Skolemization in intermediate logics with the finite model property

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Abstract

An alternative Skolemization method, which removes strong quantifiers from formulas, is presented that is sound and complete with respect to intermediate predicate logics with the finite model property. For logics without constant domains the method makes use of an existence predicate, while for logics with constant domains no additional predicate is necessary. In both cases an analogue of Hebrand's theorem is obtained and it is proved that the one-variable fragment of a logic with the finite model property is decidable once the propositional fragment of the logic is. It is also shown that universal constant domain logics with the finite model property have interpolation once their propositional fragment has. For logics without constant domains some of these results, but with far more complicated proofs, have been obtained in (Iemhoff, 2010).

Keywords: Skolemization, Herbrand's theorem, interpolation, intermediate logic
MSC: 03B10, 03B55, 03F03

1 Introduction

It is a remarkable fact that the Skolemization method, so successful in classical logic, does not apply to several well-known intermediate logics, including intuitionistic predicate logic IQC, in that it fails to be sound and complete for these logics. This failure is not a consequence of the lack of prenex normal forms outside the realm of classical logic, as one can extend the Skolemization method to infix formulas in a natural way. But even for this extended method there exist formulas, such as $\forall x \neg\neg P(x) \to \neg\neg\forall x P(x)$, for $P(x)$ being a predicate, that are underivable in many intermediate logics while their Skolemization, in this case $\forall x \neg\neg P(x) \to \neg\neg P(c)$, is. For some intermediate logics, however, there exist alternative methods to remove strong quantifiers from formulas. In (Baaz and Iemhoff, 2006b), for example, it is shown that using the existence predicate, a

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Skolemization method can be defined for existential quantifiers in intuitionistic logic. This result is in (Baaz and Iemhoff, 2008) extended to universal quantifiers, an extension not fully satisfying because it requires the presence of certain predicates less natural than the existence predicate.

In this paper we develop a Skolemization method for intermediate logics with the finite model property, with and without constant domains. For logics with constant domains this method, *parallel skolemization*, produces for any formula φ a formula φ^{ps} without strong quantifiers such that the following holds.

$$\vdash_{\mathsf{L}} \varphi \Leftrightarrow \vdash_{\mathsf{L}} \varphi^{\mathrm{ps}}.$$

Note that we consider Skolemization here with respect to derivability rather than satisfiability, as is usual in a nonclassical setting. In case the logic does not have constant domains we use an approach similar to the one in (Baaz and Iemhoff, 2006b), by considering IQCE (IQC with an existence predicate) instead of IQC. A sound and complete Skolemization method, called epskolemization, for logics in IQCE with the finite model property is defined that implies a Skolemization method for intermediate logics as well. Here the *finite* refers to the number of nodes in a model, not to the domains at the nodes, which may well be infinite.

In (Iemhoff, 2010) it is shown that the *eskolemization method* from (Baaz and Iemhoff, 2006b) is sound and complete for logics with the finite model property, thus implying the last result mentioned in the previous paragraph. However, the proof presented here is much simpler than the one in (Iemhoff, 2010), and logics with constant domains are not explicitly treated in that paper (although we think that similar results can be obtained from it as well). For these reasons we think it worthwhile to present the alternative skolemization method here.

Skolemization is often considered in combination with Herbrand's theorem, as it is this combination that provides important applications in logic and computer science. Here we provide, for the logics with (e)pskolemization, Herbrand theorems that are the usual extension to infix formulas of the standard Hebrand theorem. Finally, an application of the developed methods to interpolation is presented. It is shown that for all intermediate logics with the finite model property and constant domains that can be axiomatized by universal formulas, if their propositional fragment has interpolation, then so does the predicate logic.

Skolemization and Herbrand theorems have been studied for other nonclassical theories and logics as well. For references to these topics in the setting of substructural logics, see (Baaz and Metcalfe, 2008, 2009; Cintula and Metcalfe, 2013). Recently there has appeared a paper that uses the method introduced in this paper, but then in the setting of substructural logics (Cintula et al., 2015). It is shown that under certain semantical conditions, resembling those in (Iemhoff, 2010), first–order substrutural logics admit parallel skolemization. Other related work on Skolemization concerns the complexity of the method and the construction of deskolemization methods, see (Baaz and Leitsch, 1994; Baaz et al., 2012) for details.

This paper is structured as follows. Section 2 contains the preliminaries, in particular the definition of Kripke models for predicate logic. Section 4 introduces a semantical property that is one of the two main ingredients in the proof, in Section 5, that the skolemization method defined in Section 3 is sound and complete. The other ingredient is model extensions, which are introduced in Section 5.1. Section 6 is about Herbrand's theorem and Section 7 contains the application to interpolation discussed above. In Section 8 the methods developed in the previous sections are extended to logics without constant domains. Section 9 contains the conclusion.

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2 Preliminaries

The theories we consider are theories in intuitionistic predicate logic IQC or, in Section 8, in its extension IQCE. The former are called *intermediate theories.* We mostly use intermediate logics rather than theories as examples (the difference being that the latter do not have to be closed under substitution), but as all our results apply to theories as well logics, we present them in the most general form throughout the paper.

Except in the last section, our language $\mathcal{L}$ consists of the usual connectives, variables, constants, quantifiers, and predicate and function symbols, infinitely many of every arity. Terms are defined as usual. An occurrence of a quantifier in a formula is *strong* if it is a positive occurrence of a universal or a negative occurrence of an existential quantifier. It is *weak* otherwise.

Given a theory L we write $\vdash_{\mathsf{L}}$ to denote derivability in L, and also $\vdash$ when it is clear from the context which theory is meant.

