v,\tau_1,\ldots,\tau_n\rangle \mid \circ(\tau,\tau_1,\ldots,\tau_n) \text{ holds in } Y\} \\
\preccurlyeq''_{\xi(Y)} & = & \preccurlyeq''_Y \cup \{(\tau, \tau*\langle v,\tau_1,\ldots,\tau_n\rangle), (\tau*\langle v,\tau_1,\ldots,\tau_n\rangle, \tau_i) \mid \\
 & & \tau*\langle v,\tau_1,\ldots,\tau_n\rangle \notin Y, i\leq n\} \\
R_{\xi(Y)} & = & R_Y
\end{array}$$

$$\begin{array}{rcl}
W_{\delta(Y)} & = & W_y \cup \{\sigma'*\langle m,\tau\rangle \mid *(\sigma,\sigma',\tau) \text{ holds in } Y\} \\
\preccurlyeq''_{\delta(Y)} & = & \preccurlyeq''_Y \cup \{(\sigma'*\langle m,\tau\rangle, \tau) \mid \sigma'*\langle m,\tau\rangle \notin Y\} \\
R_{\delta(Y)} & = & R_Y \cup \{(\sigma', \sigma'*\langle m,\tau\rangle) \mid \sigma'*\langle m,\tau\rangle \notin Y\}.
\end{array}$$

Let $k = (|\{B \to C|(B \to C) \in X\}| + 1) \cdot |\{\Box B|\Box B \in X\}|$. We define an iterated version of $\beta(Y)$, $\bar{\beta}(Y)$, to be the frame $(\bigcup_{i=0}^{k} W_{Y_i}, \bigcup_{i=0}^{k} \preccurlyeq_{Y_i}, \bigcup_{i=0}^{k} R_{Y_i})$, where $Y_0 = Y$ and $Y_{i+1} = \beta(Y_i)$. It is easy to see that $i(\sigma, B \to C)$ or $p(\sigma, B \rhd C)$ can never hold in $\bar{\beta}(Y)$.

Now define frames $Y_0, Y_1, \ldots$ via: $Y_0 = (W_{Y_0}, \preccurlyeq_{Y_0}, R_{Y_0},)$, where $W_{Y_0} = \{\langle\rangle\}$, $\preccurlyeq_{Y_0} = \{(\langle\rangle, \langle\rangle)\}$ and R_{Y_0} is empty, and

$$
\begin{array}{lll}
Y_{6n+1} = \bar{\beta}(Y_{6n}) & Y_{6n+3} = \bar{\beta}(Y_{6n+2}) & Y_{6n+5} = \bar{\beta}(Y_{6n+4}) \\
Y_{6n+2} = \zeta(Y_{6n+1}) & Y_{6n+4} = \xi(Y_{6n+3}) & Y_{6n+6} = \delta(Y_{6n+5}).
\end{array}
$$

Let $W^* = \bigcup_i W_{Y_i}$, and let $\preccurlyeq$ be the transitive closure of $\bigcup_i \preccurlyeq_{Y_i}$, and let $R = \bigcup R_{Y_i}$. We show that (5.1) holds in W^*: if $\sigma R \tau \preccurlyeq^* \tau'$ and $\gamma(\tau) \neq \gamma(\tau')$ it is clear that there will be a node $x = \sigma * \langle m, \tau, \tau' \rangle$ such that $\sigma R^* x$ and $\tau \preccurlyeq^* x \preccurlyeq^* \tau'$ and $x(\preccurlyeq;R) \subseteq \tau' \tilde{R}^*$. We show that also in the case that $\sigma R \tau \preccurlyeq^* \tau'$ but $\gamma(\tau) = \gamma(\tau')$, there exists such a node x, namely $x = \tau$. It suffices to show that $\tau(\preccurlyeq;R) \subseteq \tau' \tilde{R}^*$. Therefore, assume $\tau(\preccurlyeq;R)\pi$. Thus, by construction, $\gamma(\tau') = \gamma(\tau)\tilde{R}^*\pi$. If $\gamma(\tau) = \tau'$ this gives $\tau' \tilde{R}^* \pi$. If $\gamma(\tau') \neq \tau'$, there exists $\sigma' R \tau'$. Because also $\tau'(\preccurlyeq;\tilde{R}^*)\pi$, since $\tau' \preccurlyeq \gamma(\tau')$, we can again conclude $\tau' \tilde{R}^* \pi$.

To see that (5.2) also holds, first observe that the construction is such that if $\sigma R \tau \preccurlyeq^* \tau_1, \ldots, \tau_n$, there exists a node $x = \tau * \langle v, \tau_1, \ldots, \tau_n \rangle$ such that $\tau \preccurlyeq^* x \preccurlyeq^* \tau_i$ and $\tau(\preccurlyeq;R) \subseteq x\tilde{R}^*$. Let Y_m be the first Y_i in which x occurs. It is clear that in Y_m we have $\forall y \succ^* x(\tau_i \preccurlyeq^* y)$, for some i. We show that this remains the case during the construction. We show this by induction on Y_n. The case Y_m is done. The case $(n > m)$. Assume $x \prec^* y$ holds in Y_n but not in Y_{n-1}. Without loss of generality assume that there is no $x \prec^* y' \prec^* y$. First, observe that the construction is such that $\sigma' R \tau'$ implies that $\tau' = \sigma' * D$, for some D of the form $B \rhd C, \langle m, \pi, \pi' \rangle$ or $\langle m, \pi \rangle$. Therefore, there is no x' with $x' R x$. Hence $x \prec^* y$ implies $x \prec y$. And since there is no $x \prec^* y' \prec^* y$, $x \prec'' y$. If $Y_n = \bar{\beta}(Y_{n-1})$, this implies $y = x * \langle B \to C \rangle$. We show that this cannot be, by showing that $i(x, B \to C)$ can never hold. It suffices to show that $i(x, B \to C)$ does not hold in Y_m. Note that all τ_i are already elements of some Y_j with $j < m$. This implies that $i(\tau_i, B \to C)$ does not hold in Y_m. Consider $(B \to C) \notin x$. Either $B \in x$ and $C \notin x$, in which case $i(x, B \to C)$ does not hold, or $B \notin x$. In the last case, $B \to C, B \notin \alpha_x$. Since α_x is a tight predecessor of $\alpha_{\tau_1}, \ldots, \alpha_{\tau_n}$ in the canonical model, this implies that $(B \to C) \notin \alpha_{\tau_i}$, for some i. Because $i(\tau_i, B \to C)$ does not hold in Y_m, this implies that there exists τ' such that $\tau_i \preccurlyeq \tau'$ and $B \in \tau'$ while $C \notin \tau'$. Clearly, this implies that $i(x, B \to C)$ does not hold in Y_m. Now consider the case in which $Y_n = \zeta(Y_{n-1})$. The fact that $x \prec'' y$ holds in Y_n but not in Y_{n-1}, implies that $x = \pi \preccurlyeq \sigma' * \langle m, \pi, \pi' \rangle = y$. Hence $\sigma' R x$. But we concluded before that there is no x' with $x' R x$, a contradiction. In the case that $Y_n = \xi(Y_{n-1})$, we have $x \preccurlyeq x * \langle v, \tau'_1, \ldots, \tau'_m \rangle = y$. But this implies that there exists x' with $x' R x$, contradicting our previous observation that there is no x' with $x' R x$. We leave the remaining case, $Y_n = \delta(Y_{n-1})$, to the reader. This completes the proof that (5.2) holds.

Since (5.1) and (5.2) hold, we can apply Lemma 5.1.3 to conclude that the frame $(W^*, R^*, \preccurlyeq^*)$ is a gathering *Mp Vp*-frame. To show that R^* is conversely well-founded, it suffices to show that

$$\sigma R^* \tau \text{ implies } |\{\Box B \in X | \Box B \notin \tau\}| < |\{\Box B \in X | \Box B \notin \sigma\}|,$$

a proof which we leave to the reader. The valuation

$$\sigma \Vdash p \equiv_{def} \alpha_\sigma \Vdash p, \text{ for p} \in X.$$

(see Subsection 3.4.5) makes the frame into a model on which A is not valid, which completes our proof. □

5.2 Admissible rules of preservativity logic

In this section we treat two admissible rules of iPH. If iPH would be the preservativity logic of HA it should certainly satisfy the

Reflection Rule $\Box A/A$,

as (the arithmetical version of) $\Box A/A$ is an admissible rule of HA. The next lemma shows that this is indeed the case.

5.2.1. Lemma. The Reflection Rule holds: if $\vdash_{\mathsf{iPH}} \Box A$ then $\vdash_{\mathsf{iPH}} A$.

Proof We transform a model $\mathcal{M} = (W, \preccurlyeq, R, V)$ for iPH in which A is refuted to a model $\mathcal{M}'$ for iPH in which $\Box A$ is refuted. We can assume that A is not valid in the root w of $\mathcal{M}$. The first idea would be to extend the model in such a way that $w'Rw$ for some new node w'. However, this is not always possible. Namely, it can be the case that wRv but not $w \preccurlyeq v$, for some node v. If we add $w'Rw$ then we should also require $w \preccurlyeq v$ since we have to construct a gathering model. Therefore, we cannot guarantee that w forces the same formulas in both models. To overcome this problem we extend $\mathcal{M}$ in such a way that $w'Rw'' \preccurlyeq w$ for some new nodes w', w''.

We do not spell out the construction but only sketch the idea. We start with $W \cup \{w', w''\}$ and require $w'Rw'' \preccurlyeq w$. Then in every even step we add, in the notation of Theorem 5.1.4, nodes $v * \langle m, u, u' \rangle$, $v * \langle m, u \rangle$ and $v * \langle v, u_1, \ldots, u_n \rangle$ if respectively $\star(v, u, u')$, $*(v', v, u)$ or $\circ(v, u_1, \ldots, u_n)$ holds. It is not difficult to see that we will end up with a conversely well-founded, gathering *Mp Vp*-frame, and that for all nodes v which are not in $\mathcal{M}$, there is no u in $\mathcal{M}$ such that $u \preccurlyeq v$. Therefore, we can extend the valuation of $\mathcal{M}$ to nodes in $\mathcal{M}'$ by not forcing any propositional variable in a new node. Nodes in $\mathcal{M}$ force the same formulas in both models. Hence $w' \nVdash \Box A$. □

Recall that for all arithmetical realization *, HA proves $A^*\rhd B^*$ for all its propositional admissible rules A/B (Sections 2.2 and 7.6). Hence for propositional formulas A, B,

$$(\Box A \to \Box B)/A \rhd B$$

is an admissible rule of the preservativity logic of HA. This rule does no longer hold when A, B range over arithmetical formulae. Consider for example the Rosser sentence R. Since, in HA, $(\Box R \to \Box\bot)$ is derivable, this rule would imply $R \rhd \bot$, and thus by the definition of preservativity and the fact that R is a Σ_1-formula, $\Box(R \to \bot)$ is derivable, quod non. However, $(\Box A \to \Box B)/A \rhd B$ is an admissible rule of iPH as the next lemma shows. Note that this is not in conflict with the possibility of iPH being the preservativity logic of HA.

5.2.2. Lemma. $\vdash_{\mathsf{iPH}} A \rhd B$ iff $\vdash_{\mathsf{iPH}} (\Box A \to \Box B)$.

Proof It suffices to show the following. For any model $\mathcal{M}$ with root w for which there is a node wRv such that $v \Vdash A$ and $v \nVdash B$, there is a submodel $\mathcal{M}'$ such that nodes in $\mathcal{M}$ above v force the same formulas in both models, and such that all nodes in $\mathcal{M}'$ are either equal to w of above v. Hence $w' \nVdash \Box B$ and $w \Vdash \Box A$. The proof is left to the reader.

$\Box$

It has been shown by Gargov (1984) that if a c.e. extension of HA has the Disjunction Property then so does its provability logic. We have the following.

5.2.3. Lemma. The logics iPH and iH have the Disjunction Property.

Proof Using the completeness results in Theorem 5.1.4 and Proposition 5.3.6, this is easy. $\Box$

5.3 Modal completeness of provability logic

Is this section we show that in the language $\mathcal{L}_\Box$, the logic iH is complete with respect to the class of finite brilliant *LLeMa*-frames. Since we know that the principle *Vp* spoils the finiteness of the frames, the logic iH is probably the strongest logic among the considered logics in $\mathcal{L}_\Box$ which still is complete with respect to a class of finite frames. Its completeness proof is more complicated then the completeness proof for iPH and it looks less natural. But it has the following interesting feature. We saw that the finite model property for iLLe is obtained by restricting the domain to X-saturated sets. We do not see how we can apply this method to iMa. Therefore, when we want to guarantee that the final model has the *Ma*-property, we use the construction method instead. In the completeness proof for iH we will see how these two techniques can be combined.

Proof sketch

In the completeness proof for iH we will construct via the construction method, a set of sequences $W = \{\sigma \mid \sigma \text{ is defined }\}$ by selecting nodes α_σ in the iH-canonical model $\mathcal{M}$. This is done via three procedures. Intuitively, the first procedure (**I**) extends a given frame to a frame that has the Ma-property: it adds markov-nodes (Section4.6.2) for pairs that do not already have one. The second procedure (**II**) extends it to a frame that has the Le-property: it adds leivant-nodes (Section 4.4) for pairs that do not already have one. And the third procedure (**III**) extends it to a frame with domain W' that satisfies

$$\forall B \in X, \forall \sigma \in W'(\sigma \Vdash B \text{ iff } \alpha_\sigma \Vdash B).$$

This is done in a similar way as in the construction method. The main trick to ensure that the resulting model is finite is that in **II** and **III** we not only take the iH-canonical model into account but also the model $\mathcal{N}$ constructed in the completeness proof for iLLe, Proposition 4.4.1.

The nodes in our model will be sequences, the elements of which will be formulas $\Box B$, $(B \to C)$, and pairs (l, τ), (m, τ), where τ is a sequence. In the resulting model, $\sigma * \langle (l, \tau) \rangle$ and $\sigma * \langle (m, \tau) \rangle$ will be a leivant- and a markov-node for (σ, τ) respectively.

We will define (relevant pairs) when a pair (σ, τ) needs a leivant-or a markov-node. When we have to add a leivant-node τ' for such a pair, we guarantee that τ' is a so-called minimal leivant-node, i.e. it is a leivant node for (σ, τ) but also $\forall u(\tau' R u \to \tau' \preccurlyeq u)$ holds. This is done to make W finite: such a node τ' is a leivant-node for both (σ, τ) and (σ, τ'). Thus we do not have to add a leivant-node for (σ, τ') as well. In order to be able to make this extra requirement on leivant-nodes, we show (Lemma 5.3.5) that in $\mathcal{N}$ it holds that if a pair has a leivant-node it has a minimal leivant-node as well.

To make W finite we also check if, when we add a leivant-node τ'' for a pair (σ, τ), the node τ is a markov-node. And if so, if it holds that $\tau''_\Box = \tau_\Box$. In that case we do not also have to add a markov-node for (σ, τ'') since τ suffices. To remind us of this fact we use a function h and put $h(\tau) = \tau''$ in this case. During the construction we then guarantee that indeed τ will be a markov-node for (σ, τ'').

At every step in the construction we have a finite set of sequences W_n. We define $\dot{\preccurlyeq}$ and $\dot{R}$ such that we get the desired intuitionistic and modal relations on W_n. This allows us to conclude for which pairs we still have to add markov- or leivant-nodes.

5.3.1 Notation and explication

We cannot avoid a lot of notation. Fix an Le-adequate set X. We will tacitly take all definitions in which an adequate set occurs relative to this particular set X. We will not give corresponding relations in distinct models different names. For example, we say '$w \preccurlyeq v$ *in* K', if v is above w in the model $\mathcal{K}$. Or we write 'we work in $\mathcal{K}$', to stress that in this context $\preccurlyeq$ and R refer to the intuitionistic and modal relation of $\mathcal{K}$ respectively.

The models $\mathcal{M}$ and $\mathcal{N}$

Let $\mathcal{M}$ be the iH-canonical model. Let $\mathcal{N}$ be the model constructed in the proof for iLLe, proposition 4.4.1, with adequate set X. For any node w in the canonical model $\mathcal{M}$, that is, for any saturated set w, let $[w] = w \cap X$. Note that $[w]$ is a node in $\mathcal{N}$. We write $[w]R[v]$ instead of '$[w]R[v]$ in $\mathcal{N}$', and similar for $\preccurlyeq$.

5.3.1. Remark. Note that if for two nodes w, v in the canonical model $\mathcal{M}$ we have wRv in $\mathcal{M}$ and $w_\Box \subset v_\Box$, i.e. there is a $\Box B \in v \cap X$ such that $\Box B \notin w$, then $[w]R[v]$ in $\mathcal{N}$. On the other hand, if $[w]R[v]$ in $\mathcal{N}$, then $w_\Box \subset v_\Box$. It also follows from the definition of $\mathcal{N}$ that $[w] \preccurlyeq [v]$ in $\mathcal{N}$ iff $w \cap X \subseteq v \cap X$.

The relations $\subseteq_\rightarrow$ and $\subseteq_{\not\rightarrow}$ on sequences

For sequences σ, τ, we write $\sigma \subseteq \tau$ when $\tau = \sigma * \sigma'$ for some possibly empty sequence σ'. We write $\sigma \subseteq_\rightarrow \tau$ if σ' consists of implications only or is empty, and $\sigma \subseteq_{\not\rightarrow} \tau$ otherwise.

Relevant pairs of sequences

Intuitively, the relevant leivant (markov) pairs are the only pairs which we have to give a leivant (markov)-node, in order to guarantee that all pairs of nodes have a leivant-(markov-)node in the resulting model. Call a pair (x, y) *relevant-leivant in* $\dot{W}$ if $y = x * \langle D \rangle$, where D is not an implication and not of the form $\langle l, z \rangle$, and x is not of the form $\sigma * \langle (m, \tau) \rangle$, and (x, y) does not have a leivant-node in $\dot{W}$. Call a pair (x, y) *relevant-markov in* $\dot{W}$ if $y = x * \langle D \rangle * y'$, where $x * \langle D \rangle \subseteq_\rightarrow y$, and D is not an implication and not of the form $\langle m, z \rangle$, and x is not of the form $\sigma * \langle (m, \tau) \rangle$, and (x, y) does not have a markov-node in $\dot{W}$.

The relations $\dot{\preccurlyeq}$ and $\dot{R}$ of the model $\dot{W}$

For a given set of sequences W, the model $\dot{W}$ is the model $(W, \dot{\preccurlyeq}_{\restriction W}, \dot{R}_{\restriction W}, V)$, where V is defined via

$$\dot{W}, \sigma \Vdash p \equiv_{def} \mathcal{M}, \alpha_\sigma \Vdash p,\ \ p \text{ a propositional variable in } X.$$

The relations $\dot{\preccurlyeq}_{\restriction W}$ and $\dot{R}_{\restriction W}$ are the restrictions of respectively $\dot{\preccurlyeq}$ and $\dot{R}$ to W, where $\dot{\preccurlyeq}$ and $\dot{R}$ are defined as follows. An *lm-chain* is a sequence $(x_1, .., x_n)$ of sequences, such that for all i, one of (a)-(e) is the case:

(a) $i = n$
(b) $x_i = y * \langle(m, x_{i+1})\rangle$ or $x_i = y * \langle(l, x_{i+1})\rangle$ for some y
(c) $x_i \subseteq x_{i+1}$ and $x_i \neq x_{i+1}$
(d) $h(x_i) = x_{i+1}$
(e) $h(x_{i+1}) = x_{i+2}$ and $x_{i+1} = y * \langle(m, x_i)\rangle$ for some y.