2.1 Universal theories

Universal formulas are formulas in prenex normal form that only contain universal quantifiers. A theory is *universal* when it is axiomatizable over IQC by universal formulas. Note that theories defined by universal formulas as axiom schemes do not fall under this definition, as the schemes may be instantiated by formulas that are not universal. It is required that every single axiom of the theory is itself universal. For example, the intermediate theory axiomatized by $(\neg P(\bar{x}) \vee \neg\neg P(\bar{x}))$, where P is a predicate, is universal, but the predicate version of the propositional logic KC axiomatized by $(\neg\varphi(\bar{x}) \vee \neg\neg\varphi(\bar{x}))$, where φ may be any formula, is not.

2.2 Kripke models

Kripke models are defined as usual, except that we require them to have constant domains and their frames to be trees. Following (Troelstra and van Dalen, 1988) we always assume that the elements of the domain of a (Kripke) model are constants in $\mathcal{L}$. In this way one does not have to use valuations but can define truth $\models$ in classical models and forcing $\Vdash$ in Kripke models inductively on sentences in $\mathcal{L}$.

A *Kripke model with constant domains* or *constant domain Kripke model K* is a tuple $(W, \preccurlyeq, D, I)$, where $(W, \preccurlyeq)$ is a rooted tree, D is a nonempty set (the *domain*) and I is a collection $\{I_k \mid k \in W\}$ of interpretations such that for all $k \in$

W, (D, I_k) is a classical model, such that the following persistency requirements are satisfied. For all terms t, all n-ary predicates P and all $\bar{d} = d_1, \ldots, d_n \in D$:

$$\begin{array}{rcl} k \preccurlyeq l & \Rightarrow & I_k(t) = I_l(t) \\ (D, I_k) \models P(\bar{d}) \text{ and } k \preccurlyeq l & \Rightarrow & (D, I_l) \models P(\bar{d}). \end{array} \tag{up}$$

Since the models are rooted, this implies that for any term t, $I_k(t) = I_l(t)$ holds for *all* nodes k and l.

Forcing is defined as usual, where the forcing of atomic formulas is defined by

$$k \Vdash P(\bar{d}) \equiv_{def} (D, I_k) \models P(\bar{d}).$$

It is clear that because of (up) the upwards persistency requirement for Kripke models is satisfied.

In ordinary Kripke models, the domains at the nodes may differ, but as such models are not considered in this paper, they will not be defined here. For the same reason, sometimes the words *with constant domains* are omitted and we just speak of *Kripke models*.

2.3 Finite width and finite models

A class of models has *width* $\leq n$ if no model in the class contains an anti-chain of size larger than n. It is of *width* n if it is of width $\leq n$ but not of width $\leq (n-1)$. A model is of *width* n if the class consisting of that model is. A theory L has *width* n if it is complete with respect to a class of models of width n.

A theory is a *constant domain width n theory* if the theory is complete with respect to a class of models of width n with constant domains. A theory has *finite width* or the *finite width property (fwp)* if it has width n for some $n \in \mathbb{N}$. A theory has the *constant domain finite width property (cdfwp)* when for some $n \in \mathbb{N}$ it is a constant domain width n theory. The smallest such n is denoted by w_{L}. If L has fwp but not cdfwp then w_{L} is the smallest n for which it has width n. A theory has the *constant domain finite model property (cdfmp)* when it is sound and complete with respect to a class of finite models with constant domains. Do note that the *finite* in this definition refers to the nodes of the frame of the model, and not to the domains of the model, which may well be infinite.

3 Skolemization

A Skolemization method $(\cdot)^s$, by which we mean an algorithm on formulas that produces formulas that do not contain strong quantifiers, is *sound* when $\vdash \varphi$ implies $\vdash \varphi^s$ and *complete* when the opposite holds. All Skolemization methods that we consider are sound. The *standard Skolemization method* replaces occurrences $Qx\psi(x, \bar{y})$ of quantifiers by $\psi(f(\bar{y}), \bar{y})$ for a fresh f, in case $Q = \forall$ and the occurrence is strong or $Q = \exists$ and the occurrence is weak, where $\bar{y}$ are the variables of the weak quantifiers in the scope of which $Qx\psi$ occurs. The standard Skolemization method is not complete for intuitionistic logic, as $\forall x \neg\neg P(x) \to \neg\neg\forall x P(x)$ is not derivable in the logic, whereas its Skolemized version, $\forall x \neg\neg P(x) \to \neg\neg P(c)$, is.

For theories L of finite width we define the following *parallel skolemization method* (*pskolemization* for short) that removes strong quantifiers from formulas in the following way. The last part of this section discusses the intuition behind this variant of Skolemization. Recall that $\vdash$ denotes derivability in the theory L.

Given a formula φ and a subformula $Qx\psi(x,\bar{y})$, where $\bar{y}$ are the variables of the weak quantifiers in the scope of which $Qx\psi$ occurs, we define

$$\mathsf{ps}(Qx\psi(x,\bar{y}))_\varphi \equiv_{def} \begin{cases} \bigvee_{i=1}^{w_\mathsf{L}} \psi(f_i(\bar{y}),\bar{y}) & \text{if } Q=\exists \\ \\ \bigwedge_{i=1}^{w_\mathsf{L}} \psi(f_i(\bar{y}),\bar{y}) & \text{if } Q=\forall, \end{cases}$$

where the f_i are assumed to not occur in φ. We write $\varphi \rightarrowtail \varphi'$ if φ' is the result of replacing the leftmost strong quantifier occurrence $Qx\psi$ in φ by $\mathsf{ps}(Qx\psi(x,\bar{y}))_\varphi$. Using the multiset ordering by Dershowitz and Manna (1979) it is not hard to see that up to the renaming of function symbols, for every φ there are unique $\varphi=\varphi_1,\dots,\varphi_n=\varphi'$ such that $\varphi_i \rightarrowtail \varphi_{i+1}$ and φ' does not contain strong quantifiers. This φ' is the *pskolemization* of φ and is denoted by φ^{ps}.