$x\dot{R}y \equiv_{def}$ there is an lm-chain $(x = x_1, .., x_n)$ and $x_n \subseteq_{\not\to} y$
$x\dot{\preccurlyeq}y \equiv_{def}$ $x \subseteq_{\to} y$, or $(x = x' * \langle(l, z)\rangle$ and
$(z \subseteq_{\to} y$ or $z\dot{R}y$ or $x\dot{R}y))$.

5.3.2. Remark. Note that if $x \subseteq_{\not\to} y$ then $x\dot{R}y$. And if $x'\dot{R}y$ and for x there is an lm-chain $(x = x_1, .., x_n)$ such that $x_n \subseteq x'$, then $x\dot{R}y$.

Procedure I

Start with a set of sequences V_0. Consider in step n all the relevant markov pairs (σ, τ) of $\dot{V}_n$, and choose markov-nodes $\alpha_{\sigma*\langle(m,\tau)\rangle}$ for the corresponding pairs $(\alpha_\sigma, \alpha_\tau)$ in the iH-canonical canonical model $\mathcal{M}$. Let V_{n+1} be the union of V_n and these newly defined sequences, and go to step $(n + 1)$.

The sequences in V_0 have to be such that these markov-nodes exist. In the cases in which we use this procedure, the set with which we start will have the desired properties, see *Case* (i) in Proposition 5.3.6.

Procedure II

Start with a set V_0. Choose for every relevant leivant pair (σ, τ) of $\dot{V}_0$, a node $\alpha_{\sigma*\langle(l,\tau)\rangle} = x$, such that $\alpha_\sigma R x$ in $\mathcal{M}$, and $[x]$ is a minimal leivant-node for $([\alpha_\sigma], [\alpha_\tau])$ in $\mathcal{N}$. The definition of a minimal leivant-node can be found just before Lemma 5.3.5. If $\tau = \sigma * \langle(m, \tau')\rangle$, and $x_\Box = (\alpha_\tau)_\Box$, then put $h(\tau) = \sigma * \langle(l, \tau)\rangle$. Let V be the union of V_0 and these new sequences.

5.3.3. Remark. Procedure **II** will be the only procedure in which we assign an h-image to some newly chosen sequences. Note that this implies that $h(\tau)$ only exists for some τ which are of the form $\sigma * \langle(m, \tau')\rangle$ and for which (σ, τ) has been once a relevant leivant pair. Note furthermore, that once a sequence τ has been part of a relevant leivant pair (σ, τ), after procedure **II** is performed it will never for any sequence σ', be part of a relevant leivant pair (σ', τ) anymore. This shows that h is indeed a (partial) function.

5.3.4. Remark. We cannot guarantee that for any leivant-node x for (w, v) in $\mathcal{M}$, $w_\Box \subset x_\Box$. Therefore, in contrast to procedure **I**, in procedure **II** we cannot just choose $\alpha_{\sigma*\langle(l,\tau)\rangle}$ to be a leivant-node for $(\alpha_\sigma, \alpha_\tau)$ in $\mathcal{M}$. For if we would do this, it could be that $(\alpha_\sigma)_\Box = (\alpha_{\sigma*\langle(l,\tau)\rangle})_\Box$, and we would have no guarantee that the process stops in a finite number of steps, see Remark 3.4.4. Note that the way in which we choose $\alpha_{\sigma*\langle(l,\tau)\rangle}$ in procedure **II** implies that $(\alpha_\sigma)_\Box \subset (\alpha_{\sigma*\langle(l,\tau)\rangle})_\Box$, by Remark 5.3.1.

Procedure III

Start with a set V_0. Step n consists of the following. If $\sigma \in V_n$, and there is a $\Box B \in X$, $\Box B \notin \alpha_\sigma$, for which there is no $x \in V_n$ such that $\sigma \dot{R} x$ and $B \notin \alpha_x$, then choose a node $\alpha_{\sigma*\langle\Box B\rangle}$ which is a node w such that $\alpha_\sigma R w$ in $\mathcal{M}$, w does not contain B, and $[w]$ is a minimal leivant-node for $([\alpha_\sigma], [w])$ in $\mathcal{N}$. The definition of a minimal leivant-node is just before Lemma 5.3.5. For $(B \to C) \in (\alpha_\sigma)_{\not\to}$, if there is no $x \in V_n$ such that $\sigma \dot{\preccurlyeq} x$ and $B \in x$ and $C \notin x$, choose a node $\sigma * \langle B \to C\rangle$ as is usual in the construction method. Let V_{n+1} be the union of V_n with the newly defined sequences, and go to step $(n+1)$.

5.3.2 The completeness proof

As said in the proof sketch we need one additional lemma which shows that on finite Le-frames the following stronger condition holds: for every pair wRv in a finite Le-frame there is leivant-node x for (w, v) which also is a leivant-node for itself, namely for (w, x). Such a node x for which

$$wRx \preccurlyeq v \wedge \forall u(vRu \to x \preccurlyeq u) \wedge \forall u(xRu \to x \preccurlyeq u),$$

is called a *minimal leivant-node for* (w, v).

5.3.5. Lemma. (i) A finite Le-frame satisfies

$$wRv \to \exists x(wRx \preccurlyeq v \wedge \forall u(vRu \to x \preccurlyeq u) \wedge \forall u(xRu \to x \preccurlyeq u)).$$

(ii) If wRv holds in $\mathcal{M}$ and $[w]R[v]$ holds in $\mathcal{N}$, there exists a node x such that wRx in $\mathcal{M}$, and $[x]$ is a minimal leivant-node for $([w], [v])$ in $\mathcal{N}$.

Proof (i) Let $\mathcal{F}$ be a finite Le-frame. Then $\mathcal{F}$ satisfies

$$wRv \to \exists x(wRx \preccurlyeq v \wedge \forall u(vRu \to x \preccurlyeq u)). \tag{5.4}$$

Consider wRv. We show that the pair (w, v) has a minimal leivant-node x. Define $*(y)$ via

$*(y) \qquad \forall u(yRu \to y \preccurlyeq u).$

We construct a sequence $x_1 \succcurlyeq x_2 \succcurlyeq \ldots$ of nodes, such that

$$wRx_i \preccurlyeq v \wedge \forall u(vRu \to x_i \preccurlyeq u) \wedge (x_{i+1} = x_i \to *(x_i)).$$

Let us first see why we are done then. The finiteness of the frame implies that $x_{i+1} = x_i$, for some i. Such a node x_i has the desired properties, i.e. we can take $x = x_i$.

We show how to construct the sequence $x_1 \succcurlyeq x_2 \succcurlyeq \ldots$ by induction. By (5.4) there is a node x_1 such that $wRx_1 \preccurlyeq v \wedge \forall u(vRu \to x_1 \preccurlyeq u)$. Assume x_i is already defined. Thus

$$wRx_i \preccurlyeq v \wedge \forall u(vRu \to x_i \preccurlyeq u).$$

By (5.4) there is node x_{i+1} such that

$$wRx_{i+1} \preccurlyeq x_i \wedge \forall u(x_iRu \to x_{i+1} \preccurlyeq u).$$

Observe that since $x_{i+1} \preccurlyeq x_i$, we also have

$$wRx_{i+1} \preccurlyeq v \wedge \forall u(vRu \to x_{i+1} \preccurlyeq u).$$

Further note that since $\forall u(x_iRu \to x_{i+1} \preccurlyeq u)$, if $*(x_i)$ does not hold, then $x_i \neq x_{i+1}$. Thus x_{i+1} has the desired properties. This completes the construction of the sequence.

(ii) The proof that there is a node x such that wRx and $[x]$ is a leivant-node for $([w],[v])$, is almost the same as the part of the completeness proof for iLLe, proposition 4.4.1, in which it is shown that the frame has the Le-property. Instead of sets w_σ consider sets $\{D \mid \Box D \in w\} \cup w_\sigma$. To conclude from this that x can be chosen in such a way that $[x]$ is in fact a minimal leivant-node for $([w],[v])$, is similar to the proof of (i). □

5.3.6. Proposition. $\vdash_{\mathsf{iH}} A$ iff A is valid on all finite transitive conversely well-founded brilliant $LeMa$-frames.

Proof Assume $\nvdash_{\mathsf{iH}} A$. Let b be a node in the iH-canonical model $\mathcal{M}$ which does not force A. Let X be a finite Le-adequate set which contains A. As described in the sketch of the proof we construct a finite model of the form $\dot{W}$ (Section 5.3.1) in a similar way as in the construction method. We construct the domain W of $\dot{W}$ stepwise: we define $W_n = \{\sigma \mid \sigma \text{ is defined in a step } \leq n\}$ and let $W = \bigcup_n W_n$.

Step 0. Let W_0 be the result of procedure **III**, starting with $V_0 = \{b\}$.
Step $3n+1$. Let W_{3n+1} be the result of procedure **I**, starting with set $V_0 = W_{3n}$.
Step $3n+2$. Let W_{3n+2} be the result of procedure **II**, starting with set $V_0 = W_{3n+1}$.
Step $3n+3$. The set W_{3n+3} is the result of procedure **III**, starting with $V_0 = W_{3n+2}$.

Unless stated otherwise, $\dot{\preccurlyeq}$ and $\dot{R}$ will be short for $\dot{\preccurlyeq}_{\restriction W}$ and $\dot{R}_{\restriction W}$. To see that $\dot{W}$ exists, and that it is a finite transitive conversely well-founded brilliant $LeMa$-model, it suffices to show the following claims.

(i) The construction is correct, i.e. the nodes we choose in the consecutive steps do exist.

(*ii*) $\dot{W}$ is a model, i.e. $\dot{\preccurlyeq}$ is a partial order, $(\dot{\preccurlyeq};\dot{R}) \subseteq \dot{R}$ holds and for all propositional variables $p \in X$, $\forall \sigma, \tau \in W$ (if $\sigma \dot{\preccurlyeq} \tau$ and $\sigma \Vdash p$ then $\tau \Vdash p$). The model $\dot{W}$ has a transitive, conversely well-founded and brilliant frame.

(*iii*) $\dot{W}$ has the *Ma*-property.

(*iv*) $\dot{W}$ has the *Le*-property.

(*v*) $\forall B \in X \forall \sigma \in W(\dot{W}, \sigma \Vdash B$ iff $\mathcal{M}, \alpha_\sigma \Vdash B)$.

(*vi*) The process stops in a finite number of steps, i.e. there is some n such that $\forall m \succcurlyeq n$ we have $W_m = W_n$. Since clearly every W_m is finite, this shows that W is finite.

We will omit the tedious but straightforward proofs of (*ii*) and (*v*) and only prove (*i*), (*iii*), (*iv*) and (*vi*). Unless stated otherwise, $\sigma, \tau, x, y, z, a, b$ range over sequences, B, C, D over elements of sequences. We remind the reader that $w_\Box$ denotes $w_\Box^X$.

Claim (*i*). We show that the three procedures are correct. We will need the following lemmas and remark.

5.3.7. Lemma. If $\sigma \subseteq_\rightarrow \tau$ and $\sigma \neq \tau$, then $\alpha_\sigma \preccurlyeq \alpha_\tau$ in $\mathcal{M}$.
If $\sigma \subseteq_\rightarrow \tau$ and $\sigma \neq \tau$, then $\alpha_\sigma \subset \alpha_\tau$.
If $\sigma \subseteq_\rightarrow \tau$ and $\sigma \neq \tau$, then $[\alpha_\sigma] \prec [\alpha_\tau]$ in $\mathcal{N}$.

Proof By examining procedure **III** (Section 5.3.1) it is easy to conclude the first part. Since by definition, $\preccurlyeq = \subseteq$ on canonical models, the second part follows immediately from the first one. From the second part and remark 5.3.1 the third part follows. $\Box$

5.3.8. Remark. For any two nodes w, v in the iH-canonical model $\mathcal{M}$ the following holds,

$$wR = vR \text{ iff } w_\Box = v_\Box.$$

Hence if t is a markov-node for (w, v), then $t_\Box = v_\Box$ holds.

5.3.9. Lemma. Assume that procedures **I**,**II** and **III** are well-defined till step $(n+1)$. Then for all pairs (σ, τ) in W_n that are either a relevant leivant-pair or a relevant markov-pair, we have that

$$\alpha_\sigma R \alpha_\tau \text{ in } \mathcal{M} \text{ and } [\alpha_\sigma] R [\alpha_\tau] \text{ in } \mathcal{N}.$$

Proof Let (σ, τ) be any relevant-leivant or relevant-markov pair, in W_n. By the definition of relevant pairs (Section 5.3.1), we have that $\tau = \sigma * \langle D \rangle * \sigma'$, for some D which is no implication, and σ' consists of implications only. Hence D is of the form $\Box B$, $\langle l, \tau' \rangle$ or $\langle m, \tau' \rangle$. We show that in all these cases,

$$\alpha_\sigma R \alpha_{\sigma * \langle D \rangle} \text{ in } \mathcal{M} \text{ and } [\alpha_\sigma] R [\alpha_{\sigma * \langle D \rangle}] \text{ in } \mathcal{N}. \tag{5.5}$$

Let us first see why we are done then. Since $\sigma * \langle D \rangle \subseteq_{\rightarrow} \sigma * \langle D \rangle * \sigma' = \tau$, it follows from lemma 5.3.7 that $\alpha_{\sigma * \langle D \rangle} \preccurlyeq \alpha_\tau$ in $\mathcal{M}$ and $[\alpha_{\sigma * \langle D \rangle}] \preccurlyeq [\alpha_\tau]$ in $\mathcal{N}$. As both $\mathcal{M}$ and $\mathcal{N}$ are brilliant models, this gives the desired result;

$$\alpha_\sigma R \alpha_\tau \text{ in } \mathcal{M} \text{ and } [\alpha_\sigma] R [\alpha_\tau] \text{ in } \mathcal{N}.$$

Therefore, all we have to show is that for the three possibilities of D, (5.5) holds.
Case $z = \sigma * \langle \Box B \rangle$. By examining procedure **III** (Section 5.3.1) we see that $\alpha_\sigma R \alpha_z$ in $\mathcal{M}$. Moreover, since $[\alpha_z]$ is a minimal-leivant node for $([\alpha_\sigma], [\alpha_z])$, it follows that $[\alpha_\sigma] R [\alpha_z]$ in $\mathcal{N}$, and we are done.
Case $z = \sigma * \langle (l, \tau') \rangle$. By examining procedure **II** we see that $\alpha_\sigma R \alpha_z$ in $\mathcal{M}$. Moreover, since $[\alpha_z]$ is a minimal-leivant node for $([\alpha_\sigma], [\alpha_{\tau'}])$, it follows that $[\alpha_\sigma] R [\alpha_z]$ in $\mathcal{N}$.
Case $z = \sigma * \langle (m, \tau') \rangle$. By examining procedure **I** we see that α_z is a markov-node for $(\alpha_\sigma, \alpha_{\tau'})$ in $\mathcal{M}$. Hence by the definition of markov-nodes, Section 4.6.2, $\alpha_\sigma R \alpha_z$ in $\mathcal{M}$. Thus we only have to show $[\alpha_\sigma] R [\alpha_z]$. Since $\alpha_\sigma R \alpha_z$, by Remark 5.3.1 it suffices to show that

$$(\alpha_\sigma)_\Box \subset (\alpha_z)_\Box. \tag{5.6}$$

Note that the existence of $\alpha_{\sigma * \langle (m, \tau') \rangle}$ implies that (σ, τ') must have been a relevant markov-pair in some W_m. Thus, by the definition of a relevant markov-pair (Section 5.3.1), $\tau' = \sigma * \langle D' \rangle * \sigma''$, where D' is no implication and not of the form $\langle m, \tau'' \rangle$, and σ'' consists of implications only. Hence D' is of the form $\Box B$ or $\langle l, \tau'' \rangle$. But for these two cases we just proved that $[\alpha_\sigma] R [\alpha_{\sigma * \langle D' \rangle}]$. Now it follows from remark 5.3.1 that

$$(\alpha_\sigma)_\Box \subset (\alpha_{\sigma * \langle D' \rangle})_\Box$$

By lemma 5.3.7, it follows that $\alpha_{\sigma * \langle D' \rangle} \subseteq \alpha_{\sigma * \langle D' \rangle * \sigma''} = \alpha_{\tau'}$ in $\mathcal{M}$. Thus

$$(\alpha_\sigma)_\Box \subset (\alpha_{\tau'})_\Box. \tag{5.7}$$

Since α_z is a markov-node for $(\alpha_\sigma, \alpha_{\tau'})$ in $\mathcal{M}$, we certainly have $\alpha_z R = \alpha_{\tau'} R$ in $\mathcal{M}$. By Remark 5.3.8 this implies that $(\alpha_z)_\Box = (\alpha_{\tau'})_\Box$. From this and (5.7), (5.6) follows. $\Box$

Now we are ready to show that procedure **I** is correct. With induction to n we show that for any relevant markov-pair (σ, τ) in W_n, there is a node t in the canonical model $\mathcal{M}$ which is a markov-node for $(\alpha_\sigma, \alpha_\tau)$ in $\mathcal{M}$. From Lemma 5.3.9 we know that for any relevant markov-pair $\alpha_\sigma R\alpha_\tau$ holds in $\mathcal{M}$. Therefore, it suffices to show that $\mathcal{M}$ has the *Ma*-property, i.e.

in $\mathcal{M}$: if wRv then $\exists t \in Top(wRt \wedge tR = vR)$.

The proof that $\mathcal{M}$ has the *Ma*-property is completely analogous to the proof that the iMa-canonical model has the *Ma*- property, Proposition 4.6.7.

To see that procedure **II** is correct, we show with induction to n that for any relevant leivant-pair (σ, τ) in W_n, there is a node $x \in \mathcal{M}$ such that $\alpha_\sigma Rx$ in $\mathcal{M}$ and such that $[x]$ is a minimal leivant-node for $([\alpha_\sigma], [\alpha_\tau])$ in $\mathcal{N}$. From Lemma 5.3.9 we know that $\alpha_\sigma R\alpha_\tau$ holds in $\mathcal{M}$, and $[\alpha_\sigma]R[\alpha_\tau]$ holds in $\mathcal{N}$. Apply Lemma 5.3.5.

To see that procedure **III** is correct, consider any W_n. Let $\sigma \in W_n$ for which $\Box B \notin \alpha_\sigma$. We show that there is node w in the canonical model $\mathcal{M}$ such that $\Box B \in w$, $B \notin w$ and $\alpha_\sigma Rw$ in $\mathcal{M}$, and moreover such that $[w]$ is a minimal leivant-node for $([\alpha_\sigma], [w])$. This will prove that procedure **III** is correct. In the canonical model there is a node v such that $\alpha_\sigma Rv$, $\Box B \in v$, and $B \notin v$. Hence $[\alpha_\sigma]R[v]$ by Remark 5.3.1. If $[v]$ is a minimal leivant-node for $([\alpha_\sigma], [v])$, let w be this node v. If not, by Lemma 5.3.5 there is a node u such that $\alpha_\sigma Ru$ and $[u]$ is a minimal leivant-node for $([\alpha_\sigma], [v])$. Since $[u] \preccurlyeq [v]$ in $\mathcal{N}$, it follows that $u \cap X \subseteq v \cap X$ by Remark 5.3.1. Since $B \in X$ and $B \notin v$, $B \notin u$. So in this case we can choose u for w.

Claim (iii). We show that $\dot{W}$ has the *Ma*- property, i.e.

$x\dot{R}y \rightarrow \exists z \in Top(x\dot{R}z \wedge y\dot{R} = z\dot{R})$.

First we need some lemmas and a remark.

5.3.10. Remark. By examining the steps in which W is constructed it is easy to see that if $x * \langle(m, y)\rangle \in W$, then (x, y) must have been a relevant markov-pair in some W_m. By the definition of a relevant markov-pair, section 5.3.1, this implies that x and y are not of the form $\sigma * \langle(m, \tau)\rangle$. In a similar way one can see that if $x * \langle(l, y)\rangle \in W$, then x is not of the form $\sigma * \langle(m, \tau)\rangle$ and y is not of the form $\sigma * \langle(l, \tau)\rangle$.

5.3.11. Lemma. For all n, the node $x * \langle(m, y)\rangle \in \dot{W}_n$ is a markov-node for (x, y) in $\dot{W}$. If $z = x * \langle(m, z')\rangle$ and $h(z) = y$, then z is a markov-node for (x, y) in $\dot{W}$.

Proof It is convenient to treat the last part first. Therefore, consider $z \in W_n$ such that $h(z) = y$ and $z = x * \langle(m, \tau)\rangle$ for some τ. We have to show that

z is a top node in $\dot{W}$ and $x\dot{R}z$ and $y\dot{R} = z\dot{R}$.

First note that the fact that $h(z) = y$, gives $y = x * \langle(l, z)\rangle \in \dot{W}_n$, see Remark 5.3.3. To see that z is a top node in $\dot{W}_n$, observe that since α_z is a markov-node in $\mathcal{M}$, it is a top node in $\mathcal{M}$. Therefore, it follows from procedure **III** that sequences of the form $z * \langle B \to C\rangle$ will never be defined. Now from the definition of $\dot{\preccurlyeq}$, Section 5.3.1, it follows that z is a top node in $\dot{W}$. From the definition of $\dot{R}$, it follows immediately that $x\dot{R}z$.

Thus it remains to show that $y\dot{R} = z\dot{R}$. Observe that $(\dot{\preccurlyeq};\dot{R};\dot{\preccurlyeq}) = \dot{R}$. By the definition of $\dot{\preccurlyeq}$, we have $y\dot{\preccurlyeq}z$, it follows that $z\dot{R} \subseteq y\dot{R}$ by the observation. To see that $y\dot{R} \subseteq z\dot{R}$, assume $y\dot{R}a$. Hence there is an lm-chain $(y = y_1, .., y_m)$ such that $y_m \subseteq_{\not\to} a$. As clearly $(z, y_1, .., y_m)$ is an lm-chain too, $z\dot{R}a$.

We prove the first part of the lemma. Consider $z = x * \langle(m, y)\rangle \in \dot{W}_n$. Again, we have to show that

$$z \text{ is a top node in } \dot{W} \text{ and } x\dot{R}z \text{ and } y\dot{R} = z\dot{R}.$$

To see that z is a top node in $\dot{W}$ and that $x\dot{R}z$ is analogous to the case above. Therefore, it remains to show that $y\dot{R} = z\dot{R}$.
$y\dot{R} \subseteq z\dot{R}$: Assume $y\dot{R}a$. Hence there is an lm-chain $(y = y_1, .., y_m)$ such that $y_m \subseteq_{\not\to} a$. As clearly $(z, y_1, .., y_m)$ is an lm-chain too, $z\dot{R}a$.
$z\dot{R} \subseteq y\dot{R}$: Assume $z\dot{R}a$. Let $(z = z_1, .., z_m)$ be an lm-chain such that $z_m \subseteq_{\not\to} a$. We have to show that $y\dot{R}a$. By the definition of an lm-chain, Section 5.3.1, we have for $z = z_1$ either

(a) $z = z_m$ hence $z \subseteq_{\not\to} a$
(b) $z = z' * \langle(m, z_2)\rangle$ or $z = z' * \langle(l, z_2)\rangle$ for some z'
(c) $z \subseteq z_2$ and $z \neq z_2$
(d) $h(z) = z_2$
(e) $h(z_2) = z_3$ and $z_2 = z' * \langle(m, z)\rangle$.

We show that the only cases that can occur are (b) or (d). Now observe that in case (b) $z_2 = y$, thus $(y = z_2, .., z_n)$ is an lm-chain too. And in case (d) $(y, z_1, .., z_m)$ is an lm-chain. Hence in both cases (b) and (d), $y\dot{R}a$ follows. Therefore, we are done if we can show that only the cases (b) or (d) can occur.

Case (e) cannot occur by remark 5.3.10. So, case (a) and (c) remain. In both these case there is a D such that $z * \langle D\rangle \in W$. But this contradicts the following lemma. This completes the proof. □

5.3.12. Lemma. For any $z = x * \langle(m, y)\rangle \in W$, for any D, there is no element $z * \langle D\rangle \in W$.

Proof Consider a sequence $z = x * \langle(m, y)\rangle \in W$. Arguing by contradiction, assume $z * \langle D\rangle \in W$ is the first such element defined. Clearly, there are four possibilities for D: (a) $D = (B \to C)$, (b) $D = \Box B$, (c) $D = (l, \tau)$, (d) $D = (m, \tau)$. We show that none of these cases can occur. First note, by examining procedure **I**, Section 5.3.1, that α_z is a markov-node for (α_x, α_y) in $\mathcal{M}$. Thus

$$\alpha_z R = \alpha_y R \text{ and } \alpha_z \text{ is a top node in } \mathcal{M}.$$

Hence by Remark 5.3.8

$$(\alpha_z)_\Box = (\alpha_y)_\Box \text{ and } \alpha_z \text{ is a top node in } \mathcal{M}. \tag{5.8}$$

For (a), observe that in this case, $z \subseteq_\rightarrow z{*}\langle D\rangle$. Hence by Lemma 5.3.7, $\alpha_z < \alpha_{z*\langle D\rangle}$ in $\mathcal{M}$, which contradicts the fact that α_z is a top node in $\mathcal{M}$.

Case (c) and (d) cannot occur because $z * \langle D\rangle$ is the first such element defined.

Thus case (b) remains. By examining procedure **III** we see that $\Box B \notin (\alpha_z)_\Box$. Hence by (5.8) also $\Box B \notin (\alpha_y)_\Box$. We show that this implies that $z * \langle \Box B\rangle$ cannot be defined. Assume y is defined in step m and z in step n. We show that

$$\exists y' \in W_{n-1}(z\dot{R}y' \wedge B \notin y'). \tag{5.9}$$

By examining procedure **III** one easily conclude that this implies that a sequence $z * \langle \Box B\rangle$ will never be defined. Hence we have established that case (b) cannot occur either.

First, observe that by Lemma 5.3.11, $y\dot{R} \subseteq z\dot{R}$ (note that for the part of the proof of Lemma 5.3.11 where $y\dot{R} \subseteq z\dot{R}$ is established, we do not need this lemma, so there is no circle argument here). Therefore, to prove (5.9) it suffices to show that

$$\exists y' \in W_{n-1}(y\dot{R}y' \wedge B \notin y'). \tag{5.10}$$

Clearly $m \leqslant n$, because $z = x * \langle (m, y)\rangle$. Note furthermore that $n = 3k + 1$, for some k. Observe that by remark 5.3.10, y is not of the form $\sigma * \langle (m, \tau)\rangle$. Thus $m = 0$ or $m = 3k + 2$ or $m = 3k + 3$, for some k. Hence if $m = 0$ or $m = 3k + 3$, $m < n$. And if $m = 3k+2$, $m+1 < n$. Therefore, we can prove (5.10), by showing

$$(m = 0 \vee m = 3k + 3) \rightarrow \exists y' \in W_m(y\dot{R}y' \wedge B \notin y'), \tag{5.11}$$

$$m = 3k + 2 \rightarrow \exists y' \in W_{m+1}(y\dot{R}y' \wedge B \notin y'). \tag{5.12}$$

We only show (5.12). One can prove (5.11) in a similar way. Assume $m = 3k + 2$ for some k. Note that since $y \in W_m = W_{3k+2}$, W_{m+1} is the result of procedure **III** starting with set $V_0 = W_m$. By the definition of procedure **III**, Section 5.3.1, (5.12) follows immediately. □

Now we are ready to show that every pair $x\dot{R}y$ in $\dot{W}$ has a markov-node. Consider such a pair $x\dot{R}y$. Let $(x_1, .., x_n)$ be an lm-chain, $x = x_n \subseteq_{\not\rightarrow} y$. There are x', y', D such that

$$x_n \subseteq x' \subseteq_{\not\rightarrow} x' * \langle D\rangle = y' \subseteq_\rightarrow y.$$

The pair (x', y) cannot be a relevant markov pair in $\dot{W}$, otherwise W would not be the union of all W_n. By the definition of a relevant markov-pair this implies that either (a) D is an implication, (b) $D = (m, \tau)$, for some τ, (c) $x' = \sigma * \langle (m, \tau)\rangle$, for some σ, τ, (d) the pair (x', y) has a markov-node in $\dot{W}$. Since $x' \subseteq_{\not\rightarrow} x' * \langle D\rangle$, D cannot be an implication. Thus case (a) cannot occur. Since $x' * \langle D\rangle \in W$, by

Lemma 5.3.12, case (c) cannot occur either. Therefore, case (b) and (d) remain. First, we show that in both these cases the pair (x', y) has a markov-node, i.e. that

$$\exists z \in W(x'\dot{R}z \text{ and } z\dot{R} = y\dot{R} \text{ and } z \text{ is a top node in } \dot{W}). \tag{5.13}$$

And then we show that

$$\forall z \in W(\text{if } z \text{ is a markov-node for } (x', y) \text{ it is one for } (x, y) \text{ too}). \tag{5.14}$$

This will complete the proof.

In showing (5.13), we may restrict attention to case (b), as it follows trivially for case (d). It suffices to show that y is a markov-node for (x', y), i.e. that

$$x'\dot{R}y \text{ and } y\dot{R} = y\dot{R} \text{ and } y \text{ is a top node in } \dot{W}. \tag{5.15}$$

Since $x' \subseteq_{\not\to} y$, we have $x'\dot{R}y$ by Remark 5.3.2. By Lemma 5.3.11, y' is a markov-node for (x', τ) in $\dot{W}$, and thus, by the definition of a markov-node, a top node in $\dot{W}$. Since, by Lemma 5.3.12,we have $y = y'$, we have shown (5.15) and hence (5.13).

To show (5.14), consider a markov-node z for (x', y), i.e.

$$z \text{ is a top node in } \dot{W},\ x'\dot{R}z \text{ and } z\dot{R} = y\dot{R}.$$

We have to show that z is a markov-node for (x, y), i.e.

$$z \text{ is a top node in } \dot{W},\ x\dot{R}z \text{ and } z\dot{R} = y\dot{R}.$$

Thus we only have to infer $x\dot{R}z$. But this follows immediately from Remark 5.3.2 and the fact that $(x = x_1, .., x_n)$ is an lm-chain and that $x_n \subseteq x'$.

Claim (iv). One can show that any pair xRy in $\dot{W}$ has a leivant-node in a completely similar way. Instead of (x', y), consider the pair (x', y'). And instead of lemma 5.3.11 for markov-nodes, use the following corresponding lemma for leivant-nodes.

5.3.13. Lemma. For all n, $x * \langle (l, y) \rangle \in \dot{W}_n$ is a leivant-node for (x, y) in $\dot{W}_n$.

Proof Immediate from the definition of $\dot{W}$. □

Claim (vi). We show that $\exists n \forall m \geq n(W_n = W_m)$. First, we prove some lemmas. Let $l(x)$ be the length of the sequence x.

5.3.14. Lemma. $\forall x \in W \exists n \forall (x * \langle D \rangle) \in W$ (if $x * \langle D \rangle \notin W_n$, then D is either of the form $\langle (l, \tau) \rangle$ or $\langle (m, \tau) \rangle$).

Proof By examining procedure **III** one can conclude that if x is defined in step n, no sequence of the form $x * \langle B \to C \rangle$ or $x * \langle \Box B \rangle$ will be defined after step $n + 3$ anymore. □

5.3.15. Lemma. $\forall x \in W \exists n \forall x * y \in W$ (if $l(y) \leqslant 1$, then $x * y \in W_n$).

Proof Arguing by contradiction, assume there is a sequence $x \in W$ such that

$$\forall n \exists y (x * y \in W \wedge l(y) \leqslant 1 \wedge x * y \notin W_n).$$

This implies that

$$\forall n \exists m > n \exists y (l(y) \leqslant 1 \wedge x * y \notin W_n \wedge x * y \in W_m). \tag{5.16}$$

Let us start with three observations.

First, by (5.16) and Lemma 5.3.14 there is an infinite sequence $y_1, y_2, ..$ in W, where y_i is defined in a step before the one in which y_{i+1} is defined, and y_i is either of the form $\sigma * \langle (l, \tau) \rangle$ or $\sigma * \langle (m, \tau) \rangle$.

Second, by examing the way in which W is constructed in consecutive steps, it is not difficult to see that if $x * \langle (l, z) \rangle$ or $x * \langle (m, z) \rangle$ is defined in step k, z must be defined in step k', for some $k - 2 \leqslant k' \leqslant k$.

Third, if $x * \langle (l, z) \rangle \in W$, (x, z) must have been relevant-leivant in some $\dot{W}_k$. Hence by the definition of relevant pairs (Section 5.3.1), z is not of the form $\sigma * \langle (l, \tau) \rangle$. Similar for $x * \langle (m, z) \rangle$.

These observations imply that w.l.o.g. we can assume $y_{2i+1} = x * \langle (l, y_{2i}) \rangle$ and $y_{2i+2} = x * \langle (m, y_{2i+1}) \rangle$. Hence, by procedure **I**, the node $\alpha_{y_{2i+2}}$ is a markov-node, in $\mathcal{M}$, for the pair $(\alpha_x, \alpha_{y_{2i+1}})$. Hence $\alpha_{y_{2i+2}} R = \alpha_{y_{2i+1}} R$. Remark 5.3.8 implies that we have

$$(\alpha_{y_{2i+2}})_\Box = (\alpha_{y_{2i+1}})_\Box . \tag{5.17}$$

From procedure **II** we conclude that $[\alpha_{y_{2i+1}}]$ is a leivant-node for $([\alpha_x], [\alpha_{y_{2i}}])$ in $\mathcal{N}$. Hence $[\alpha_{y_{2i+1}}] \preccurlyeq [\alpha_{y_{2i}}]$ in $\mathcal{N}$. Thus by Remark 5.3.1,

$$(\alpha_{y_{2i+1}})_\Box \subseteq (\alpha_{y_{2i}})_\Box . \tag{5.18}$$

Combining the equations (5.17) and (5.18) we arrive at the following chain of $\subseteq$ and $=$:

$$.. \, (\alpha_{y_{2i+2}})_\Box = (\alpha_{y_{2i+1}})_\Box \subseteq (\alpha_{y_{2i}})_\Box = \; ..$$

The finiteness of the sets $(\alpha_{y_j})_\Box$ implies that

$$(\alpha_{y_{2i+1}})_\Box = (\alpha_{y_{2i}})_\Box , \text{ for some } i.$$

Then we should have $h(y_{2i}) = y_{2i+1}$. Lemma 5.3.11 shows that y_{2i} is a markov-node for (x, y_{2i+1}). Therefore, the pair (x, y_{2i+1}) will never be a relevant-markov pair in any W_k, and that contradicts the existence of the node y_{2i+2}. This completes the proof. □

5.3.16. Lemma. $\forall x \in W \forall m \exists n \forall x * y \in W$ (if $l(y) \leqslant m$, then $x * y \in W_n$).

Proof By lemma 5.3.15 and the fact that every W_k is clearly finite, for every n there is a number, denoted with $f(n)$, such that

$$\forall x \in W_n \forall x * y \in W \text{ (if } l(y) \leqslant 1, \text{ then } x * y \in W_{f(n)}).$$

In other words,

$$\forall x \in W_n \forall x * y \in W \text{ (if } x * y \notin W_{f(n)}, \text{ then } l(y) \geq 2).$$

We will show that

$$\forall x \in W_n \forall m \forall x * y \in W \text{ (if } x * y \notin W_{f^m(n)}, \text{ then } l(y) \geq m + 1).$$

This will prove the lemma.
Therefore, consider some $x \in W_n$ and $x * y \in W$ such that $x * y \notin W_{f^m(n)}$. Let $y = \langle D_1, .., D_k \rangle$ and assume $k \leqslant m$. We derive a contradiction. Observe that for any $z * \langle D \rangle \in W$, if $z * \langle D \rangle \notin W^{m+1}_{f(n)}$, then $z \notin W^{m}_{f(n)}$. Hence $x * \langle D_1, .., D_{k-1} \rangle \notin W^{m-1}_{f(n)}$. And again, $x * \langle D_1, .., D_{k-2} \rangle \notin W^{m-2}_{f(n)}$, etcetera. Whence $x \notin W^{m-k}_{f(n)} \supseteq W_n$, which contradicts the fact that $x \in W_n$. Thus $k > m$, and therefore $l(y) \geq m + 1$. This proves the lemma. □

Finally, we are ready to show that

$$\exists n \forall m \geq n (W_m = W_n).$$

First, observe by examining procedure **III** that if $x \subseteq_{\to} y$ and $x \neq y$, then $(\alpha_y)_{\not\to} \subset (\alpha_x)_{\not\to}$. And that if $x \subseteq_{\not\to} y$, then $(\alpha_y)_{\not\Box} \subset (\alpha_x)_{\not\Box}$. Let n_0 be the number of implications in X, and let n_1 be the number of formulas in $b_{\not\Box}$. From the observation above it follows that no sequence $x \in W$ can contain more than n_0 consecutive implications, or more that n_1 elements which are no implication. Hence $l(x) \leqslant (n_0 + 1).n_1 + n_0$, for all $x \in W$. Now apply Lemma 5.3.16 to $x = \langle\rangle \in W_0$ and $m = (n_0 + 1).n_1 + n_0$; thus there exists a number n such that for all $y \in W$, if $l(y) \leqslant m$ then $y \in W_n$. Hence $W = W_n$. Thus the finiteness of W is established. □

5.3.17. Corollary. iH is complete with respect to the class of finite *LLeMa*-frames in which every node is either a top node or above a minimal leivant-node.