We use the convention that in strong quantifier occurrences $Qx\psi(x,\bar{y})$ the $\bar{y}$ always denote the variables of the weak quantifiers in the scope of which $Qx\psi$ occurs.

A theory has *pskolemization* if for all formulas φ and φ', where φ' is the result of replacing a strong quantifier occurrence $Qx\psi$ in φ by $\mathsf{ps}(Qx\psi(x,\bar{y}))_\varphi$:

$$\vdash \varphi \Leftrightarrow \vdash \varphi'.$$

In particular, if a theory has pskolemization then for all formulas φ:

$$\vdash \varphi \Leftrightarrow \vdash \varphi^{\mathrm{ps}}.$$

It is instructive to compare pskolemization to standard Skolemization by considering the simple example $\exists x\forall y\varphi(x,y)$ where φ is quantifier-free. The Skolemization of this formula is $\exists x\varphi(x,fx)$ while the pskolemization for a logic of width n is $\exists x \bigwedge_{i=1}^{n} \varphi(x,f_ix)$. The idea is that every branch of a Kripke model of the logic has its own skolem function. For the standard method, a simple proof of the completeness of Skolemization for such formulas is semantical: a counter model to $\exists x\forall y\varphi(x,y)$ produces a counter model to $\exists x\varphi(x,fx)$ by interpreting fx as the y such that $\varphi(x,y)$ does not hold in the original model. In the case of pskolemization, a Kripke counter model to $\exists x\forall y\varphi(x,y)$ with branches $b_1,\dots,b_n$ produces a counter model to $\exists x \bigwedge_{i=1}^{n} \varphi(x,f_ix)$ by interpreting f_i as the y such that $\varphi(x,y)$ does not hold along b_i. Here we use that the models we consider in the setting of pskolemization have constant domains. The next two sections contain the technical details behind this informal argument.

4 Quantifier witnesses

In the previous section a simple semantical proof of the completeness of Skolemization was sketched. An analogue of this proof idea for Kripke models will be used to prove the completeness of pskolemization in the next section, where it is first shown that for any Kripke model K, a model K' is defined such that for every strong quantifier occurrence $Qx\psi(x,\bar{y})$ in φ:

$$K, k \Vdash Qx\psi(x,\bar{a}) \text{ if and only if } K', k \Vdash \mathsf{ps}(Qx\psi(x,\bar{a}))_\varphi.$$

For this to work, the existence, in K, of certain nodes and elements of the domain has to be guaranteed. These are the quantifier witnesses defined as follows.

Given a formula $Qx\psi(x,\bar{y})$, a Kripke model K with constant domains, root r_K and at least one element d_K in its domain D, has *quantifier witnesses for* $Qx\psi(x,\bar{y})$ if the following holds:

- if $Q = \exists$, then for any $\bar{a} \subseteq D$ and any branch b along which $\exists x\psi(x,\bar{a})$ is forced, there exists a lowest node $k = \mathsf{nd}(b, \exists x\psi(x,\bar{a}))$ for which there is a $d = \mathsf{wt}(b, \exists x\psi(x,\bar{a})) \in D$ such that $k \Vdash \psi(d,\bar{a})$; and if $\exists x\psi(x,\bar{a})$ is nowhere forced along b, we put $\mathsf{nd}(b, \exists x\psi(x,\bar{a})) = r_K$ and $\mathsf{wt}(b, \exists x\psi(x,\bar{a})) = d_K$;
- if $Q = \forall$, then for any $\bar{a} \subseteq D$ and any branch b along which $\forall x\psi(x,\bar{a})$ is not forced, there exists a highest node $k = \mathsf{nd}(b, \forall x\psi(x,\bar{a}))$ for which there is a $d = \mathsf{wt}(b, \forall x\psi(x,\bar{a})) \in D$ such that $k \nVdash \psi(d,\bar{a})$; and if $\forall x\psi(x,\bar{a})$ is forced everywhere along b, we put $\mathsf{nd}(b, \forall x\psi(x,\bar{a})) = r_K$ and $\mathsf{wt}(b, \forall x\psi(x,\bar{a})) = d_K$;
- the witnesses are chosen such that if $\mathsf{nd}(b, Qx\psi(x,\bar{a}))$ lies on another branch c, then $\mathsf{nd}(c, Qx\psi(x,\bar{a})) = \mathsf{nd}(b, Qx\psi(x,\bar{a}))$ and $\mathsf{wt}(c, Qx\psi(x,\bar{a})) = \mathsf{wt}(b, Qx\psi(x,\bar{a}))$.

K has *quantifier witnesses* if it has quantifier witnesses for every quantified formula $Qx\psi(x,\bar{y})$.

The idea behind this definition is that for a model with quantifier witnesses for $Qx\psi(x,\bar{y})$, along every branch b, there exists an element $d = \mathsf{wt}(b, Qx\psi(x,\bar{a}))$ such that along b, $Qx\psi(x,\bar{y})$ is forced exactly where $\psi(d,\bar{y})$ is forced. Figure 1 contains examples of models that have and do not have quantifier witnesses for a certain formula.

Lemma 4.1 Any finite Kripke model with constant domains has quantifier witnesses.

Proof Suppose the finite Kripke model is of width n and let $b_1,\ldots,b_n$ be its branches, r_K its root and d_K an element in the domain. Consider a formula $\exists x\psi(x,\bar{y})$ and elements $\bar{a}$ of the domain. Abbreviate $\exists x\psi(x,\bar{a})$ by φ. We define witnesses for this formula along the branches one-by-one. So consider b_i and suppose that for $j < i$, the witnesses have already been defined. If φ is forced nowhere along b_i, then put $\mathsf{nd}(b_i,\varphi) = r_K$ and $\mathsf{wt}(b_i,\varphi) = d_K$. These witnesses clearly have the required properties.