Proof By examining the way in which the finite model in the completeness proof for iH above, is constructed. □

We now show that also for iH we have a completeness proof with respect to gathering frames; it is complete with respect to the class of finite gathering *LLeMa*-frames. We know already that we lose the brilliancy of the frames in this case, see Section 3.4.6.

5.3.18. Proposition. $\vdash_{\mathsf{iH}} A$ iff A is valid on all finite gathering conversely well-founded Ma-frames.

Proof Similar to the proof of the completeness of iLLe with respect to gathering frames, proposition 4.4.2. In the notation of this proof; the property

$$wRv \to w(R';\preccurlyeq)v$$

is sufficient to preserve the Ma-property of the frame. The model $\mathcal{M}$ has the property

$$wRv \to \exists t \in Top(wRt \wedge tR = vR),$$

And the model $\mathcal{M}'$ has the property

$$wRv \to \exists t \in Top(w(R';\preccurlyeq)t \wedge \forall u(t(R';\preccurlyeq)u \leftrightarrow v(R';\preccurlyeq)u)),$$

thus certainly

$$wR'v \to \exists t \in Top(w(R';\preccurlyeq)t \wedge \forall u(t(R';\preccurlyeq)u \leftrightarrow v(R';\preccurlyeq)u)).$$

□

Proposition 5.3.18 is not a strengthening of proposition 5.3.6, since we loose the brilliancy when we restrict ourselves to gathering frames. This was already pointed out in Section 4.4.

But we do have a real strengthening of the completeness result in Proposition 5.3.6. This is an immediate corollary of Lemma 5.3.5.

5.3.19. Corollary. iH is complete with respect to the class of finite brilliant L-frames which have the Ma-property and satisfy

$$wRv \to \exists x(wRx \preccurlyeq v \wedge \forall u(vRu \to x \preccurlyeq u) \wedge \forall u(xRu \to x \preccurlyeq u)).$$

5.4 Nonconservativity

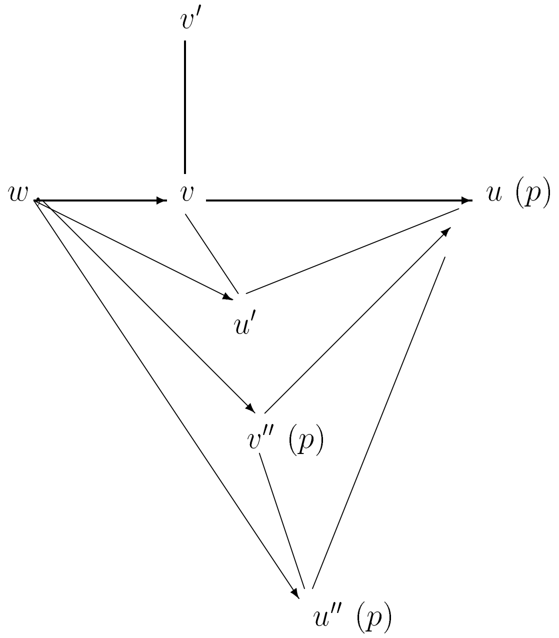
In Chapter 3 we showed that the logic iH is contained in iPH. There we also promised to show that iPH contains strictly more, which is the content of the following lemma.

5.4.1. Proposition. The logic iH is contained in iPH, and iPH is not conservative (with respect to formulas in $\mathcal{L}_\Box$) over iH.

Proof The first part of the proposition is proved in Section 3.3. For the second part, consider the formula

$$\Box(p \vee \neg\neg\Box\bot) \to \Box(p \vee \Box\bot). \tag{5.19}$$

It is not difficult to see that this formula is derivable in iPH, see Section 3.3. We show that this formula is not derivable in iH. Consider the following model.



(The arrows denote the modal relation, the lines the intuitionistic relation. If there is a line between x and y and x is below y, then this means that $x \preccurlyeq y$, e.g. $u'' \preccurlyeq v''$ and v'' is a top node.)

We leave it to the reader to verify that the transitive brilliant closure of this model is conversely well-founded and has the *Le*- and the *Ma*-property. Observe that $v \nVdash p \vee \Box\bot$, whence $w \nVdash \Box(p \vee \Box\bot)$, and that $w \Vdash \Box(p \vee \neg\neg\Box\bot)$. This shows that w does not force (5.19). By Proposition 5.3.6 this implies that iH does not derive (5.19). □

Part II

Intuitionistic Propositional Logic

Chapter 6

Concepts

In this chapter we introduce the notions studied in part II of the thesis. Sections 6.1 and 6.2 discuss admissible rules and intermediate logics, the main subjects of the next chapters. Intermediate logics only occur in Chapter 8. Section 6.3 contains preliminaries.

6.1 Admissible rules

The admissible rules of a theory are the rules under which the theory is closed. It is well-known that, in contrast to classical propositional logic, intuitionistic propositional logic IPC, has admissible rules which are not derivable. Probably the first such rule known for this logic is the rule

$$\neg A \to (B \vee C)/(\neg A \to B) \vee (\neg A \to C),$$

stated by Harrop (1960). Extensions of this rule which are also admissible but not derivable followed (Mints 1976) (Citkin 1977) but the question whether there were other admissible rules for IPC than the ones known remained open.

In 1975 Friedman posed the problem whether it is decidable if a rule is an admissible rule for IPC or not. In (Rybakov 1997) this question was answered in the affirmative. Moreover, Rybakov showed that the admissible rules of IPC do not have a finite basis. Informally speaking this means that there is no finite set of admissible rules which in some sense 'generates' all the admissible rules of IPC. However, this does not exclude the possibility that there is a representation of the admissible rules via a simple infinite basis or in some other clarifying way.

Some ten years ago both de Jongh and Visser isolated the same simple c.e. set of rules $\mathcal{V}$ which they conjectured to be a basis for the admissible rules of IPC. In Chapter 7 we prove their conjecture. This is the main result of Part II of the thesis. In that chapter we also present a proof system for the admissible rules. Furthermore, we give semantic criteria for admissibility which are similar to the ones found by Rybakov (1997). Since Visser (1999) proved that the admissible rules of IPC are the same as the propositional admissible rules of Heyting Arithmetic HA this provides us with a proof system and a basis for the propositional admissible rules of HA as well.

There also is another connection with Heyting Arithmetic. Namely, we will see that our results plus certain results by Visser (1999), imply that HA proves the admissibility of its admissible rules. This means that for every propositional admissible rule A/B and for every substitution σ, HA proves the statement 'if $\mathsf{HA} \vdash \sigma(A)$, then $\mathsf{HA} \vdash \sigma(B)$'. In part I of the thesis (Chapter 2) it is explained what this means for the provability logic of HA.

One of the results (Proposition 7.3.1) we use in our characterization of the admissible rules is almost a reformulation of results by Ghilardi. Therefore, we devote one section (Section 7.3) to the recapitulation of the theorems from (Ghilardi 1998) that we use in this paper.

6.2 Intermediate logics

In contrast to classical propositional logic CPC, intermediate logics[1] can have nonderivable admissible rules. For instance, in (Rybakov 1997) it is shown that intuitionistic propositional logic IPC has countably many nonderivable admissible rules. There are several very natural questions concerning intermediate logics and their admissible rules which become trivial once all the admissible rules of the logic are derivable, but which appear to be rather complicated otherwise. An example of such a question is which intermediate logics are maximal. This means the following.

Let us call a logic T with the Disjunction Property *maximal with respect to a set of admissible rules* $\mathcal{R}$ if all the rules in $\mathcal{R}$ are admissible for T and there is no intermediate logic with the Disjunction Property which is a proper extension of T for which all rules in $\mathcal{R}$ are admissible. If $\mathcal{R}$ is the set of all admissible rules of T we just say that T is *maximal*. Clearly, if T is maximal with respect to some set of admissible rules, it is maximal. A maximal logic T is characterized by its admissible rules plus the Disjunction Property: the only logic with the Disjunction Property that contains T and for which all admissible rules of T are admissible is T itself. The requirement that the logic contains T is redundant, because if all the admissible rules of T are admissible then so are the rules $\top/A$ for all theorems A of T, and whence the logic contains T. Note that if all rules in $\mathcal{R}$ are derivable in T then T is maximal with respect to $\mathcal{R}$ once it has no proper extensions with the Disjunction Property. For in this case any extension of T derives all rules in $\mathcal{R}$. We use the terms 'characterized by its admissible rules plus the Disjunction Property' and 'maximal' interchangeably.

[1] The logics between IPC and CPC are also called superintuitionistic logics, e.g. in (Chagrov and Zakharyaschev 1997)

It may appear to the reader that a better definition of maximality (in this sense) would be one without a restriction to logics with the Disjunction Property. However, this restriction is more an empirical than a natural one (or is empirically natural ...): the only interesting results we encountered on maximality with respect to admissible rules, were in the sense of maximality as defined above and not in the broader sense.

In Chapter 8 we will show that some well-known intermediate logics are maximal. In particular, we will see IPC is maximal. To show this, we use a result from Chapter 7, namely that there exists a certain countable basis $\mathcal{V}$ for the admissible rules of IPC. The logic IPC is not characterized by its admissible rules; for example all admissible rules of IPC are admissible for CPC. Kreisel and Putnam (1957) showed that neither is IPC characterized by the Disjunction Property. However, the fact that IPC is maximal shows that IPC is characterized by the combination of the two properties.

We will see that the characterization of IPC is optimal. By optimal we mean that there is no proper subset $\mathcal{R}$ of $\mathcal{V}$ such that IPC is already maximal with respect to $\mathcal{R}$. We show that for any finite subset X of $\mathcal{V}$ there is a proper intermediate logic for which X is admissible. The logic in question is even maximal with respect to X. For this we use the countably many proper intermediate logics $\mathsf{D_0}, \mathsf{D_1}, \mathsf{D_2}, \ldots$ with the Disjunction Property which were constructed in (Gabbay and de Jongh 1974). We show that there is a correspondence between finite subsets of $\mathcal{V}$ and these logics. Any such $\mathsf{D_n}$ is maximal with respect to a finite subset X of $\mathcal{V}$ and for any finite subset X of $\mathcal{V}$ there is a number n such that $\mathsf{D_n}$ is maximal with respect to X. Furthermore, it will turn out to be a trivial observation that any cofinal subset of the basis is equivalent, in terms of the admissible rules which are derivable from it, to the basis itself. Therefore, there is no proper subset of $\mathcal{V}$ with respect to which IPC is maximal. Moreover, it shows that the Gabbay-de Jongh logics are all maximal.

With the characterization of IPC we do not claim a completely new result since a similar result, a characterization of IPC in terms of the Kleene slash, was already obtained by de Jongh (1970) (Section 8.3). However, not only is the reduction of the one characterization to the other not trivial, but the connection with the admissible rules is new and interesting. We show that these characterizations are effectively reducible to each other. Hence the effectiveness of the characterization in terms of the Kleene slash (de Jongh 1970) implies the effectiveness of the characterization in terms of the admissible rules.

There are many interesting open questions concerning maximality of logics. To name a few: Are there any logics which are not maximal with respect to their admissible rules? If so, can any such logic be extended to an intermediate logic which is maximal with respect to its admissible rules? Given a set of rules $\mathcal{R}$ which are derivable in CPC there is, by definition, an intermediate logic for which all rules in $\mathcal{R}$ are admissible. But is there an intermediate logic which is maximal with respect to $\mathcal{R}$?

6.3 Preliminaries

In this section we will define what an admissible rule is and what a basis for admissible rules is, and we will fix some notation concerning Kripke models. In Section 6.3 we define two specials models needed in Chapter 8. Since we will mostly work in the context of intuitionistic propositional logic IPC we will not define these notions in full generality. For example, what we will call an *admissible rule* is in fact a *propositional* admissible rule. For a general setting and for interesting results about admissible rules in the context of other logics see (Rybakov 1997) and (Visser 1999).

Unless stated otherwise, formulas are meant to be formulas in a (fixed) language for intuitionistic propositional logic. The letters A, B, C, D, E, F will always range over formulas and p, q, r, s, t over propositional variables. We write $\vdash$ for derivability in IPC.

An $\mathfrak{L}$-*substitution* σ is a map which assigns to every propositional variable a formula in the language $\mathfrak{L}$. For a propositional formula A, we write $\sigma(A)$ for the result of applying σ to A, i.e. for the result of substituting $\sigma(p_i)$ for p_i in A. When $\mathfrak{L}$ is our fixed language of propositional logic mentioned above, we say 'substitution' instead of '$\mathfrak{L}$-substitution'.

An intermediate logic is a consistent theory in the language of propositional logic, containing IPC, which is closed under substitution. For intermediate logics T we will write $\vdash_T$ for derivations in T. If we only know that T is a theory we write $T \vdash$ instead.

A *rule* is an expression of the form

$$\frac{A_1 \ \ldots \ A_n}{B}.$$

We sometimes write $A_1, \ldots, A_n/B$ for this expression. We say that an expression

$$\frac{A'_1 \ldots A'_n}{B'},$$

is a substitution instance of such a rule when there is a substitution σ such that $\sigma(A_i) = A'_i$ and $\sigma(B) = B'$. Let T be some theory in a language $\mathfrak{L}$. We say that a rule A/B is an *admissible rule of* T, and write $A \mathrel{|\!\sim}_T B$, if

for all $\mathfrak{L}$-substitutions σ: if $T \vdash \sigma(A)$ then $T \vdash \sigma(B)$.

In this case we also say that A *admissibly derives* B *in* T. We write $\mathrel{|\!\sim}$ for $\mathrel{|\!\sim}_{\mathsf{IPC}}$.

Bases

For a set of rules $\mathcal{R}$ and a set of formulas $\mathcal{A}$, we say that B is *derivable in* T *by the set of rules* $\mathcal{R}$ *from assumptions* $\mathcal{A}$ when there is a sequence of formulas

$(B_1, \ldots, B_n)$, where $B_n = B$, such that for every $i \leq n$ either $B_i \in \mathcal{A}$ or there are $B_{i_1}, \ldots, B_{i_m}$ with $i_j < i$ such that either

$$\vdash_T (B_{i_1} \wedge \ldots \wedge B_{i_m}) \to B,$$

or

$$\frac{B_{i_1} \ldots B_{i_m}}{B_i},$$

is a substitution instance of some rule in $\mathcal{R}$.

We call a set of rules $\mathcal{R}$ a *basis* (in T) for some other set of rules $\mathcal{R}' \supseteq \mathcal{R}$ if for every rule $A_1 \ldots A_n/B$ in $\mathcal{R}'$, B is derivable in T by the rules $\mathcal{R}$ from the assumptions $A_1, \ldots, A_n$. Given T, we say that a set $\mathcal{R}$ of admissible rules of T is a *basis for the admissible rules of* T when $\mathcal{R}$ is a basis for the set of admissible rules of T.

Subbases

If a theory T has the Disjunction Property,

DP if $T \vdash A \vee B$ then $T \vdash A$ or $T \vdash B$,

then it follows that if $A \mathrel{|\!\!\sim}_T B$ and $C \mathrel{|\!\!\sim}_T D$, then also $A \vee C \mathrel{|\!\!\sim}_T B \vee D$. However the rule $(A \vee C)/(B \vee D)$ does not have to be derivable from the rules A/B and C/D in T. Therefore, in the context of theories which possess the Disjunction Property, the notion of a basis for the admissible rules seems too restrictive. This accounts for the notion of a *subbasis for the admissible rules*, introduced below. That is, for theories with the Disjunction Property, we think that the right notion of a basis (for the admissible rules), is in fact that what we will call a subbasis here: a set $\mathcal{R}$ of admissible rules of T is a *subbasis for the admissible rules of* T if the following is a basis for the admissible rules of T: the collection of rules of the form

$$\frac{A \vee p}{B \vee p}$$

where the rule A/B is in $\mathcal{R}$ and p does not occur in A or B.

6.3.1 Models

In this section we fix some notation and terminology concerning Kripke models. Most of the notions we introduce are standard, the only exception is the notion of a tight predecessor. In the last section we define what basic models and Jaskowski models are.

A *frame* is a pair $(W, \preccurlyeq)$, where W is a set and $\preccurlyeq$ is a partial order on W. A *(Kripke) model* K is a triple $(W, \preccurlyeq, \Vdash)$, where $(W, \preccurlyeq)$ is a frame and $\Vdash$ is the so-called forcing relation defined as usual (Section 3.4.2). A formula A is *valid* in a model K, $K \models A$, if it is valid in all nodes. If no confusion is possible we use the same notation $\preccurlyeq$ and $\Vdash$ for the partial order and forcing relation of different models.

For two nodes w, v we say that *w is below v* when $w \preccurlyeq v$. In this case we also say that *v is above w*. We write $w \prec v$ or $w \succ v$ if $w \neq v$, and $w \preccurlyeq v$ or $w \succcurlyeq v$ respectively. In contrast to intuitionistic modal logic we now call a node v a *successor of w* if $w \prec v$, in which case we also call w a *predecessor of v*. A node y is called *an immediate successor* of x if $x \prec y$ and there is no z for which $x \prec z \prec y$. A *maximal* node is a node which has no nodes above it except itself. We call a model *rooted* when it contains a node which is below all other nodes in the model.

We say that $K' = (W', \preccurlyeq', \Vdash')$ is a *submodel of* $K = (W, \preccurlyeq, \Vdash)$ if W' is a subset of W, and $\preccurlyeq', \models'$ are the restrictions of the corresponding relations of K to W'. We say that K' is a *finite submodel* when W' is finite. We write K_w for K' if $W' = \{x \in W \mid w \preccurlyeq x\}$. A submodel of the form K_w is called the *submodel generated by w*. Note that submodels are completely characterized by their domain. Therefore, we will from now on notationally confuse a submodel with its domain.

For Kripke models $K_1, \ldots, K_n$, we let $(\sum_i K_i)'$ denote the Kripke model which is the result of attaching one new node at which no propositional variables are forced, below all nodes in $K_1, \ldots, K_n$ (Smoryński 1973).

The extension property

We repeat from (Ghilardi 1998) the following definitions. We say that two rooted Kripke models are *variants* of each other when they have the same domain and partial order, and their forcing relations only possibly differ at the roots. A class of Kripke models is called *stable* if for every model K in the class and every node w of K, K_w is in the class as well. A class of rooted Kripke models has the *extension property* when for every finite set of Kripke models $K_1, \ldots, K_n$ in this class there is a variant of $(\sum_i K_i)'$ which is in this class as well. A theory T has the *extension property up to n* if for every family of at most n *rooted* models $K_1, \ldots, K_n$ of T, there is a variant of $(\sum_i K_i)'$ which is a model of T as well. A theory T has the *extension property* if it has the extension property up to n, for all n.

When $\mathcal{K}$ is a class of Kripke models we say that A is valid in $\mathcal{K}$, notation $\mathcal{K} \models A$, when A is valid in every model of $\mathcal{K}$.

Tight predecessors (in propositional logic)

Consider a Kripke model $K = (W, \preccurlyeq, \Vdash)$, some node u in K and a set U of nodes in K. We say that u is a *tight predecessor of U*, if

$$\forall x \in U (u \preccurlyeq x) \wedge \forall x \succ u \exists y \in U (y \preccurlyeq x).$$

In the sequel we will actually only consider tight predecessors of finite sets of nodes. We often write 'a tight predecessor of $u_1, \ldots, u_n$' instead of 'a tight predecessor of $\{u_1, \ldots, u_n\}$'.
Observe that a set does not necessarily have a tight predecessor but that every node in a Kripke model is a tight predecessor of some set, namely, of the set of all its successors. Note the similarity with the notion of a tight predecessor in modal logic (Section 4.6).

Jaskowski models

A modified Jaskowski frame (Smoryński 1973) is one of the frames $J_1, J_2, \ldots$ defined via:

J_1 consists of one node

J_{n+1} is the result of attaching one node below $(n+1)$ copies of J_n.

(In (Smoryński 1973), J_i is denoted with J_i^*.) A *Jaskowski model* is a model based on a modified Jaskowski frame.

Basic models

A *basic model* is a model for which the following holds:

- the only nodes that force propositional variables are maximal nodes,
- every maximal node forces exactly one propositional variable and no two maximal nodes force the same propositional variable.

For example, if $1, \ldots, n$ are the maximal nodes of a frame F, then the model given by the valuation ($x \Vdash p_i$ iff $x = i$) is a basic model on F. A *basic Jaskowski model* is a basic model based on a modified Jaskowski frame. It is easy to see that the following fact about basic models holds.

6.3.1. Fact. Let F be a frame in which no two nodes have exactly the same maximal nodes above them. Consider the basic model on F. There are formulas A_x such that $y \Vdash A_x$ iff $x \preccurlyeq y$. Namely, if $1, \ldots, n$ are the maximal nodes above x and $i \Vdash p_i$, then the formula $A_x = \neg\neg(p_1 \vee \ldots \vee p_n)$ has the desired properties.

Chapter 7

The admissible rules of IPC

In this chapter we give a basis for the admissible rules of intuitionistic propositional logic. We proceed as follows. In the first section we define a proof system, called AR, which derives expressions of the form $A \rhd B$, where A and B are propositional formulas. In Section 7.4 we then show that AR is a proof system for the admissible rules: AR derives $A \rhd B$ iff $A \mathrel{|\!\sim} B$. The proof of this fact has two main ingredients: In Section 7.2 we characterize AR in terms of Kripke models. We define what an AR-model is and show that AR derives $A \rhd B$ if and only if B is valid in all AR-models on which A is valid. Note that in the light of Section 7.4 this is a semantical characterization of the admissible rules. In Section 7.3 we derive a semantical characterization (in terms of classes of finite Kripke models) of the admissible rules from results by Ghilardi (1998). In Section 7.4 we show that these two characterizations are 'the same', which leads to the result mentioned above. Finally, in the last section we show how this provides us with a basis for the admissible rules.

7.1 A proof system

As explained above, we define a system AR that is a proof system which derives expressions of the form $A \rhd B$ where A and B are propositional formulas. To keep the definition of this system readable, we will use the following abbreviation,

$$(A)(B_1, \ldots, B_n) \equiv_{def} (A \to B_1) \vee \ldots \vee (A \to B_n).$$

Furthermore, we adhere to the same reading conventions as in the case of preservativity logic (Section 3.4).

Axioms:

$$V \quad ((A \to B \vee C) \vee D) \rhd ((A)(E_1, \ldots, E_n, B, C) \vee D),$$
$$\text{for } A = \textstyle\bigwedge_{i=1}^{n}(E_i \to F_i)$$
$$I \quad A \rhd B, \qquad \text{where } \mathsf{IPC} \vdash (A \to B)$$

Rules:

$$\mathit{Conj}\ \frac{C \rhd A \quad C \rhd B}{C \rhd A \wedge B} \qquad\qquad \mathit{Cut}\ \frac{A \rhd B \quad B \rhd C}{A \rhd C}$$

Note that V is not an axiom in the strict sense. It consists in fact of the infinitely many principles V_n which are

$$V_n \quad ((\bigwedge_{i=1}^{n}(E_i \to F_i) \to B \vee C) \vee D) \rhd ((\bigwedge_{i=1}^{n}(E_i \to F_i))(E_1, \ldots, E_n, B, C) \vee D).$$

De Jongh and Visser observed that the rules corresponding to V_n (Section 7.5) are admissible and conjectured them to be a basis.

As noted before, if $A \mathrel{|\!\sim} C$ and $B \mathrel{|\!\sim} C$ then also $(A \vee B) \mathrel{|\!\sim} C$. This property of the admissible rules is not reflected in the rules of AR. That is, there is no rule

$$\mathit{Disj}\ \frac{A \rhd C \quad B \rhd C}{(A \vee B) \rhd C}$$

However, it turns out that AR satisfies this rule. This is the next lemma, which we will need in the completeness proof for AR to come.

7.1.1. Lemma. If $\mathsf{AR} \vdash A \rhd C$ and $\mathsf{AR} \vdash B \rhd C$ then $\mathsf{AR} \vdash (A \vee B) \rhd C$.

Proof. It is easy to prove (with an induction to the length of derivation) that $\mathsf{AR} \vdash A \rhd B$ implies $\mathsf{AR} \vdash (A \vee C) \rhd (B \vee C)$. Hence $\mathsf{AR} \vdash A \rhd B$ also implies $\mathsf{AR} \vdash (C \vee A) \rhd (C \vee B)$.

Now assume $\mathsf{AR} \vdash A \rhd C$ and $\mathsf{AR} \vdash B \rhd C$. From the previous observation it follows that $\mathsf{AR} \vdash (A \vee B) \rhd (C \vee B)$ and $\mathsf{AR} \vdash (C \vee B) \rhd (C \vee C)$. Clearly, also $\mathsf{AR} \vdash (C \vee C) \rhd C$. Applying Cut (twice) gives the desired result. □

7.2 Completeness of the proof system

In this section we characterize AR in terms of Kripke models. The Kripke models we use have special properties, they are the so-called AR-models defined as follows.

AR-models

We call a Kripke model K an AR-*model* when it is a rooted model in which every finite set of nodes $\{u_1, \ldots, u_n\}$ has a tight predecessor u, i.e. a node u such that

$$u \preccurlyeq u_1, \ldots, u_n \wedge \forall u' \succ u \,(u_i \preccurlyeq u',\ \text{for some } i \in \{1, \ldots, n\}).$$

(We write '$x \preccurlyeq y_1, \ldots, y_n$' for '$x \preccurlyeq y_1 \wedge x \preccurlyeq y_2 \wedge \ldots \wedge x \preccurlyeq y_n$'.)

We will prove that AR derives $A \rhd B$ if and only if B is valid in every AR-model in which A is valid. The proof uses a lemma which we present separately in advance. Before stating it, let us remind the reader that a set of formulas x is called IPC-*saturated* if it is a consistent set such that for all A and B, if $x \vdash A \vee B$, then $A \in x$ or $B \in x$. In particular, x is closed under deduction in IPC.

7.2.1. Lemma. Let Θ be some set of formulae. Every IPC-saturated set $x \subseteq \Theta$ can be extended to an IPC-saturated set $y \subseteq \Theta$ such that for no IPC-saturated set y' it holds that $y \subset y' \subseteq \Theta$.

Proof. Let x and Θ be as in the lemma. We construct a sequence $y_0 \subseteq y_1 \subseteq \ldots$, such that for all i, $*(y_i)$ holds, where the property $*(\cdot)$ is defined as

$*(z)$ for all n, for all $A_1, \ldots, A_n$: if $z \vdash A_1 \vee \ldots \vee A_n$, then $A_i \in \Theta$ for some $i = 1, \ldots, n$.

We construct the sequence of sets as follows. Let $C_0, C_1, \ldots$ be an enumeration of all formulae in which every formula occurs infinitely often. We put $y_0 = x$. Clearly $*(y_0)$ holds. Suppose y_i is already defined. Then we put

$$y_{i+1} \equiv_{def} \begin{cases} y_i \cup \{C_i\} & \text{if } *(y_i \cup \{C_i\}) \text{ does hold} \\ y_i & \text{if } *(y_i \cup \{C_i\}) \text{ does not hold.} \end{cases}$$

Now we take $y = \bigcup_i y_i$. First, we have to see that this is indeed an IPC-saturated set. And second we have to show that there are no proper supersets of y which are IPC-saturated and are contained in Θ.

To see that y is IPC-saturated, suppose $y \vdash A \vee B$. Hence $y_i \vdash A \vee B$, for some i. There are $i \leq j \leq k$ such that $C_j = A$ and $C_k = B$. If $*(y_j \cup \{C_j\})$ or $*(y_k \cup \{C_k\})$ holds, then clearly A or B is in y. We show that indeed one of $*(y_j \cup \{C_j\})$ and $*(y_k \cup \{C_k\})$ has to hold. Arguing by contradiction, assume this is not the case. Thus there are $A_1, \ldots, A_n, B_1, \ldots, B_m$ such that $y_j, C_j \vdash \bigvee_{i=1}^n A_i$ and $y_k, C_k \vdash \bigvee_{i=1}^m B_i$ but none of $A_1, \ldots, A_n, B_1, \ldots, B_m$ is in Θ. Since $y_i \subseteq y_j \subseteq y_k$ and $y_i \vdash C_j \vee C_k$, this implies that $y_k \vdash \bigvee_{i=1}^n A_i \vee \bigvee_{i=1}^m B_i$, which contradicts the fact that $*(y_k)$ holds.

To see that there are no IPC-saturated proper supersets of y which are contained in Θ, consider an IPC-saturated set $y \subseteq y' \subseteq \Theta$. We show that $y = y'$. Consider a formula $A \in y'$, and suppose $C_i = A$. It is easy to see that since $y_i \subseteq y' \subseteq \Theta$ and the fact that y' is saturated, $*(y_i \cup \{C_i\})$ holds. Hence $A \in y$. Therefore $y = y'$. □

Now we are ready to prove the following lemma.

7.2.2. Proposition. $\mathsf{AR} \vdash A \rhd B$ iff B is valid on all AR-models on which A is valid.

Proof. The direction from left to right. We have to see that if $\mathsf{AR} \vdash A \rhd B$ and A is valid on an AR-model, then B is valid on this model as well. This can be shown by induction to the length of the derivation of $A \rhd B$ in AR.

The case that $A \rhd B$ is an instance of the axiom scheme I is easy. In the induction step we have to consider the two rules. All of them are straightforward.

Therefore, we only consider V. We have to show that for any conjunct of implications $A = \bigwedge_{i=1}^{n}(E_i \to F_i)$, if $(A \to B \vee C) \vee D$ is valid on all AR-models, then so is $(A)(B, C, E_1, \ldots, E_n) \vee D$. Therefore, assume that indeed for such a formula A, $(A \to B \vee C) \vee D$ is valid on an AR-model K. Let v be the root of K. We show that $(A)(B, C, E_1, \ldots, E_n) \vee D$ is valid in K at v, whence that $(A)(B, C, E_1, \ldots, E_n) \vee D$ is valid in K.

Arguing by contradiction, assume $(A)(B, C, E_1, \ldots, E_n) \vee D$ is not valid at v. Hence $(A \to B \vee C)$ is valid at v. Moreover, $\neg A$ is not valid at v. Therefore, there is a nonempty set U of nodes, such that

$$\forall x (x \Vdash A \text{ iff for some } u \in U,\ u \preccurlyeq x).$$

Since $(A)(B, C, E_1, \ldots, E_n)$ is not valid at v, there are, for some $m \leq n+2$, nodes $u_{i_1}, \ldots, u_{i_m} \in U$ such that

$$\forall D \in \{B, C, E_1, \ldots, E_n\} \exists u \in \{u_{i_1}, \ldots, u_{i_m}\}\ u \nVdash D.$$

Since we consider an AR-model the set $\{u_{i_1}, \ldots, u_{i_m}\}$ has a tight predecessor. That means that there is a node u such that

$$u \preccurlyeq u_{i_1}, \ldots, u_{i_m} \wedge \forall u' \succ u (u_{i_j} \preccurlyeq u', \text{ for some } j \leq m).$$

If A is valid at u then B or C has to be valid at u, which contradicts the fact that for both B and C there is a node in $u_{i_1}, \ldots, u_{i_m}$ which does not validate the formula. On the other hand, if A is not valid at u, then since A is valid at all nodes $u' \succ u$, E_j has to be valid at u, for some j. But this is a contradiction as well, since for every $j \in \{1, \ldots, n\}$ there is a node in $u_{i_1}, \ldots, u_{i_m}$ which does not validate E_j.

The direction from right to left. Assume $\mathsf{AR} \nvdash A \rhd B$. We construct an AR-model K in which A is valid while B is not.

First we construct an IPC-saturated set of formulas v in such a way that

$$A \in v,\ B \notin v, \text{ for all } C \rhd D\text{: if } \mathsf{AR} \vdash C \rhd D \text{ and } C \in v, \text{ then } D \in v. \tag{7.1}$$

This v will be the root of the model K we are going to construct. The existence of v is proved in the following Claim.

Claim If $\mathsf{AR} \nvdash A \rhd B$, then there is an IPC-saturated set v such that $A \in v$ and $B \notin v$, which has the property that if for some C, D, $\mathsf{AR} \vdash C \rhd D$ and $C \in v$, then $D \in v$ as well.

Proof of Claim. Assume $\mathsf{AR} \nvdash A \rhd B$. We construct a sequence of finite sets $\{A\} = x_0 \subseteq x_1 \subseteq \ldots$ such that for all i, $\mathsf{AR} \nvdash (\bigwedge x_i) \rhd B$, and if $\mathsf{AR} \vdash (\bigwedge x_i) \rhd C$, then $C \in x_j$ for some j. The set v we look for will be the set $\bigcup x_i$.

Let $C_0, C_1, \ldots$ be an enumeration of all formulas in which every formula occurs infinitely often. Given the set x_i, we show how to construct x_{i+1}.

$$x_{i+1} \equiv_{def} \begin{cases} x_i & \text{if } \mathsf{AR} \nvdash (\bigwedge x_i) \rhd C_i \\ x_i \cup \{C_i\} & \text{if } \mathsf{AR} \vdash (\bigwedge x_i) \rhd C_i,\ C_i \text{ is not a disjunction} \\ x_i \cup \{D_j, C_i\} & \text{if } \mathsf{AR} \vdash (\bigwedge x_i) \rhd C_i,\ C_i = D_1 \vee D_2,\ j = 1, 2 \\ & \text{is the least such that } \mathsf{AR} \nvdash (\bigwedge x_i \wedge D_j) \rhd B. \end{cases}$$

It is easy to see that each of these sets x_i has the desired properties, assuming it is well-defined. Thus it remains to show that they are indeed well-defined, i.e. that given x_i, x_{i+1} exists. Therefore, suppose $\mathsf{AR} \vdash (\bigwedge x_i) \rhd C_i$ and $C_i = (D_1 \vee D_2)$. We have to see that either $\mathsf{AR} \nvdash (\bigwedge x_i \wedge D_1) \rhd B$ or $\mathsf{AR} \nvdash (\bigwedge x_i \wedge D_2) \rhd B$. Arguing by contradiction, assume this is not the case. But then we can derive the contradiction that $\mathsf{AR} \vdash (\bigwedge x_i) \rhd B$ in the following way (we do not state all the rules used, but only the crucial ones).

$$\begin{array}{lll} \mathsf{AR} \vdash & (\bigwedge x_i \wedge D_1) \rhd B & \\ & (\bigwedge x_i \wedge D_2) \rhd B & \\ & (\bigwedge x_i \wedge (D_1 \vee D_2)) \rhd B & \text{(Lemma 7.1.1)} \\ & (\bigwedge x_i) \rhd (\bigwedge x_i \wedge (D_1 \vee D_2)) & \text{(assumption on } x_i) \\ & (\bigwedge x_i) \rhd B. & (\textit{Cut}) \end{array}$$

Now we take $v = \bigcup_i x_i$. It is easy to see that v has the desired properties. This proves the Claim.

Thus we know that there exists an IPC-saturated set v which satisfies (7.1). Next we construct our model K as follows. Its domain consists of all IPC-saturated sets which extend v. Its partial order $\preccurlyeq$ is the subset relation $\subseteq$. And the forcing relation is defined via

$$w \Vdash p \text{ iff } p \in w, \text{ for propositional variables } p.$$

It is easy to see that this indeed defines a Kripke model, that the model is rooted, and that A is valid in this model but B is not. Thus it only remains to show that K is an AR-model.

Therefore, consider nodes $u_1, \ldots, u_n \in K$. We have to show that there is a node u such that

$$u \preccurlyeq u_1, \ldots, u_n \wedge \forall u' \succ u(u_i \preccurlyeq u', \text{ for some } i \leq n).$$

First note that $u_1 \cap \ldots \cap u_n$ is not saturated in general. Therefore, although $u_1 \cap \ldots \cap u_n$ contains v, it does not have to be a node in K. Let now

$$\Delta = \{E \to F \mid (E \to F) \in u_1 \cap \ldots \cap u_n \wedge E \notin u_1 \cap \ldots \cap u_n\}.$$

Then we have

Claim The set $\{C \mid v \cup \Delta \vdash C\}$ is IPC-saturated.

Proof of Claim. Suppose that $v \cup \Delta \vdash C_1 \vee C_2$ holds. This implies that there is a conjunct $D = \bigwedge_{i=1}^m (E_i \to F_i)$ of implications in Δ, such that it holds that $v \vdash (D \to C_1 \vee C_2)$. Thus $(D \to C_1 \vee C_2) \in v$, because v is saturated. Since the expression $(D \to C_1 \vee C_2) \rhd (D)(C_1, C_2, E_1, \ldots, E_m)$ is derivable in AR, the way v is constructed implies that then also $(D)(C_1, C_2, E_1, \ldots, E_m) \in v$. And thus one of $(D \to C_1), (D \to C_2), (D \to E_1), \ldots, (D \to E_m)$ is in v. Since no E_i is in $u_1 \cap \ldots \cap u_n$, this implies that v does not contain any of $(D \to E_i)$. Therefore v contains either $(D \to C_1)$ or $(D \to C_2)$. Hence $v \cup \Delta$ derives either C_1 or C_2. This proves the Claim.