If φ is forced along b_i, we distinguish two cases. First consider the case that the lowest node along b where φ is forced is of the form $\mathsf{nd}(b_j,\varphi)$, for some $j < i$. Then put $\mathsf{nd}(b_i,\varphi) = \mathsf{nd}(b_j,\varphi)$ and $\mathsf{wt}(b_i,\varphi) = \mathsf{wt}(b_j,\varphi)$. In the remaining case, choose the lowest node where φ holds along b. Such a node k exists because the model is finite. Choose an element $d \in D$ such that $k \Vdash \psi(d,\bar{a})$ and put

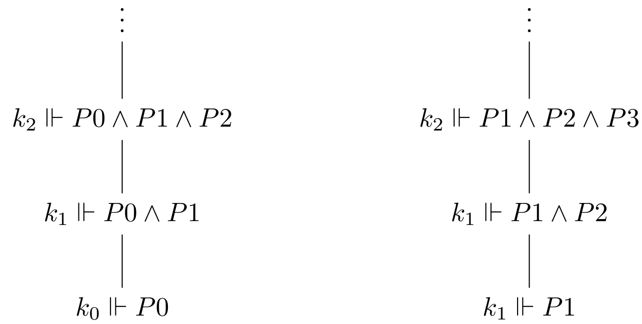


Figure 1: Two Kripke models with a constant domain, $\mathbb{N}$, in which $\forall x P(x)$ is forced nowhere. The model at the right has a quantifier witness for $\forall x P(x)$ (namely k_0 and 0) while the model at the left does not.

$\mathsf{nd}(b_i, \varphi) = k$ and $\mathsf{nd}(b_j, \varphi) = d$. It is not hard to see that these satisfy the quantifier witness requirements.

The proof for universal formulas is similar, using that the model is conversely well-founded. □

5 Completeness

In this section we prove the completeness of pskolemization. As mentioned above, we give a semantical proof of this fact, and the following construction to extend Kripke models for a certain language to models for a richer language, is its main ingredient.

5.1 Model extensions

Consider a theory L of width n in language $\mathcal{L}$. Given a model $K = (W, \preccurlyeq, D, I)$ of width n for L that has quantifier witnesses for $Qx\psi(x, \bar{y})$, we show how to extend it to a model $K' = (W, \preccurlyeq, D, I')$ for $\mathcal{L}' = \mathcal{L} \cup \{f_1, \ldots, f_n\}$, where the f_i are the skolem functions occurring in $\mathsf{ps}(Qx\psi(x, \bar{y}))_\varphi$. Let $b_1, \ldots, b_n$ be the branches of K. For every k, I'_k equals I_k on terms in $\mathcal{L} \cup D$, and for $\bar{a} \in D$:

$$I'_k(f_i)(\bar{a}) = \mathsf{wt}(b_i, Qx\psi(x, \bar{a})).$$

Remark 5.2 We leave it to the reader to verify that forcing in K is equal to forcing in K' for all formulas that do not contain the function symbols $f_1, \ldots, f_n$.

Lemma 5.3 For every strong quantifier occurrence $Qx\psi(x, \bar{y})$ in φ:

$$K, k \Vdash Qx\psi(x, \bar{a}) \text{ if and only if } K', k \Vdash \mathsf{ps}(Qx\psi(x, \bar{a}))_\varphi. \tag{1}$$

Proof First observe that the definition of quantifier witnesses implies that for every branch b_i through k:

$$K, k \Vdash Qx\psi(x, \bar{a}) \text{ if and only if } K', k \Vdash \psi(\mathsf{wt}(b_i, \mathcal{X}), \bar{a}). \tag{2}$$

To prove (3) we treat the existential and universal quantifier separately.

$\exists$: The direction from left to right follows from (2). For the opposite direction, suppose $K', k \Vdash \psi(f_j(\bar{a}), \bar{a})$ for some j, that is, $K', k \Vdash \psi(\mathsf{wt}(b_j, \mathcal{X}), \bar{a})$. By Remark 5.2, $K, k \Vdash Qx\psi(x, \bar{a})$ follows.

$\forall$: If $K, k \nVdash Qx\psi(x, \bar{a})$, then $K, k \nVdash \psi(\mathsf{wt}(b_i, \mathcal{X}), \bar{a})$ for all branches b_i through k. Hence $K', k \nVdash \psi(f_i(\bar{a}), \bar{a})$. Thus $K', k \nVdash \mathsf{ps}(\mathcal{X})_\varphi$. For the other direction, suppose $K', k \nVdash \psi(f_j(\bar{a}), \bar{a})$ for some j. This implies that $K, k \nVdash \forall x\psi(x, \bar{a})$, that is, $K, k \nVdash Qx\psi(x, \bar{a})$. $\square$

Lemma 5.4 If $\varphi \rightarrowtail \varphi'$, then for every model K with quantifier witnesses:

$$K, k \Vdash \varphi \text{ if and only if } K', k \Vdash \varphi'. \tag{3}$$

Proof With formula induction, using Lemma 5.3. $\square$

Theorem 5.5 Every theory that is sound and complete with respect to a class of Kripke models of width n with quantifier witnesses and constant domains, has pskolemization. In particular, for all formulas φ:

$$\vdash \varphi \Leftrightarrow \vdash \varphi^{\text{ps}}.$$

Proof The direction from left to right is easy. The other direction follows by contraposition from repeated application of Lemma 5.4. $\square$

Corollary 5.6 Every intermediate theory with cdfmp has pskolemization.

A logic is *tabular* when it is the logic of a single finite Kripke frame. A logic is a *constant domain tabular* logic if for some finite frame it consists of all formulas that hold in all models with constant domain on that frame.

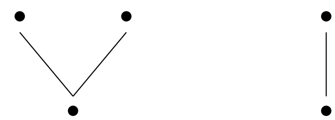
Corollary 5.7 Every constant domain tabular logic has pskolemization.

The *logic of constant domains* CD is the intermediate predicate logic axiomatized over IQC by the scheme

$$\mathsf{D} \quad \forall x(\varphi(x) \vee \psi) \rightarrow (\forall x\varphi(x) \vee \psi),$$

where x does not occur free in ψ. CD characterizes the class of Kripke models with constant domains. Given a propositional logic L, let $\mathsf{CD} + \mathsf{L}$ denote the smallest intermediate predicate logic containing CD and all formulas in L as axiom schemes.

Shimura (1993) has proven that for any tabular propositional intermediate logic L, the logic $\mathsf{CD} + \mathsf{L}$ is strongly Kripke complete and has cdfmp. This follows from the fact (Lemma 3.5 in that paper) that for such logics the canonical model with constant domains has a finite frame. Well-known examples of tabular propositional logics are GSc and Sm, with respective frames:



Shimura's result and Corollary 5.7 imply the following.

Corollary 5.8 For any tabular propositional intermediate logic L, the predicate logic $\mathsf{CD} + \mathsf{L}$ has pskolemization. In particular, $\mathsf{CD} + \mathsf{GSc}$ and $\mathsf{CD} + \mathsf{Sm}$ have pskolemization.

6 Herbrand's theorem

Herbrand's theorem states that for every quantifier free formula $\varphi(\bar{x})$:

$$\vdash_{\mathsf{CQC}} \exists \bar{x} \varphi(\bar{x}) \Leftrightarrow \vdash_{\mathsf{CQC}} \bigvee_{i=1}^{n} \varphi(\bar{s}_i) \text{ for some sequences of terms } \bar{s}_1, \ldots, \bar{s}_n.$$

In combination with the Skolemization method it provides a powerful tool in the study of classical logic. As for Skolemization, there exists a natural extension of the theorem that applies to infix formulas without strong quantifiers. This is the variant we will use, which is defined as follows.

Given a formula φ, a formula φ' is an *Herbrand expansion* of φ if it is the result of replacing, from inside out, every positive occurrence of a formula $\exists x \psi(x)$ by a disjunction $\bigvee_{i=1}^{m} \psi(s_i)$ for some terms $s_1, \ldots, s_m$, and every negative occurrence of a formula $\forall x \psi(x)$ by a conjunction $\bigwedge_{i=1}^{n} \psi(t_i)$ for some terms $t_1, \ldots, t_n$. The *dual Herbrand expansion* of φ is defined similarly, by switching "$\exists x \psi(x)$" and "$\forall x \psi(x)$". For example, $\bigwedge_{i=1}^{m} P(t_i) \to \bigvee_{j=1}^{n} Q(s_j)$ is an Herbrand expansion of $\forall x P(x) \to \exists z Q(x)$ and dual Herbrand expansion of $\exists x P(x) \to \forall z Q(x)$.

Observe that in an Hebrand expansion all the weak quantifiers of a formula are removed. Thus the Hebrand expansion of a formula without strong quantifiers does not contain any quantifiers. It is not hard to see that any Herbrand expansion of a formula implies the formula, while the formula implies all its dual Herbrand expansions. In universal theories the following holds as well.

Lemma 6.1 In any universal intermediate theory L, for any formula φ without strong quantifiers: if φ is provable in L, then so is at least one Hebrand expansion of φ.

Proof Suppose that φ is derivable in L. Then for some finite conjunction ψ of axioms from L, $\psi \to \varphi$ is derivable in IQC. $\psi \to \varphi$ does not contain strong quantifiers. This implies that some Hebrand expansion $\psi' \to \varphi'$ of $\psi \to \varphi$ is derivable in IPC, where φ' is an Herbrand expansion of φ and ψ' is a dual Herbrand expansion of ψ (folklore, but for a proof see (Baaz and Iemhoff, 2008)). Thus $\vdash_{\mathsf{IPC}} \psi \to \psi'$. Hence $\vdash_{\mathsf{L}} \varphi'$, which is what we had to show. $\square$

For theories with cdfmp Lemma 6.1 leads to a correspondence between derivability in predicate theories and their propositional fragment, as given in the following theorem. The correspondence could be viewed as an analogue of the situation in classical universal theories, where one uses standard Skolemization and the Herbrand Theorem instead of parallel Skolemization and Lemma 6.1.

Theorem 6.2 In every universal intermediate theory with cdfmp, for all formulas φ:

$$\varphi \text{ is provable} \Leftrightarrow \text{ at least one Hebrand expansion of } \varphi^{\mathrm{ps}} \text{ is provable.}$$

Theorem 6.3 The one-variable fragment of every universal intermediate theory with cdfmp and a decidable propositional fragment is decidable.

Proof As formulas in the one-variable fragment of any intermediate theory contain at most one variable, the pskolemization of such formulas contain no Skolem functions, only Skolem constants. Thus so do the Herbrand expansions of such pskolemizations. Derivability of such Herbrand expansions is therefore decided in the propositional fragment of the theory, which is decidable by assumption. This proves that the derivability of formulas in the one-variable fragment is decidable. □

7 Interpolation

Recall that a logic L has *interpolation* if whenever $\vdash_{\mathsf{L}} \varphi \to \psi$, there exists a formula ι in the common language of φ and ψ such that $\varphi \to \iota$ and $\iota \to \psi$ hold in L. In the case of propositional logic, the common language consists of the atoms that occur in both φ and ψ and all connectives, and in the case of predicate logic it consists all predicates, functions and constants that occur in both φ and ψ and all variables, connectives and quantifiers.

Theorem 7.1 For any universal intermediate logic with pskolemization, if the propositional fragment has interpolation, then so does the full logic.