By the previous claim and the fact that $v \cup \Delta \subseteq u_1 \cap \ldots \cap u_n$, it follows from Lemma 7.2.1 that $\{C \mid v \cup \Delta \vdash C\}$ can be extended to an IPC-saturated set $u \subseteq u_1 \cap \ldots \cap u_n$ such that there are no saturated sets u' with $u \subset u' \subseteq u_1 \cap \ldots \cap u_n$. We show that this is the set we look for, i.e. if $u' \succ u$ for some saturated set u', then $u_i \preccurlyeq u'$, for some $i \in \{1, \ldots, n\}$.

Suppose not, that is, let $u \subset u'$ for some saturated set u' and assume that no u_i is contained in u'. We derive a contradiction. For all $i \leq n$, we (can) choose a formula $A_i \in u_i$ outside u'. Then the formula $A_1 \vee \ldots \vee A_n$ is in $u_1 \cap \ldots \cap u_n$ but not in u'. From the construction of u, and the fact that u' is a superset of u, it follows that u' is not contained in $u_1 \cap \ldots \cap u_n$. Thus there is a formula $E \in u'$ which is not in this intersection. Now $(E \to A_1 \vee \ldots \vee A_n)$ is an element of Δ, thus also of u. Hence $A_1 \vee \ldots \vee A_n$ should be in u', a contradiction. This finally proves the proposition. □

7.3 Results by Ghilardi

In the proof of the characterization of the admissible rules in terms of $\rhd$ we will use, besides the semantical completeness of AR (Section 7.2), the following fact which follows from results proved by Ghilardi (1998).

7.3.1. Proposition. If $A \mathrel{|\!\sim} B$, then B is valid in every stable class of finite rooted Kripke models which has the extension property (see Section 6.3.1) and in which A is valid.

This section is devoted to the recapitulation of the results by Ghilardi which lead to the proposition above. First we have to introduce some terminology.

Terminology

Let $\bar{p}$ be a sequence of propositional variables. We say that a formula A is a *formula in* $\bar{p}$, when all the propositional variables in A are among the variables in the sequence $\bar{p}$. We say that a Kripke model is a Kripke model *over* $\bar{p}$, when the forcing relation of the model is only defined for formulas in $\bar{p}$. If $\bar{p}$ is the sequence of all the propositional variables that occur in A, then $Mod(A)$ denotes all *finite* models of A over $\bar{p}$.

Following Fine (Fine 1974) (Fine 1985), Ghilardi defines equivalence relations $\sim_n$ and preorders $\leq_n$ between rooted Kripke models. Let K, K' be two rooted Kripke models with roots b and b' respectively.

$$
\begin{array}{lll}
K \sim_0^{\bar{p}} K' & \equiv_{def} & b \Vdash p \text{ iff } b' \Vdash p, \text{ for all atoms } p \text{ in } \bar{p}. \\
K \sim_{n+1}^{\bar{p}} K' & \equiv_{def} & \forall k \in K \exists k' \in K'((K)_k \sim_n (K')_{k'}) \text{ and vice versa.} \\
K \leq_0^{\bar{p}} K' & \equiv_{def} & b' \Vdash p \text{ implies } b \Vdash p, \text{ for all atoms } p \text{ in } \bar{p}. \\
K \leq_{n+1}^{\bar{p}} K' & \equiv_{def} & \forall k \in K \exists k' \in K'((K)_k \sim_n (K')_{k'}).
\end{array}
$$

When it is clear from the context to which sequence of variables we refer we omit this in the notation.

Moreover Ghilardi uses a measure of complexity, $c(\cdot)$, on propositional formulas defined as follows. Put $c(A) = 0$ if A is a propositional variable, $c(A \circ B) = max\{c(A), c(B)\}$, for $\circ = \wedge, \vee$, and $c(A \to B) = 1 + max\{c(A), c(B)\}$.

The proof of Proposition 7.3.1

In the proof of Proposition 7.3.1 we will use four results by Ghilardi which we will state below. The first three have to do with the relation $\leq_n$.

7.3.2. Proposition. (Ghilardi 1998) For two finite rooted Kripke models K and K' over $\bar{p}$ it holds that $K \leq_n K'$ iff for all formulas A in $\bar{p}$ with $c(A) \leq n$, $K' \models A$ implies $K \models A$.

7.3.3. Proposition. (Ghilardi 1998) Let $\mathcal{K}$ be a class of finite rooted Kripke models over $\bar{p}$ for which there exists a number n such that for all Kripke models K over $\bar{p}$ it holds that

$$\text{if there is a } K' \in \mathcal{K} \text{ with } K \leq_n K', \text{ then } K \in \mathcal{K}.$$

Then $\mathcal{K}$=$Mod(A)$ for some formula A in $\bar{p}$.

7.3.4. Proposition. (Ghilardi 1998) If a stable class $\mathcal{K}$ of finite rooted Kripke models over $\bar{p}$ has the extension property then so does the class of models

$$\{K \mid K \text{ is a finite rooted model over } \bar{p} \text{ and } \exists K' \in \mathcal{K}(K \leq_n K')\}.$$

The heart of Proposition 7.3.1 is the following theorem.

7.3.5. Theorem. (Ghilardi 1998) Let A be a formula in $\bar{p}$. If $Mod(A)$ has the extension property then there is a substitution σ such that $\vdash \sigma(A)$ and for all formulas D in $\bar{p}$, $A \vdash D \leftrightarrow \sigma(D)$.

Now the proof of Proposition 7.3.1 runs as follows. Suppose $A \mathrel{|\!\sim} B$ and let $\mathcal{K}$ be a stable class of finite rooted Kripke models with the extension property in which A is valid. Assume that all the propositional variables in A and B are among $\bar{p}$. Then let $\mathcal{K}'$ be the class of all Kripke models of $\mathcal{K}$, but then considered as Kripke models over $\bar{p}$. Note that $\mathcal{K}'$ is again a stable class of finite rooted Kripke models with the extension property in which A is valid. Let n be some number such that $c(A) \leq n$, and let

$$\mathcal{K}'' = \{K \mid K \text{ is a finite rooted model over } \bar{p} \text{ and } \exists K' \in \mathcal{K}'(K \leq_n K')\}.$$

By Proposition 7.3.2, A is valid in the class $\mathcal{K}''$ because it is valid in $\mathcal{K}'$. And by Proposition 7.3.3 we know that $\mathcal{K}'' = Mod(C)$ for some formula C in $\bar{p}$. Since, by Proposition 7.3.4, we also know that $\mathcal{K}''$ has the extension property, we can apply Theorem 7.3.5 to conclude that there is a substitution σ such that

$$\mathsf{IPC} \vdash \sigma(C) \text{ and } C \vdash B \leftrightarrow \sigma(B).$$

Clearly, the fact that A is valid in $Mod(C)$ implies that $C \vdash A$. Hence $\mathsf{IPC} \vdash \sigma(A)$. But this implies that $\sigma(B)$ is derivable, because $A \mathrel{|\!\sim} B$. Thus certainly $C \vdash \sigma(B)$, and whence $C \vdash B$. Therefore, B is valid in $Mod(C)$. It is easy to see that this implies that B is valid in $\mathcal{K}$ as well. □

7.4 Characterizations of admissibility

We are now ready to give the promised characterizations of the admissible rules of IPC. One is in terms of $\rhd$, a proof system for the admissible rules. The other two are in terms of Kripke models. Let us state them before we consider their proofs.

7.4.1. Theorem. $A \mathrel{|\!\sim} B$ iff $\mathsf{AR} \vdash A \rhd B$.

7.4.2. Corollary. $A \mathrel{|\!\sim} B$ iff B is valid in every AR-model in which A is valid.

7.4.3. Corollary. $A \mathrel{|\!\sim} B$ iff B is valid in every stable class of finite rooted Kripke models with the extension property in which A is valid.

The last corollary is Proposition 7.3.1. The second characterization is a corollary of the first one in combination with Proposition 7.2.2 and Lemma 7.4.4. The latter is also needed in the proof of the first characterization. Lemma 7.4.4 shows that there is a natural correspondence between AR-models and stable classes of finite rooted Kripke models with the extension property. Therefore, the two corollaries are in some sense the same. We first treat this lemma and then we prove Theorem 7.4.1.

7.4.4. Lemma. For all n and all finite sequences of propositional variables $\bar{p}$ we have the following correspondence:
(a) For every AR-model K there is a stable class $\mathcal{K}$ of finite rooted Kripke models with the extension property such that

$$\text{for all } A \text{ in } \bar{p} \text{ with } c(A) \leq n\text{: } K \models A \text{ iff } \mathcal{K} \models A.$$

(b) For every stable class $\mathcal{K}$ of finite rooted Kripke models with the extension property there is an AR-model K such that

$$\text{for all } A\text{: } K \models A \text{ iff } \mathcal{K} \models A.$$

Proof. Let n be some number and let $\bar{p}$ be some finite sequence of propositional variables. First of all, let $\mathcal{A}$ be the set of all formulas A in $\bar{p}$ with $c(A) \leq n$. This set is, modulo provable equivalence, finite.

To show part (a) of the lemma, suppose K is an AR-model. Let $\mathcal{K}$ be the class of all Kripke models K' such that K' is a finite rooted submodel of K, and such that

$$\forall A \in \mathcal{A} \forall x \in K' (K', x \Vdash A \text{ iff } K, x \Vdash A). \tag{7.2}$$

It is easy to see that $\mathcal{K}$ is stable. We show that $\mathcal{K}$ has the extension property.

Consider models $K_1, \ldots, K_n$ in $\mathcal{K}$, with roots $u_1, \ldots, u_n$ respectively. Let u be a tight predecessor of $u_1, \ldots, u_n$ in K. That means that

$$u \preccurlyeq u_1, \ldots, u_n \wedge \forall u' \succ u (u_i \preccurlyeq u', \text{ for some } i \in \{1, \ldots, n\}).$$

Let K' be the submodel the domain of which is the union of $\{u\}$ and the domains of $K_1, \ldots, K_n$. It is easy to see K' satisfies (7.2). Hence K' is in $\mathcal{K}$. This shows that $\mathcal{K}$ has the extension property.

It remains to show that

$$\text{for all } A \in \mathcal{A}\text{: } K \models A \text{ iff } \mathcal{K} \models A.$$

The direction from left to right follows from the definition of $\mathcal{K}$. The direction from right to left is shown by contraposition, i.e. by showing that for all $A \in \mathcal{A}$ it holds that whenever $K \not\models A$ there is a $K' \in \mathcal{K}$ such that $K' \not\models A$ (it suffices to show that $\mathcal{K}$ is not empty, but the proof is the same). This again follows from the following standard result. We include the proof for the sake of completeness.

Claim For every Kripke model K, for every node w in K, there is a finite rooted submodel K' of K with root w, such that

$$\forall A \in \mathcal{A} \forall x \in K'(K', x \Vdash A \text{ iff } K, x \Vdash A). \tag{7.3}$$

Proof of Claim. Let $\mathcal{A}$, $K = (W, \preccurlyeq, \Vdash)$ and w be as in the claim. Now we choose step by step, starting with w, a finite subset of W a copy of which will be the domain W_w of our new model $K' = (W_w, \preccurlyeq_w, \Vdash_w)$. Put $\alpha_{\langle\rangle} = w$. Suppose α_σ is defined. We choose elements $\alpha_{\sigma * \langle B \to C \rangle}$ in W, for all elements $(B \to C) \in \{(D \to E) \in \mathcal{A} \mid K, \alpha_\sigma \nVdash D \to E\}$. The node $\alpha_{\sigma * \langle B \to C \rangle}$ is an element $v \in W$ such that $\alpha_\sigma \preccurlyeq v$, $K, v \Vdash B$ and $K, v \nVdash C$. Note that such elements can always be found.

Now define $W_w = \{\sigma \mid \sigma \text{ is defined }\}$, and define the partial order and the forcing relation on K as

$$\begin{array}{lll} \sigma \preccurlyeq_w \tau & \equiv_{def} & \alpha_\sigma \preccurlyeq \alpha_\tau. \\ \sigma \Vdash_w p & \equiv_{def} & \alpha_\sigma \Vdash p, \text{ for p} \in \bar{p}. \end{array}$$

Clearly, K' is finite, as $\mathcal{A}$ is finite too. It is also easy to infer that (7.3) is satisfied. This proves the claim, and thereby part (a) of the correspondence.

To show part (b) of the lemma, let $\mathcal{K}$ be a stable class of finite rooted Kripke models with the extension property. The model K we are going to construct will consist of equivalence classes of nodes of models in $\mathcal{K}$.

Replace every model in $\mathcal{K}$ by an isomorphic copy, in such a way that the domains of distinct models are disjoint. Let us define for nodes $k \in K$ and $k' \in K'$

$$k \cong k' \equiv_{def} (K)_k \text{ and } (K')_{k'} \text{ are isomorphic.}$$

(Remember that K_k is the submodel of K generated by k, see Section 6.3.1.) We write $k \Vdash A$ when A is valid at k in the unique model in $\mathcal{K}$ to which k belongs.

Now we define the domain of K as the set of all $\cong$-equivalence classes $[k]$ of nodes k of models in $\mathcal{K}$. The partial order and the forcing relation on K are defined via

$$\begin{array}{lll} [k] \preccurlyeq [k'] & \equiv_{def} & \exists l \in [k]\ \exists l' \in [k']\ (l, l' \text{ are nodes in the same model} \\ & & \text{and } l \preccurlyeq l' \text{ holds in this model.)} \\ [k] \Vdash p & \equiv_{def} & k \Vdash p. \end{array}$$

Since every two $\cong$-equivalent nodes force the same propositional variables the notion of forcing is well-defined. We have to see that K is in fact an AR-model and that

$$\text{for all } A\text{: } K \models A \text{ iff } \mathcal{K} \models A. \tag{7.4}$$

We show that K is an AR-model and leave the proof of (7.4) to the reader.

Consider nodes $[k_1], \ldots, [k_n]$ in K. Assume k_i is a node in the model $K_i \in \mathcal{K}$. Since $\mathcal{K}$ has the extension property there is (an isomorphic copy of) a variant of $(\sum(K_i)_{k_i})'$ in $\mathcal{K}$. Let b be the root of this variant. It is easy to see that $[b]$ is a tight predecessor of $[k_1], \ldots, [k_n]$ in K. This proves part (b) of the lemma. □

7.4.5. Corollary. The following are equivalent

(a) B is valid in every AR-model in which A is valid.

(b) B is valid in every stable class of finite rooted Kripke models with the extension property in which A is valid.

Now we are ready to give the

Proof of Theorem 7.4.1. First the direction from right to left. (De Jongh and Visser) We have to show that for all instances A/B of V and I, A admissibly derives B, and we have to see that the three rules of AR preserve admissibility. That is, when reading $\mid\!\sim$ for $\rhd$, if the assumptions of a rule are valid then so is the conclusion. For the two rules this is trivial. Therefore, it remains to treat the axioms. For instances A/B of I it clearly is the case that $A \mid\!\sim B$. Thus all we have to show is that for every instance A/B of the scheme V it holds that if A is derivable in IPC then so is B.

Therefore, consider such instance A/B of V. Let $X = \bigwedge_{i=1}^{n}(E_i \to F_i)$ and let $A = X \to C \vee D$ and $B = (X)(C, D, E_i, \ldots, E_n)$. Arguing by contradiction, suppose A is derivable but B is not. This implies that none of the formulas $(X \to C), (X \to D), (X \to E_1), \ldots, (X \to E_n)$ is derivable. Thus there are Kripke models $K_1, \ldots, K_{n+2}$ at which X is valid but at which respectively $C, D, E_1, \ldots, E_n$ are not valid. Consider the model $(\sum K_i)'$ and call its root b. Since A is derivable A is valid at b. Note furthermore that none of the formulas $C, D, E_1, \ldots, E_n$ can be valid at b. Therefore, the conjunction X cannot be valid at b. But it cannot be not valid either. For if so, there is some $i \leq n$ for which there is a node above b at which E_i is valid while F_i is not valid. As X is valid at all nodes except b the only possibility for this is the node b itself. Thus one of the formulas $E_1, \ldots, E_n$ would be valid at b, which cannot be.

The direction from left to right follows immediately from Proposition 7.3.1, Corollary 7.4.5 and Proposition 7.2.2. □

7.5 A basis for the admissible rules

Let R_{V_i} denote the rule corresponding to V_i (see Section 7.1), i.e. let

$$R_{V_i} \quad (\bigwedge_{i=1}^{n}(E_i \to F_i) \to B \vee C) \vee D/(\bigwedge_{i=1}^{n}(E_i \to F_i))(E_1, \ldots, E_n, B, C) \vee D.$$

Further, let

$$R_{V_i}^{-} \quad (\bigwedge_{i=1}^{n}(E_i \to F_i) \to B \vee C)/(\bigwedge_{i=1}^{n}(E_i \to F_i))(E_1, \ldots, E_n, B, C).$$

Let $\mathcal{V}$ be the set $\{R_{V_1}, R_{V_2}, \ldots\}$ and let $\mathcal{V}^-$ be the set $\{R^-_{V_1}, R^-_{V_2}, \ldots\}$. We need one more lemma to establish that the sets of rules $\mathcal{V}$ and $\mathcal{V}^-$ are respectively a basis and a subbasis for the admissible rules of IPC.

7.5.1. Lemma. If $\mathsf{AR} \vdash A \rhd B$ then the rule A/B is derivable in IPC from the set of rules $\mathcal{V}$.

Proof. We prove the proposition by induction on the length n of the derivation of $A \rhd B$ in AR. For $n = 0$ there is nothing to prove.

For $n > 0$, suppose the last rule applied in the derivation of $A \rhd B$ is the Conjunction rule. This implies that there are B_1, B_2 such that $B = B_1 \wedge B_2$, and such that $A \rhd B_1$ and $A \rhd B_2$ are derivable, and moreover have derivations of length smaller than n. By the induction hypothesis, A/B_1 and A/B_2 are derivable in IPC from $\{R_{V_1}, R_{V_2}, \ldots\}$. And thus $A/B_1 \wedge B_2$ is derivable in IPC from $\{R_{V_1}, R_{V_2}, \ldots\}$ as well. The case that the last rule applied in the derivation of $A \rhd B$ is the Cut Rule is completely similar. $\square$

7.5.2. Theorem. $\mathcal{V}$ is a basis for the admissible rules of IPC.

Proof. Immediate from Lemma 7.5.1 and Theorem 7.4.1. $\square$

7.5.3. Corollary. $\mathcal{V}^-$ is a subbasis for the admissible rules of IPC.

7.6 The connection with Heyting Arithmetic

In this section we explain what the results of this chapter mean for the provability and preservativity logic of HA.

Visser (1999) showed that the admissible rules of IPC are the same as the propositional admissible rules of HA. Therefore, Corollaries 7.5.2 and 7.5.3 give us

7.6.1. Corollary. $\mathcal{V}$ and $\mathcal{V}^-$ are respectively a basis and a subbasis for the propositional admissible rules of HA.

In Theorem 7.4.1 we saw that

> A/B is a propositional admissible rule of IPC iff $\mathsf{AR} \vdash A \rhd B$.