Proof Assume $\vdash \varphi \to \psi$. This implies $\vdash (\varphi \to \psi)^{\mathrm{ps}}$ since the logic has pskolemization. Let φ_s, ψ_s be such that $(\varphi_s \to \psi_s) = (\varphi \to \psi)^{\mathrm{ps}}$. Some Herbrand expansion $\varphi_h \to \psi_h$ of $\varphi_s \to \psi_s$ is derivable by Lemma 6.1 and the proof of the lemma shows that we can assume that φ_h is a dual Herbrand expansion of φ_s and that ψ_h is an Herbrand expansion of ψ_s.
As the propositional fragment has interpolation, there is a formula ι in the common language of φ_h and ψ_h such that $\varphi_h \to \iota$ and $\iota \to \psi_h$ hold in L. Therefore $\varphi_s \to \iota$ and $\iota \to \psi_s$ hold in L as well.
From the definition of Skolemization it follows that every skolem function can occur either in φ_s or in ψ_s but not in both. Next we construct a finite sequence of formulas $\iota = \iota_1, \ldots, \iota_n$ such that ι_n contains no skolem symbols and $\varphi_s \to \iota_i$ and $\iota_i \to \psi_s$ hold for all i. Given ι_i, consider the leftmost term of the form $f(\bar{t})$ in it, where f is a skolem function. Let x_{i+1} be a variable not occurring in ι_i and define

$$\iota_{i+1} \equiv_{\mathit{def}} \begin{cases} \exists x_{i+1} \iota_i[x_{i+1}/f(\bar{t})] & \text{if } f \text{ occurs in } \varphi_s \\ \forall x_{i+1} \iota_i[x_{i+1}/f(\bar{t})] & \text{if } f \text{ occurs in } \psi_s. \end{cases}$$

We show that for all $i \geq 1$, $\varphi_s \to \iota_i$ holds and leave the proof for $\iota_i \to \psi_s$ to the reader. The case for $i = 1$ is clear. Therefore suppose $\varphi_s \to \iota_i$ holds and consider the leftmost term in ι_i of the form $f(\bar{t})$, where f is a skolem function. Thus $\iota_i = \iota_i(f(\bar{t}))$ and clearly derives $\exists x_{i+1}\iota_i[x_{i+1}/f(\bar{t})]$, which implies that $\varphi_s \to \iota_{i+1}$ holds.

If n is equal to the number of skolem functions in φ_s and ψ_s together, then ι_n cannot contain any skolem functions. As $\varphi \to \iota_n$ and $\iota_n \to \psi$ are derivable in L by Theorem 5.5, ι_n is the desired interpolant for $\varphi \to \psi$. $\Box$

Corollary 7.2 Every universal intermediate logic with cdfmp and a propositional fragment that has interpolation, has interpolation.

As an illustration of the idea used in Theorem 7.1, consider a universal intermediate logic L with pskolemization and a propositional fragment that has interpolation, and let

$$\varphi \to \psi = \exists x(Px \wedge Qx) \vee \exists y(Py \wedge Q'y) \to \exists z(Pz \vee Rz),$$

where P, Q, R are unary predicates. Clearly, $\varphi \to \psi$ is provable in L, and thus so is its pskolemization:

$$(\varphi \to \psi)^{\mathrm{ps}} = \varphi_s \to \psi_s = \bigvee_{i=1}^{2}(Pc_i \wedge Qc_i) \to \exists z(Pz \vee Rz).$$

A possible provable Herbrand expansion of $(\varphi \to \psi)^{\mathrm{ps}}$ is

$$\varphi_h \to \psi_h = \bigvee_{i=1}^{2}(Pc_i \wedge Qc_i) \to \bigvee_{j=1}^{2}(Pc_j \vee Rc_j).$$

An interpolant for $\varphi_h \to \psi_h$ is $\iota = (Pc_1 \vee Pc_2)$. Then, following the proof of Theorem 7.1, $\iota_1 = \iota$, $\iota_2 = \exists x(Px \vee Pc_2)$ and $\iota_3 = \exists y \exists x(Px \vee Py)$, which indeed is an interpolant for $\varphi \to \psi$.

Because CD itself is not a universal formula, Corollary 7.2 may be of limited value. One way to widen the field of application is to weaken the requirement on constant domains, which is the subject of the remaining part of this paper. As we will see, it will come at a cost, namely the addition of a specific predicate, the existence predicate, to the language of theories. Therefore we do not (yet) succeed in obtaining applications to interpolation of our results, as we do not obtain an analogue of Corollary 7.2 for theories with nonconstant domains. Still, we think the generalization of parallel Skolemization to theories with nonconstant domains is of independent interest, and it may lead to other applications in the future.

8 The existence predicate

The restriction, in the results above, to constant domains is a severe one since many interesting intermediate theories do not have constant domains. In this section we extend the results of the previous sections to such theories. The main tool is the increase in expressive power of the language of intuitionistic logic through the addition of an existence predicate. In this way many logics that are not constant domain logics in the original sense, become sound and complete with respect to a certain class of models with constant domains. Therefore the Skolemization method developed above can be applied to such logics as well.

We consider an extension, IQCE, of IQC, the language of which is $\mathcal{L}$ extended by a unary predicate, E, the *existence predicate*. This logic, introduced by Scott (1979), allows one to distinguish between existing and not (yet) existing terms. There are several variants of the logic, depending on the requirements on the quantifiers. In our approach, quantifiers range over existing objects only. This means, for example, that one is allowed to infer $\exists x\varphi(x)$ only if a term t exists such that both Et and $\varphi(t)$ hold. In (Baaz and Iemhoff, 2006a), a Gentzen calculus for IQCE is provided that is a variant of the Gentzen calculus $\mathsf{G3i}$ in (Troelstra and Schwichtenberg, 1996). The difference between the two calculi lies only in the quantifier rules, which in the case of IQCE are (assuming that y does not occur free in Γ and ψ):

$$\frac{\Gamma \Rightarrow E(t) \quad \Gamma \Rightarrow \varphi(t)}{\Gamma \Rightarrow \exists x\varphi(x)} \qquad\qquad \frac{\Gamma, E(y), \varphi(y) \Rightarrow \psi}{\Gamma, \exists x\varphi(x) \Rightarrow \psi}$$

$$\frac{\Gamma, \forall x\varphi(x), \varphi(t) \Rightarrow \psi \quad \Gamma, \forall x\varphi(x) \Rightarrow E(t)}{\Gamma, \forall x\varphi(x) \Rightarrow \psi} \qquad \frac{\Gamma, E(y) \Rightarrow \varphi(y)}{\Gamma \Rightarrow \forall x\varphi(x)}$$

For theories $\mathcal{T}$ over IQC and sentences φ not containing the existence predicate, it holds that

$$\mathcal{T} \vdash_{\mathsf{IQC}} \varphi \;\Leftrightarrow\; \mathcal{T}^e \vdash_{\mathsf{IQCE}} \varphi, \tag{4}$$

where $\mathcal{T}^e$ is the theory over IQCE corresponding to $\mathcal{T}$. Roughly, $\mathcal{T}^e$ is a version of $\mathcal{T}$ in which all terms are assumed to exist. For details, see (Iemhoff, 2010). Skolemization methods and Herbrand theorems for theories over IQCE are via (4) inherited by theories over IQC. In the remainder of this section we provide such methods.

A semantics for IQCE is given by *Kripke existence models*, which are regular Kripke models with constant domains in which the existence predicate is interpreted as a unary predicate, nonempty at the root, and forcing is defined as usual, except for the quantifiers, in which case it is defined as

$$\begin{array}{l} K,k \Vdash \exists x\varphi(x) \equiv_{def} K,k \Vdash Ed \wedge \varphi(d) \text{ for some } d \in D \\ K,k \Vdash \forall x\varphi(x) \equiv_{def} K,k \Vdash Ed \to \varphi(d) \text{ for all } d \in D. \end{array}$$

IQCE is sound and strongly complete with respect to this semantics (Baaz and Iemhoff, 2006b). In particular, φ is derivable in IQCE if and only if φ holds in all Kripke existence models.

8.1 Skolemization

In Baaz and Iemhoff (2006b, 2009) we showed that for IQCE there exists a sound and complete skolemization method $(\cdot)^e$ for existential quantifiers. Here we can combine this method with the method of pskolemization as follows. Given a formula φ and a subformula $Qx\psi(x,\bar{y})$, where $\bar{y}$ are the variables of the weak quantifiers in the scope of which $Qx\psi$ occurs, we define (writing Et for $E(t)$):

$$\mathsf{eps}(Qx\psi(x,\bar{y}))_\varphi \equiv_{def} \begin{cases} \bigvee_{i=1}^{w_\mathsf{L}} Ef_i(\bar{y}) \wedge \psi(f_i(\bar{y}),\bar{y}) & \text{if } Q = \exists \\ \bigwedge_{i=1}^{w_\mathsf{L}} Ef_i(\bar{y}) \to \psi(f_i(\bar{y}),\bar{y}) & \text{if } Q = \forall, \end{cases}$$

where the f_i are assumed to not occur in φ As previously, we write $\varphi \rightarrowtail \varphi'$ if φ' is the result of replacing the leftmost strong quantifier occurrence $Qx\psi$ in φ by $\mathsf{eps}(Qx\psi(x,\bar{y}))_\varphi$. It is clear that, up to the renaming of function symbols, for every φ there are unique $\varphi = \varphi_1,\ldots,\varphi_n = \varphi'$ such that $\varphi_i \rightarrowtail \varphi_{i+1}$ and φ' does not contain strong quantifiers. This φ' is the *epskolemization* φ^{eps} ("e" for existence) of φ.

A theory has *epskolemization* if for all formulas φ and φ', where φ' is the result of replacing a strong quantifier occurrence $Qx\psi$ in φ by $\mathsf{eps}(Qx\psi(x,\bar{y}))_\varphi$:

$$\vdash \varphi \Leftrightarrow \vdash \varphi'.$$

In particular, if a theory has epskolemization, then for all formulas φ:

$$\vdash \varphi \Leftrightarrow \vdash \varphi^{\text{eps}}.$$

8.2 Quantifier witnesses

The notion of quantifier witnesses is adapted to Kripke existence models as follows. Given a formula $Qx\psi(x,\bar{y})$, a Kripke existence model with root r_K and at least one element d_K in its domain D that exists at all nodes, has *quantifier witnesses for* $Qx\psi(x,\bar{y})$ if the following holds:

- if $Q = \exists$, then for any $\bar{a} \subseteq D$ and any branch b along which $\exists x\psi(x,\bar{a})$ is forced, there exists a lowest node $k = \mathsf{nd}(b, \exists x\psi(x,\bar{a}))$ for which there is a $d = \mathsf{wt}(b, \exists x\psi(x,\bar{a})) \in D$ such that $k \Vdash Ed \wedge \psi(d,\bar{a})$; and if $E\bar{a}$ or $\exists x\psi(x,\bar{a})$ is nowhere forced along b, we put $\mathsf{nd}(b, \exists x\psi(x,\bar{a})) = r_K$ and $\mathsf{wt}(b, \exists x\psi(x,\bar{a})) = d_K$;
- if $Q = \forall$, then for any $\bar{a} \subseteq D$ and any branch b along which $\forall x\psi(x,\bar{a})$ is not forced, there exists a highest node $k = \mathsf{nd}(b, \forall x\psi(x,\bar{a}))$ for which there is a $d = \mathsf{wt}(b, \forall x\psi(x,\bar{a})) \in D$ such that $k \Vdash Ed$ and $k \nVdash \psi(d,\bar{a})$; and if $\forall x\psi(x,\bar{a})$ is forced everywhere along b, we put $\mathsf{nd}(b, \forall x\psi(x,\bar{a})) = r_K$ and $\mathsf{wt}(b, \forall x\psi(x,\bar{a})) = d_K$;
- the witnesses are chosen such that if $\mathsf{nd}(b, Qx\psi(x,\bar{a}))$ lies on another branch c, then $\mathsf{nd}(c, Qx\psi(x,\bar{a})) = \mathsf{nd}(b, Qx\psi(x,\bar{a}))$ and $\mathsf{wt}(c, Qx\psi(x,\bar{a})) = \mathsf{wt}(b, Qx\psi(x,\bar{a}))$.