In combination with the result in (Visser 1999) that states that the propositional admissible rules of HA and IPC are the same, this gives

> A/B is a propositional admissible rule of HA iff $\mathsf{AR} \vdash A \rhd B$.

It is easy to see that the logic AR is equivalent to the logic axiomatized by the preservativity principles (Section 2.2) $P1$, $P2$, Dp and all the instances $A \rhd B$ of Vp, where A and B are propositional formulas, characterizes the admissible rules of IPC (use Lemma 7.1.1).

From the definition of preservativity it follows that if $A \rhd B$ is in the provability logic of HA, then A/B is an admissible rule of HA (Chapter 2). Finally, in combination with the fact that AR is part of the preservativity logic of HA (Visser 1994), this leads to

> for propositional formulas A, B:
>
> A/B is a propositional admissible rule of HA iff
>
> $A \rhd B$ is in the preservativity logic of HA.

This shows that HA recognizes its propositional admissible rules.

Chapter 8

A characterization of IPC

In this chapter we show that IPC is characterized by its admissible rules: In Chapter 7 we gave a countable basis $\mathcal{V}$ for the admissible rules of IPC. Here we show (Section 8.1) that the only intermediate logic with the Disjunction Property for which all rules in this basis are admissible, is IPC. In Section 8.2 we prove that the characterization is optimal. We show that for any finite subset X of $\mathcal{V}$ there is a proper intermediate logic for which X is admissible. In Section 8.3 we show that the characterization is effective.

8.1 The characterization

In Chapter 7 we gave a simple c.e. description of the admissible rules (Theorem 7.5.2) which implied (Corollary 7.5.3) that the set $\mathcal{V}$ is a subbasis for the admissible rules of IPC. Let us recall the definition of this subbasis (Section 7.5): $\mathcal{V}$ is the collection of rules

$$R_{V_n} \quad (\bigwedge_{i=1}^{n}(E_i \to F_i) \to B \vee C) \;/\; (\bigwedge_{i=1}^{n}(E_i \to F_i))(B, C, E_1, \ldots, E_n).$$

where we use the abbreviation,

$$(A)(B_1, \ldots, B_m) \equiv_{def} (A \to B_1) \vee \ldots \vee (A \to B_m).$$

The rest of this section is devoted to the proof that these admissible rules together with the Disjunction Property characterize IPC, i.e. we will show that for any intermediate logic which is not equal to IPC either the Disjunction Property does not hold or one of the rules $R_{V_1}, R_{V_2}, \ldots$ is not admissible. It is convenient to have the Disjunction Property built-in into the admissible rules. Therefore, we need the following definition.

Definition of the rules P_n

A theory T has the property P_n if for all substitutions σ,

$$\begin{aligned}&\text{if } \vdash_T \sigma(\textstyle\bigwedge_{i=1}^n(p_i \to q_i) \to r \vee s) \text{ then}\\ &\quad \vdash_T \sigma(\textstyle\bigwedge_{i=1}^n(p_i \to q_i) \to r) \text{ or } \vdash_T \sigma(\textstyle\bigwedge_{i=1}^n(p_i \to q_i) \to s) \text{ or}\\ &\quad \vdash_T \sigma(\textstyle\bigwedge_{i=1}^n(p_i \to q_i) \to p_1) \text{ or } \dots \text{ or } \vdash_T \sigma(\textstyle\bigwedge_{i=1}^n(p_i \to q_i) \to p_n).\end{aligned}$$

We will show that an intermediate logic is equal to IPC iff it has the property P_n, for all $n \geq 0$. The characterization mentioned above is an immediate corollary of this.

Note that a logic has P_0 if and only if it has the Disjunction Property. A logic has P_n for all $n \geq 0$ if and only if it has the Disjunction Property and for all $n \geq 1$ the rule V_n is admissible.

We need the following fact by Smoryński.

8.1.1. Fact. (Smoryński 1973) IPC is complete with respect to Jaskowski models.

8.1.2. Lemma. If an intermediate logic has the extension property it is the logic IPC.

Proof The lemma follows from the following two claims.

Claim If T is an intermediate logic with the extension property, then every basic Jaskowski model is a model of T.

Proof of the Claim Let T be an intermediate logic with the extension property (Subsection 6.3). Let K be a basic Jaskowski model (Section 6.3.1). We show that K_x is a model of T by induction to the depth of the node x. The maximal nodes of K clearly are models of T since every classical model is a model of T. Suppose x is another node in K and let $x_1, \dots, x_n$ be the immediate successors of x, i.e. the nodes y such that $x \prec y$ and such that there is no node $x \prec z \prec y$. By the induction hypothesis the models $K_{x_1}, \dots, K_{x_n}$ are models of T. Observe that K_x is the model $(\sum K_{x_i})'$ (Section 6.3.1). Because every propositional variable is valid at at most one node in K there is no other variant of $(\sum K_{x_i})'$ then the model itself. Since T has the extension property this implies that K_x is a model of T. This proves the Claim.

Claim If T is an intermediate logic such that every basic Jaskowski model is a model of T, then $T = \mathsf{IPC}$.

Proof of the Claim We show that $T \subseteq \mathsf{IPC}$ by proving that if $\nvdash_{\mathsf{IPC}} A$ holds, then $\nvdash_T A$ holds as well. If $\nvdash_{\mathsf{IPC}} A$ then there is a Jaskowski model K in which A is not valid (Fact 8.1.1). Let K' be a basic model based on the frame of K. By assumption K' is a model of T.

Now we define a substitution σ via $\sigma(p) = \bigvee_{K,x \Vdash p} A_x$, where the formulas A_x are given by Fact 6.3.1. To see that $\sigma(A)$ is not valid at K', observe that for every node x and for every formula B we have that $K, x \Vdash B$ iff $K', x \Vdash \sigma(B)$. Therefore, $\nvdash_T \sigma(A)$. Hence $\nvdash_T A$. $\square$

In the following lemma we need the notion of a *saturated set*. A T-saturated set x is a set of formulas such that $A \in x$ or $B \in x$ whenever $x \vdash_T A \vee B$. In particular, a T-saturated set is closed under deduction in T.

8.1.3. Lemma. If an intermediate logic has the property P_n for every $n \geq 0$, then it has the extension property.

Proof Let T be an intermediate logic with the Disjunction Property, for which, for all n, R_{V_n} is admissible. Consider models $K_1, \ldots, K_n$ of T with roots $x_1, \ldots, x_n$ respectively. From now on we confuse a node with the set of formulas it forces.

Claim There exists a T-saturated set $x \subseteq x_1 \cap \ldots \cap x_n$ such that for all T-saturated sets $x \subset y$ there is some $i \leq n$ such that $x_i \subseteq y$.

Proof of the Claim Consider

$$\Delta = \{(E \to F) \mid E \notin x_1 \cap \ldots \cap x_n \text{ and } F \in x_1 \cap \ldots \cap x_n\}.$$

Clearly, $\Delta \subseteq x_1 \cap \ldots \cap x_n$. Observe that the set $x_0 = \{A \mid \Delta \vdash_T A\}$ is T-saturated because for all m, the property P_m holds. Now we construct a sequence of sets $x_0 = z_0 \subseteq z_1, \ldots$ as follows. Let $C_0, C_1, \ldots$ enumerate all formulas, with infinite repetition. Define the property $*(\cdot)$ on sets via

$*(y)$ for all m, for all $A_1, \ldots, A_m$: if $y \vdash_T A_1 \vee \ldots \vee A_m$, then $A_i \in x_1 \cap \ldots \cap x_n$, for some $i = 1, \ldots, m$.

Note that $*(z_0)$ holds. If $*(z_i \cup \{C_i\})$ does not hold then put $z_{i+1} = z_i$. If $*(z_i \cup \{C_i\})$ holds do the following: if C_i is no disjunction, put $z_{i+1} = z_i \cup \{C_i\}$; if $C_i = D \vee E$, let z_{i+1} be $z_i \cup \{D\}$ if $*(z_i \cup \{D\})$ holds and $z_i \cup \{E\}$ otherwise. It is easy to see that at least one of $*(z_i \cup \{D\})$ and $*(z_i \cup \{E\})$ has to hold. Therefore, $*(z_i)$ holds for all i. Let $x = \bigcup_i z_i$. Clearly, x is T-saturated and $x \subseteq x_1 \cap \ldots \cap x_n$. Finally, we have to see that for all T-saturated sets $x \subset y$ there is some $i \leq n$ for which $x_i \subseteq y$. Arguing by contradiction assume $y \supset x$ and $x_i \nsubseteq y$ for all $i \leq n$. From the construction of x it is easy to see that $y \nsubseteq x_1 \cap \ldots \cap x_n$. Thus there are formulas $E \in y$, $E \notin x_1 \cap \ldots \cap x_n$ and $A_i \in x_i$, $A_i \notin y$, for all $i \leq n$. Hence $(E \to A_1 \vee \ldots \vee A_n) \in \Delta$. Thus $A_1 \vee \ldots \vee A_n \in y$, quod non. This proves the Claim.

Now we define a variant of $(\sum K_i)'$ by requiring ($b \Vdash p$ iff $p \in x$) at the root b of $(\sum K_i)'$, for propositional variables p.

Claim For all formulas B: $b \Vdash B$ iff $B \in x$.

Proof of the Claim We prove this by formula-induction. The case of the propositional variables and the connectives $\wedge$ and $\vee$ is trivial. Consider a formula $B = (C \to D)$. If $(C \to D) \in x$ then it is easy to see that indeed $b \Vdash (C \to D)$. We prove that $x \Vdash B$ implies $B \in x$ by contraposition. Therefore, assume

$(C \to D) \notin x$. It is not difficult to see that this implies the existence of a T-saturated set $y \supseteq x$ such that $C \in y$ and $D \notin y$. From the construction of x it follows that $x = y$ or $x_i \subseteq y$ for some $i = 1, \ldots, n$. In the first case the induction hypothesis gives $b \Vdash C$ and $b \nVdash D$, thus $b \nVdash (C \to D)$. In the other case it follows that for some i, $x_i \nVdash (C \to D)$. Thus again we can conclude that $b \nVdash (C \to D)$. This proves the claim.

By the last claim the defined extension is a model of T. This proves that T has the extension property. □

These two lemmas lead to the following characterization of IPC:

8.1.4. Theorem. For any intermediate logic T it holds that $T = \mathsf{IPC}$ iff T has the property P_n for every $n \geq 0$.

8.1.5. Corollary. For any intermediate logic T it holds that $T = \mathsf{IPC}$ iff T has the Disjunction Property and all the rules R_{V_n} are admissible. Thus IPC is maximal with respect to $\mathcal{V}$ and hence maximal.

8.2 Optimality of the characterization

In the previous section we saw that the properties $P_1, P_2, \ldots$ characterize IPC. In this section we show that no finite subset of $P_1, P_2, \ldots$ characterizes IPC. This proves that our characterization is optimal. Note that it is not interesting to consider infinite subsets of $P_0, P_1, P_1, \ldots$, since having the property P_{m+1} implies having the property P_m.

We use logics $\mathsf{D_n}$ ($n \geq 1$) given by Gabbay and de Jongh (1974). The logic $\mathsf{D_n}$ is axiomatized by

$$D_n \quad \bigwedge_{i=0}^{n+1}((A_i \to \bigvee_{j \neq i} A_j) \to \bigvee_{j \neq i} A_j) \to \bigvee_{i=0}^{n+1} A_i.$$

We need the following theorem.

8.2.1. Theorem. (Gabbay and de Jongh 1974) The intermediate logic $\mathsf{D_n}$ is a proper extension of IPC with the Disjunction Property. $\mathsf{D_n}$ is complete with respect to the class of finite trees in which every point has at most $(n+1)$ immediate successors.

Knowing this, it is easy to prove the following lemma.

8.2.2. Lemma. The logic $\mathsf{D_n}$ has the property P_{n+1} and it does not have the property P_{n+2}.

Proof To see that $\mathsf{D_n}$ has the property P_{n+1}, suppose $\mathsf{D_n}$ derives the formula $(A \to D \vee E)$, where $A = \bigwedge_{i=1}^{n+1}(B_i \to C_i)$. Suppose also that $\mathsf{D_n}$ does not derive $(A)(B_1, \ldots, B_{n+1}, D, E)$. By the Disjunction Property and the completeness of $\mathsf{D_n}$ this implies that there are models K_i, such that $K_i \models A$ and, for $i \leq n+1$, $K_i \not\models B_i$ and $K_{n+2} \not\models D$ and $K_{n+3} \not\models E$. Furthermore, the frame of every K_i is a finite tree in which every node does not have more than $(n+1)$ immediate successors. Consider $(((\sum_{i=1}^{n+1} K_i)' + K_{n+2})' + K_{n+3})'$. Clearly, the frame of this model is again a finite tree in which every node does not have more than $(n+1)$ immediate successors. In this model A is valid while $(D \vee E)$ is not, contradicting the assumption that $\mathsf{D_n}$ derives $(A \to D \vee E)$.

To see that $\mathsf{D_n}$ does not have the property P_{n+2}, consider the axiomatization of $\mathsf{D_n}$. It is easy to see, using the completeness of $\mathsf{D_n}$, that $\mathsf{D_n}$ does not derive

$$(\bigwedge_{i=0}^{n+1}((A_i \to \bigvee_{j \neq i} A_j) \to \bigvee_{j \neq i} A_j))((A_0 \to \bigvee_{j \neq 0} A_j), \ldots, (A_{n+1} \to \bigvee_{j \neq n+1} A_j)).$$

This completes the proof of the lemma. □

8.2.3. Corollary. No finite subset of $P_1, P_2, \ldots$ characterizes IPC.

In fact, $\mathsf{D_n}$ is characterized by P_{n+1} in the same way as IPC is characterized by all the $P_0, P_1, \ldots$, see Corollary 8.2.7. The proof of this proposition is analogous to the one of Theorem 8.1.4: the next lemma is the analogue of Lemma 8.1.2 and the following one is the analogue of Lemma 8.1.3.

8.2.4. Lemma. If an intermediate logic has the extension property up to $(n+1)$, then it is contained in $\mathsf{D_n}$.

Proof Let T be an intermediate logic that has the extension property up to $(n+1)$. Suppose $\mathsf{D_n} \nvdash A$. It easily follows from Theorem 8.2.1 that $\mathsf{D_n}$ is complete with respect to the class of the finite trees in which every point has at most $(n+1)$ immediate successors, and in which no two nodes have exactly the same maximal nodes above them. To be precise, the last property reads:

$$\forall x \forall y \exists z (x \neq y \to \neg \exists z' (z \prec z') \wedge ((x \preccurlyeq z \wedge y \not\preccurlyeq z) \vee (y \preccurlyeq z \wedge x \not\preccurlyeq z))).$$

Let M be a model based on such a frame F in which A is not valid. Let M' be a basic model on F (see Section 6.3.1). By the same reasoning as before it follows that M' is a model of T. Define the substitution σ via $\sigma(p) = \bigvee_{M,x \Vdash p} A_x$, where the formulas A_x are given by Fact 6.3.1. Clearly,

$$M, x \Vdash B \text{ iff } M', x \Vdash \sigma(B).$$

Thus $\nvdash_T \sigma(A)$. Hence $\nvdash_T A$. This shows that the logic T is contained in the logic $\mathsf{D_n}$. □

8.2.5. Lemma. If an intermediate logic has the property P_n it has the extension property up to n.

Proof Let T be an intermediate logic that has the property P_n. The proof that T has the extension property up to n is completely similar to the proof of Lemma 8.1.3, except for one point, which we will explain. The rest of the proof we leave to the reader.

In the first Claim of Lemma 8.1.3 we define a set Δ and observe that, in the notation of this lemma, the set $x_0 = \{A \mid \Delta \vdash_T A\}$ is T-saturated because for all m, P_m holds. In this case, having only P_n, this is the only place in the proof where we have to be careful. Assume $x_0 \vdash_T A \vee B$. Hence there are $E_1, \ldots, E_m \notin x_1 \cap \ldots \cap x_n$ and $F_1, \ldots, F_m \in x_1 \cap \ldots \cap x_n$ such that

$$\vdash_T \bigwedge_{i=1}^{m}(E_i \to F_i) \to A \vee B.$$

For $i \leq n$, let $G_i = \bigvee\{E_j \mid j \leq m, E_j \notin x_i\}$ and let $F = \bigwedge_{i=1}^{m} F_i$. Observe that $G_i \notin x_i$ and that $(G_i \to F) \in \Delta$. Clearly,

$$\vdash_T \bigwedge_{i=1}^{n}(G_i \to F) \to A \vee B.$$

And thus, since T has P_n, we can conclude

$$\vdash_T (\bigwedge_{i=1}^{n}(G_i \to F))(G_1, \ldots, G_n, A, B).$$

Since $\bigwedge_{i=1}^{n}(G_i \to F) \in x_1 \cap \ldots \cap x_n$ while $G_i \notin x_1 \cap \ldots \cap x_n$, we have either

$$\vdash_T \bigwedge_{i=1}^{n}(G_i \to F) \to A \text{ or } \vdash_T \bigwedge_{i=1}^{n}(G_i \to F) \to B.$$

And because $x_0 \vdash_T \bigwedge_{i=1}^{n}(G_i \to F)$ either $x_0 \vdash_T A$ or $x_0 \vdash_T B$. And this proves that x_0 is T-saturated. □

8.2.6. Proposition. Any intermediate logic T which has P_{n+1} is contained in $\mathsf{D_n}$.

8.2.7. Corollary. For any intermediate logic $T \supseteq \mathsf{D_n}$ it holds that $T = \mathsf{D_n}$ iff T has P_{n+1}. Thus $\mathsf{D_n}$ is maximal with respect to $R_{V_{n+1}}$ and hence maximal.

Since the union of the $\mathsf{D_n}$ is equivalent to IPC, Theorem 8.1.4 follows from the previous proposition. However, we preferred to give a separate proof of the theorem in advance.

8.3 Effectiveness

In (de Jongh 1970) the following characterization of IPC in terms of the Kleene slash $|$ (Kleene 1962) is given: IPC is the only intermediate logic T satisfying

$$\text{if } A \,|_T\, A \text{ and } \vdash_T (A \to B \vee C), \text{ then } \vdash_T (A \to B) \text{ or } \vdash_T (A \to C).$$

We remind the reader that the Kleene slash is defined as follows. (We use the abbreviation $\Gamma \Vdash_T A \equiv_{def} (\Gamma|_T A$ and $\Gamma \vdash_T A)$.)

$$\begin{array}{lll} \Gamma \,|_T\, p & \equiv_{def} & \Gamma \vdash_T p \text{ for } p \text{ a propositional variable or } \bot \\ \Gamma \,|_T\, A \wedge B & \equiv_{def} & \Gamma \,|_T\, A \text{ and } \Gamma \,|_T\, B \\ \Gamma \,|_T\, A \vee B & \equiv_{def} & \Gamma \Vdash_T A \text{ or } \Gamma \Vdash_T B \\ \Gamma \,|_T\, A \to B & \equiv_{def} & \Gamma \Vdash_T A \text{ implies } \Gamma \,|_T\, B. \end{array}$$

De Jongh (1970) also proved that the characterization in terms of the Kleene slash is an effective one: given any intermediate logic $T \neq \mathsf{IPC}$ we can obtain formulae A, B, C such that $A \,|_T\, A$, $\vdash_T (A \to B \vee C)$ but $\nvdash_T (A \to B)$, $\nvdash_T (A \to C)$ in an effective way. We show that the characterization in terms of the admissibles rules treated in this chapter, is effective as well, by giving an effective reduction from the characterization in terms of the Kleene slash to the one in terms of the admissible rules.