K has *quantifier witnesses* if it has quantifier witnesses for every quantifier $Qx\psi(x,\bar{y})$.

It is not difficult to see that the analogues of Lemmas 5.3 and 5.4 hold. Using these analogues we can prove the following variants of Theorem 5.5 and Corollary 5.6.

Theorem 8.3 Every theory in IQCE that is sound and complete with respect to a class of Kripke existence models of finite width with quantifier witnesses, has epskolemization. In particular, for all formulas φ:

$$\vdash \varphi \Leftrightarrow \vdash \varphi^{\mathrm{eps}}.$$

Corollary 8.4 Every theory in IQCE with fmp has epskolemization.

The above result is implied by the result (Corollary 1) in (Iemhoff, 2010) that for logics with fmp the eskolemization method introduced in (Baaz and Iemhoff, 2006b) is sound and complete. Roughly, eskolemization is epskolemization without the disjunctions and conjunctions. Thus the mentioned result in (Iemhoff, 2010) is stronger than the one in Corollary 8.4. However, the proof of it in that paper is far more complicated than the simple one presented here. We therefore think that even given the stronger result, epskolemization is of independent interest.

8.5 Herbrand's theorem

The notion of an Hebrand expansion also has to be adapted in the presence of an existence predicate. In extensions of IQCE, given a formula φ, a formula φ' is a *Herbrand expansion* of φ if it is the result of replacing every positive occurrence of a formula $\exists x\psi(x)$ by a disjunction $\bigvee_{i=1}^{m}(Es_i \wedge \psi(s_i))$ for some terms $s_1, \ldots, s_m$, and every negative occurrence of a formula $\forall x\psi(x)$ by a conjunction $\bigwedge_{i=1}^{n}(Et_i \rightarrow \psi(t_i))$ for some terms $t_1, \ldots, t_n$. The *dual Herbrand expansion* of φ is defined similarly, by switching the expressions "$\exists x\psi(x)$" and "$\forall x\psi(x)$".

In (Baaz and Iemhoff, 2008; Iemhoff, 2010) it is shown that in IQCE, derivability of φ implies derivability of at least one Herbrand expansion of φ. As in Lemma 6.1 this can be used to show the following.

Lemma 8.6 In any universal theory L in IQCE: if φ is provable in L, then so is at least one Hebrand expansion of φ.

Theorem 8.7 In every universal theory in IQCE with fmp, for all formulas φ:

$$\varphi \text{ is provable} \Leftrightarrow \text{ at least one Hebrand expansion of } \varphi^{\mathrm{eps}} \text{ is provable.}$$

Using that for every universal theory $\mathcal{T}$ in IQC with fmp the theory $\mathcal{T}^e$ in IQCE is also universal and has fmp (this can be concluded from the construction of $\mathcal{T}^e$ as given in (Iemhoff, 2010)) we obtain the following.

Corollary 8.8 In any universal theory $\mathcal{T}$ in IQC with fmp, for all sentences φ not containing the existence predicate:

$$\mathcal{T} \vdash_{\mathsf{IQC}} \varphi \Leftrightarrow \mathcal{T}^e \vdash_{\mathsf{IQCE}} \varphi \Leftrightarrow$$
$$\text{at least one Hebrand expansion of } \varphi^{\mathrm{eps}} \text{ is provable in } \mathcal{T}^e.$$

The same remarks concerning the results in (Baaz and Iemhoff, 2006b) that follow Corollary 8.4 apply to Corollary 8.8.

The proof of the following theorem is analogous to that of its counterpart Theorem 6.3 for constant domain logics.

Theorem 8.9 The one-variable fragment of every universal intermediate theory in IQC with fmp and a decidable propositional fragment is decidable.

9 Conclusion

It has been shown that for certain intermediate logics and intermediate theories alternative skolemization methods and Herbrand theorems can be developed that, like the standard method, provide a connection between derivability in a theory and its propositional fragment. Crucial for this to hold is that the theory is complete with respect to a class of models that have quantifier witnesses, a technical notion that is satisfied, for example, by logics with the finite model property. First, theories have been treated for which the models in the class in addition have constant domains. For these theories the alternative Skolemization method is but a simple variant of the standard method in which per strong quantifier instead of one skolem term finitely many skolem terms are used. In case the theory does not have constant domains, the extension IQCE of IQC is used to obtain a similar method. Here the existence predicate of IQCE is applied in the same way as in (Baaz and Iemhoff, 2006b), where it was used to obtain a skolemization method for existential quantifiers in IQC. In the constant domain as well as the not constant domain case a corresponding Herbrand theorem can be obtained easily. For the latter case, similar results but with far more complicated proofs have been obtained in (Iemhoff, 2010).

For universal constant domain logics with the finite model property a consequence of the above is that whenever the propositional fragment of the logic has interpolation, so does the full logic. Whether we can obtain a similar result for logics that do not have constant domains we do not know. Another corollary of the above is the decidability of the one-variable fragment of all logics with the finite model property and a decidable propositional fragment.

In general, the obtained results show that useful alternatives to Skolemization can be obtained for nonclassical theories by allowing quantified subformulas to be replaced by more complex formulas than in the standard method. Whether these methods can be of use in the study of nonclassical theories, the future will tell.

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