Let us call a triple of formulas A, B, C a *J-example* or an *I-example* of $T \neq \mathsf{IPC}$ if respectively

$$A \,|_T\, A,\ \vdash_T (A \to B \vee C),\ \nvdash_T (A \to B),\ \nvdash_T (A \to C),$$

or for $A = \bigwedge(D_i \to E_i)$,

$$\vdash_T (A \to B \vee C),\ \nvdash_T (A \to B),\ \nvdash_T (A \to C),\ \nvdash_T (A \to D_i).$$

The following proposition shows that there exist effective reductions from one characterization to the other.

8.3.1. Proposition. For any intermediate logic $T \neq \mathsf{IPC}$ there is an effective way of creating an I-example from a J-example, and vice versa.

Proof During the proof $\vdash$, $|$ stand for $\vdash_T$, $|_T$ respectively. The second part of the proposition is easy: any I-example $A = \bigwedge(D_i \to E_i), B, C$ of $T \neq \mathsf{IPC}$ is a J-example because $\nvdash (A \to D_i)$ for all i, implies $A \mid A$.

For the other part, suppose A, F, G is an I-example of $T \neq \mathsf{IPC}$. We are going to construct, in an inductive way, formulas $A_1, A_2, \ldots$ which are all equivalent to A in T. Every A_i is a conjunction of propositional variables, disjunctions and implications such that for the implications $(B \to C)$ either $A_i \mid (B \to C)$ or

$A_i \nvdash B$, and for the disjunctions B, $A_i \mid B$. Note that A is such a formula. Let $A_1 = A$. During the construction we will often use, without mentioning, the fact that if $E \mid F$ and $\vdash E \leftrightarrow E'$ then $E' \mid F$.

If A_i is a conjunction in which one the conjuncts is a disjunction (note that this captures the case that A_i is a disjunction), let $(B \vee C)$ be the first such reading from left to right. Thus $A_i = D \wedge (B \vee C) \wedge E$ for some D, E. By assumption $A_i \mid (B \vee C)$. Hence $A_i \Vdash B$ or $A_i \Vdash C$. In the first case put $A_{i+1} = D \wedge B \wedge E$, in the second case $A_{i+1} = D \wedge C \wedge E$. Now consider the case that A_i is a conjunction of implications and propositional variables. If every conjunct either is a propositional variable or an implication $(B \to C)$ such that $A_i \nvdash B$, put $A_{i+1} = A_i$. If not, let $(B \to C)$ be the first implication, reading from left to right, such that $A_i \vdash B$. Thus $A_i = D \wedge (B \to C) \wedge E$ for some D, E. By assumption $A_i \mid (B \to C)$. We inductively define A_{i+1}.

$\star$ If $B = p$, put $A_{i+1} = D \wedge C \wedge E$. Note that $A_{i+1} \mid C$ since $A_i \mid C$ which again follows from $A_i \mid (B \to C)$ and $A_i \Vdash B$.

$\star$ If $B = B_1 \wedge B_2$ observe that $A_i \vdash B$ implies $\vdash A_i \leftrightarrow D \wedge (B_j \to C) \wedge E \leftrightarrow D \wedge C \wedge E$. Hence $D \wedge (B_j \to C) \wedge E \vdash B_j$. If for some $j = 1, 2$, $D \wedge (B_j \to C) \wedge E \mid (B_j \to C)$, let $A_{i+1} = D \wedge (B_j \to C) \wedge E$. It cannot be that for no j, $D \wedge (B_j \to C) \wedge E \mid (B_j \to C)$. For if so, then $D \wedge (B_j \to C) \wedge E \Vdash B_j$. Hence $D \wedge (B \to C) \wedge E \Vdash B$, and so $D \wedge (B \to C) \wedge E \mid C$. Whence $D \wedge C \wedge E \mid C$ and thus $D \wedge (B_j \to C) \wedge E \mid (B_j \to C)$, a contradiction.

$\star$ If $B = B_1 \vee B_2$ observe that $\vdash A_i \leftrightarrow D \wedge (B_1 \to C) \wedge (B_2 \to C) \wedge E$ and that $A_i \mid (B_j \to C)$. Put $A_{i+1} = D \wedge (B_1 \to C) \wedge (B_2 \to C) \wedge E$.

$\star$ Finally $B = (B_1 \to B_2)$. If $A_i \nVdash B_1$ or $A_i \mid B_2$ then $A_i \Vdash B$ and therefore $A_i \mid C$. Put $A_{i+1} = D \wedge C \wedge E$. If $A_i \Vdash B_1$ and not $A_i \mid B_2$ then $\vdash A_i \leftrightarrow D \wedge B_1 \wedge (B_2 \to C) \wedge E$ and clearly $A_i \mid B_1$ and $A_i \mid (B_2 \to C)$. Put $A_{i+1} = D \wedge B_1 \wedge (B_2 \to C) \wedge E$. This ends the construction of the A_i.

It is easy to check that the A_i have the desired properties. Moreover, the construction shows that eventually $A_i = A_{i+1}$. Hence A_i is a conjunction of propositional variables and implications $\bigwedge_{i=1}^n p_i \wedge \bigwedge_{i=1}^m (B_i \to C_i)$ such that $A_i \nvdash B_i$. Let $A' = \bigwedge_{i=1}^m (B_i \to C_i)$ and let σ be the substitution which is the identity on all variables except $p_1, \ldots, p_n$, on which it is $\top$. Hence $\sigma(A_i)$ is equivalent to $\sigma(A')$. Since A_i is equivalent with A in T,

$$\vdash (A_i \to F \vee G),\ \nvdash (A_i \to F),\ \nvdash (A_i \to G).$$

Clearly, we have

$$\vdash (\sigma(A') \to \sigma(F) \vee \sigma(G)),$$

In general, nonderivability is not preserved under substitution but this particular choice of σ leads to

$$\nvdash (\sigma(A') \to \sigma(F)),\ \nvdash (\sigma(A') \to \sigma(G)),\ \nvdash (\sigma(A') \to \sigma(B_i)).$$

Hence $\sigma(A'), \sigma(F), \sigma(G)$ is an I-example of $T \neq \mathsf{IPC}$. $\square$

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Index

List of symbols

Samenvatting

Dit proefschrift bestaat uit twee delen. In het eerste deel wordt de intuïtionistische bewijsbaarheids logica bestudeerd en in het tweede deel de intuïtionistische propositie logica. Hieronder schetsen we heel kort, in een notendop, waarover deze gebieden gaan, waarbij we technische termen zullen vermijden. Dit is bedoeld voor niet-wiskundigen die een indruk willen krijgen van de betekenis van bovengenoemde termen. In de twee daarop volgende secties bespreken we, ook in het kort maar op meer technische wijze, de inhoud van dit proefschrift.

8.4 Een notendop

Intuïtionistische bewijsbaarheidslogica en intuïtionistische propositielogica zijn beide gebaseerd op intuïtionistische logica. Intuïtionistische logica is een tegenhanger van klassieke logica. Klassieke logica gaat over logische waarheden. Een uitdrukking $\exists x A(x)$ betekent: er is een x zodat $A(x)$ geldt (hierbij is $A(x)$ een bewering over x, bijvoorbeeld 'x is een even getal dat niet de som is van twee priemgetallen').[1] Nu is er, in de wiskunde, een nauw verband tussen waarheden en constructies; de waarheid van een bewering wordt aangetoond via een bewijs van die bewering, en een bewijs is een constructie. Het intuïtionisme sluit aan bij deze verwantschap. Hier betekent $\exists x A(x)$: we kunnen een object x construeren zodat $A(x)$ geldt. Dus in intuïtionistische logica zeg je dat $\exists x A(x)$ geldt als je daadwerkelijk een even getal hebt geconstrueerd dat niet de som is van twee priemgetallen[2], terwijl je om in klassieke logica te weten dat $\exists x A(x)$ geldt alleen maar hoeft uit te sluiten dat álle even getallen de som van twee priemgetallen zijn. Het is een subtiel verschil, maar het is een verschil.

De overeenkomsten en verschillen tussen klassieke en intuïtionistische waarheden vertellen je veel over de manier waarop de geldigheid van een bewering wordt ingezien. Sommige wiskundigen beschouwen alleen die uitspraken als waar die waar

[1] Een getal is even als het deelbaar is door 2; een priemgetal is alleen deelbaar door 1 en zichzelf, bijvoorbeeld 3, 5, 7 en 11 zijn priem.

[2] Het is een bekend open probleem in de wiskunde of zulke getallen bestaan of niet. Men vermoedt van niet, dit vermoeden heet de Goldbach Conjecture.

zijn volgens de intuïtionistische wijze van redeneren. Zo iemand was bijvoorbeeld de nederlandse wiskundige L.E.J. Brouwer, de grondlegger van het zogenaamde intuïtionisme. Voor anderen is het werken met deze vorm van redeneren een manier om het idee erachter, namelijk het constructieve karakter van waarheden beter te begrijpen.

De twee genoemde gebieden, intuïtionistische propositielogica en intuïtionistische bewijsbaarheidslogica, zijn respectievelijk propositielogica en bewijsbaarheids logica gebaseerd op de intuïtionistische wijze van redeneren. Nu we een idee hebben van die intuïtionistische basis bespreken we in het kort de gebieden zelf. Propositielogica bestudeert de meest simpele manier van redeneren. Het beperkt zich tot eenvoudige uitdrukkingen, zoals $(p \to q)$ (uit p volgt q), of $(p \wedge q)$ (p en q), waarbij p en q voor beweringen staan. In intuïtionistische propositielogica worden de eigenschappen van dit soort beweringen bestudeerd. Hoewel dit systeem heel simpel is zijn er toch allerlei interessante vragen over te stellen. Het beantwoorden van die vragen is een van de manieren om inzicht te verkrijgen in de intuïtionistische, constructieve wijze van redeneren.

Bewijsbaarheidslogica gaat over ingewikkelder uitdrukkingen en complexere systemen. In de wiskunde heb je formele systemen die bepaalde wiskundige structuren beschrijven. Zo'n formeel systeem kan bijvoorbeeld over de natuurlijke getallen $0, 1, 2, 3, \ldots$ gaan. De bekende wiskundige Gödel liet echter in 1931 zien dat die systemen ook over zichzelf kunnen praten (zoals een schilderij zichzelf tot onderwerp kan hebben). Het kan beweringen als 'dit systeem bewijst dit-en-dat' bewijzen, en daarmee bewijst het iets over zichzelf. Het interessante is echter dat ze sommige eigenschappen van zichzelf wel kunnen zien (kunnen bewijzen) en andere niet. In de bewijsbaarheidslogica wordt bestudeerd wat zo'n systeem wel en niet van zichzelf kan begrijpen. In intuïtionistische bewijsbaarheidslogica zijn de systemen waarvoor deze vraag wordt bekeken gebaseerd op intuïtionistische logica.

Sommige begrippen uit de wiskunde zijn pas goed te begrijpen wanneer zij precies zijn gedefiniëerd. Dit komt doordat een informele uitleg de subtiliteit van een notie vaak verdoezelt. Dit geldt zeker voor bewijsbaarheidslogica, en eigenlijk ook voor logica in het algemeen. Desalniettemin geeft het bovenstaande wellicht een indruk van de gebieden waarover dit proefschrift gaat.

8.5 Eerste deel

Het eerste deel van het proefschrift is gewijd aan de bewijsbaarheidslogica van Heyting Rekenkunde, HA. Er is nog geen axiomatisering bekend van deze logica. De belangrijkste bijdragen van het proefschrift aan dit gebied zijn de volgende.

Zoals gezegd, bestaat de bewijsbaarheidslogica van een theorie uit de schema's die de theorie over zijn bewijsbaarheidspredicaat kan bewijzen. Wanneer een principe in de logica zit is het dus zeker waar voor de theorie (mits de theorie gezond is).

Dus geven zulke principes eigenschappen van de theorie weer die uitdrukbaar zijn in de bewijsbaarheidslogica. Nu hebben constructieve theorieën vaak twee speciale eigenschappen die uitdrukbaar zijn in bewijsbaarheidslogica, namelijk de Disjunctie Eigenschap en toelaatbare regels. Heyting Rekenkunde heeft die eigenschappen ook. Wil men de bewijsbaarheidslogica van deze theorie bepalen dan is het dus zeker nodig om vast te stellen of ze die eigenschappen van zichzelf bewijst of niet, met andere woorden of de principes die met deze eigenschappen corresponderen tot de logica behoren of niet.

Voor de Disjunctie Eigenschap is het antwoord op deze vraag sinds de zeventiger jaren bekend; Friedman (1975) bewees dat HA zijn eigen Disjunctie Eigenschap niet kan bewijzen. Bovendien bewees Leivant (1975) dat een zwakkere versie van die eigenschap, Leivant's Principe, wel een principe van de bewijsbaarheidslogica is. Het antwoord met betrekking tot de toelaatbare regels wordt door resultaten in Visser (1999) en Hoofdstuk 7 van het proefschrift gegeven. In Hoofdstuk 2 van het proefschrift wordt uitgelegd dat genoemde resultaten impliceren dat HA van al zijn toelaatbare regels *bewijst* dat ze toelaatbaar zijn. Dit in tegenstelling tot de hierboven beschreven situatie voor de Disjunctie Eigenschap, waarbij dit niet het geval is.

Hiermee corresponderen de principes van de bewijsbaarheidslogica van HA die nu bekend zijn met drie soorten eigenschappen van HA, namelijk

1. de karakteristieke eigenschappen van het bewijsbaarheidspredicaat van Peano Rekenkunde,

2. een verzwakte vorm van de Disjunctie Eigenschap,

3. de (propositionele) toelaatbare regels.

Zoals gezegd zijn de Disjunctie Eigenschap en het bestaan van niet afleidbare toelaatbare regels precies twee van de karakteristieke eigenschappen van constructieve theorieën die uitdrukbaar zijn in de taal van de bewijsbaarheidslogica. Het feit dat van de principes die corresponderen met deze eigenschappen nu duidelijk is of ze wel of niet tot de logica behoren geeft aan dat, zo niet alles, dan toch een welomschreven deel van die logica in kaart is gebracht. Bovendien heeft de logica, voorzover die nu bekend is, een transparante vorm. Punt 1. en 2. bestaan samen uit een paar eenvoudige principes, maar voor punt 3. is dat niet op voorhand duidelijk. In de afwezigheid van een beschrijving/axiomatisering van de toelaatbare regels geeft 3. wel inzicht in de logica, maar het leidt niet tot bruikbare axioma's voor de bewijsbaarheidslogica. Het zegt alleen dat alle principes die corresponderen met een toelaatbare regel van HA tot de bewijsbaarheidslogica behoren, maar die collectie zou heel wild kunnen zijn. In het tweede deel van het proefschrift wordt echter een karakterisering van de toelaatbare regels gegeven, waaruit volgt dat er een mooie axiomatisering van genoemde collectie is. Dit toont aan dat (het deel van) de bewijsbaarheidslogica die we nu hebben een elegante vorm heeft.

Omdat de logica die we nu hebben misschien de hele bewijsbaarheidslogica van HA is leek het zinvol om de principes modaal te karakteriseren. Dit beslaat de rest van het eerste deel van het proefschrift. Hier werken we voor het merendeel in een extentie van bewijsbaarheidslogica van de hand van Visser (1994), preservatielogica. Een van de redenen hiervoor is dat de meeste principes een elegantere formulering hebben in deze logica. In plaats van de notie van bewijsbaarheid staat daar het begrip preservatie centraal. Een formule A preserveert een formule B, notatie $A \rhd B$, als B volgt uit alle Σ_1-zinnen waaruit A volgt. In het bijzonder is B bewijsbaar precies dan als $\top$ (waar) de formule B preserveert. Dit laat zien dat preservatielogica inderdaad een extentie van bewijsbaarheidslogica is. Voor uitbreidingen van Peano Rekenkunde is de preservatielogica gelijk aan de interpreteerbaarheidslogica. Daarom kun je preservatielogica beschouwen als een constructieve variant van interpreteerbaarheidslogica. In Hoofdstuk 5 van het proefschrift wordt bewezen dat de logica bestaande uit alle nu bekende principes volledig is met betrekking tot een zekere klasse van frames. We tonen aan dat die frames noodzakelijkerwijs oneindig zijn. Verder worden de principes ook apart gekarakteriseerd (Hoofdstuk 4), en wordt daaruit afgeleidt dat ze onafhankelijk zijn. Doordat sommige van deze principes frame-eigenschappen hebben die ongebruikelijk zijn in modale logica, kunnen deze hoofdstukken ook gezien worden als een studie in intuïtionistische modale logica. Hoofdstuk 3 van het eerst deel is een introductie in de preservatielogica.

8.6 Tweede deel

Het tweede deel van het proefschrift gaat over intuïtionistische propositielogica IPC, met name over haar toelaatbare regels. In Hoofdstuk 7 wordt een hypothese van Dick de Jongh en Albert Visser uit de tachtiger jaren bewezen, namelijk dat een bepaalde verzameling $\mathcal{V}$ van regels een basis vormt voor de toelaatbare regels van IPC.

In Hoofdstuk 8 wordt een verband gelegd tussen toelaatbare regels en intermediaire logica's. Intuïtionistische propositielogica heeft twee echt constructieve eigenschappen, namelijk de Disjunctie Eigenschap en een bepaalde collectie toelaatbare regels. Men kan zich afvragen in hoeverre deze eigenschappen de logica karakteriseren, dat wil zeggen of er echte intermediaire logica's met dezelfde eigenschappen bestaan. Het is reeds lang bekend dat IPC niet gekarakteriseerd wordt door alleen de Disjunctie Eigenschap. Evenmin wordt IPC gekarakteriseerd door haar toelaatbare regels. In Hoofdstuk 8 wordt echter bewezen dat ze wel door de combinatie van die twee eigenschappen gekarakteriseerd wordt: IPC is de enige intermediaire logica met de Disjunctie Eigenschap waarvoor alle regels in $\mathcal{V}$ toelaatbaar zijn. Verder wordt in genoemd hoofdstuk bewezen dat elke deelcollectie van $\mathcal{V}$ op dezelfde wijze met een zekere intermediaire logica correspondeert als $\mathcal{V}$ met IPC; de intermediaire logica's uit (Gabbay and de Jongh 1974) worden elk gekarakteriseerd door een deel van de verzameling $\mathcal{V}$ plus de Disjunctie Eigenschap.

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