On the existence of alternative Skolemization methods

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In memory of Grigori Mints

Abstract

It is shown that no intermediate predicate logic that is sound and complete with respect to a class of frames, admits a strict alternative Skolemization method. In particular, this holds for intuitionistic predicate logic and several other well–known intermediate predicate logics. The result is proved by showing that the class of formulas without strong quantifiers as well as the class of formulas without weak quantifiers is sound and complete with respect to the class of constant domain Kripke models.

Keywords: Skolemization, Herbrand's theorem, intermediate logics, Kripke models
MSC: 03B10, 03B55, 03F03

1 Introduction

The insight that certain quantifier combinations can be reduced in complexity by introducing fresh function symbols, goes back to Thoralf Skolem's work at the beginning of the twentieth century (Skolem, 1920). This insight has been used in the meta-mathematical study of logics, but it also has practical applications, since it provides, in combination with Herbrand's Theorem, a connection between propositional and predicate logic that is one of the key ingredients in automated theorem proving and logic programming. Because of the elegance and usefulness of the Skolemization method, one might hope to be able to use it also in nonclassical settings, such as intermediate predicate logics. Grigori Mints was one of the first to study Skolemization and Herbrand's Theorem in nonclassical logic, and from the references it can be seen that it remained a point of interest for him throughout his life (Mints, 1962, 1966, 1972, 1994, 2000).

As it turns out, to many nonclassical logics, including intuitionistic predicate logic, the Skolemization method does not apply. This gave rise to the search for alternative methods, that, in combination with Herbrand's Theorem, result in a connection between between propositional and predicate intermediate logics, similar to the one for classical logic. In the case of intuitionistic logic, a partial solution that only applies to the fragment without universal quantifiers has been obtained, by extending the logic with an existence predicate (Baaz and Iemhoff, 2006, 2011), and in (Iemhoff, 2010) it has been shown that for intermediate logics with the finite model

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property this *existence Skolemization method* applies to the full logic. In (Baaz and Iemhoff, 2008) an alternative method for full intuitionistic predicate logic IQC has been developed, but at the cost of extending the language considerably. There have appeared various results on the Skolemization method and Herbrand's Theorem in substructural logics, and in some cases, when the latter does not hold, an alternative *approximate Herbrand Theorem* has been obtained (Baaz and Zach, 2000; Baaz et al., 2001; Baaz and Metcalfe, 2008, 2009; Cintula et al., 2015; Cintula and Metcalfe, 2013). For intermediate logics with the finite model property, an alternative Skolemization method called *parallel Skolemization* has been developed (Baaz and Iemhoff, 2016), and in (Cintula et al., 2015) a similar method has been developed for substructural logics.

In this paper we take the opposite approach and try to establish, given an intermediate logic, what alternative Skolemization methods cannot exist for it. For this, we first need to define what an alternative Skolemization method is, as will be done in Section 5, where the notion of a strict method will be defined as well. In Section 6 it will be shown that no intermediate logic that is sound and complete with respect to a class of frames, admits a strict alternative Skolemization method. In particular, this holds for IQC, $\mathsf{QD_n}$, QKC, QLC, and all tabular logics.

As the reader will see, none of the theorems in this short paper are complex. In fact, the proof of the main result is surprisingly simple. Nevertheless, what is obtained improves our understanding of Skolemization in nonclassical logics to such an extend that I think it worthwhile to publish it separately in this note.

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2 Preliminaries

We consider intermediate predicate logics, which are predicate logics between intuitionistic predicate logic IQC and classical predicate logic CQC. The language $\mathcal{L}$ consists of infinitely many variables, which are denoted by $x, y, z, x_i, y_i, \dots$, infinitely many predicate symbols, function symbols (of every arity infinitely many), and the connectives $\wedge, \vee, \rightarrow$, the truth constants $\top, \bot$, the quantifiers $\forall, \exists$, and $\neg\varphi$ is defined as $\varphi \rightarrow \bot$. Constants are included in the language and treated as nullary function symbols. Terms and formulas are defined as usual. We use $\bar{x}$ as an abbreviation of $x_1, \dots, x_n$, where the n will always be clear from the context. For example, $\forall x_1 \exists y_1 \forall x_2 \exists y_2 \varphi(\bar{x}, \bar{y})$ is short for $\forall x_1 \exists y_1 \forall x_2 \exists y_2 \varphi(x_1, x_2, y_1, y_2)$. Given a logic L, $\vdash_{\mathsf{L}}$ denotes valitidy in L.

Important in this paper is the distinction between strong and weak quantifiers, where the former are exactly those quantifier occurrences that become universal under classical prenexification: A quantifier occurrence in φ is *strong* if it is a positive occurrence of a universal quantifier or a negative occurrence of an existential quantifier, and it is called *weak* otherwise. Let $\mathcal{F}_{ns}$ and $\mathcal{F}_{nw}$ denote the set of formulas without strong and weak quantifiers, respectively. Identifying a logic with its set of theorems, the *strong quantifier free fragment* of a logic consists of those theorems of the logic that do not contain strong quantifiers, and likewise for weak quantifiers.

3 Kripke models

Kripke models are defined as in Section 5.11 of (Troelstra and van Dalen, 1988), although we use slightly different notation. First, we define, given a set D, the notion of an *interpretation I in D*, which is such that for every n-ary relation symbol R and every n-ary function symbol f in the language, $I(R) \subseteq D^n$ and $I(f)$ is a function from D^n to D. Interpretation I in D is extended to all terms by letting it be the identity on variables, and by inductively defining for an n-ary function symbol f and terms $t_1, \dots, t_n$: $I(f(t_1, \dots, t_n)) = I(f)(I(t_1), \dots, I(t_n))$. Given a term $t(x_1, \dots, x_m)$ and a sequence $d_1, \dots, d_m$ of elements in D, we denote by $I(t)(d_1, \dots, d_m)$ the result of replacing x_i in $I(t)$ by d_i. Note that $I(t)(d_1, \dots, d_m) \in D$. Given a set D, let $\mathcal{L}(D)$ be the language to which the elements of D are added as constants.

A *Kripke model* is defined to be a tuple $(K, \preccurlyeq, \mathcal{D}, \mathcal{I}, \Vdash)$, where

- K is a set and $\preccurlyeq$ a partial order on it with a least element, the *root*;
- $\mathcal{D} = \{D_k \mid k \in K\}$ is a collection of sets;
- $\mathcal{I} = \{I_k \mid k \in K\}$, where I_k is an interpretation in D_k;
- $\Vdash$ is a relation between elements of K and atomic formulas in $\mathcal{L}(D_k)$.

Moreover, such a Kripke model must satisfy the following persistency requirements for any relation symbol R and any function symbol f in the language, where the graph of an n-ary function $f : D^n \to D$ is defined as $\{(\bar{e}, d) \in D^{n+1} \mid f(\bar{e}) = d\}$ and denoted by $\mathrm{graph}(f)$:

- $k \preccurlyeq l$ implies $D_k \subseteq D_l$;
- $k \preccurlyeq l$ implies $I_k(R) \subseteq I_l(R)$;
- $k \preccurlyeq l$ implies $\mathrm{graph}(I_k(f)) \subseteq \mathrm{graph}(I_l(f))$;
- for any n-ary predicate φ, any $\bar{d} = d_1, \dots, d_m$ in D, and terms $t_1(\bar{x}), \dots, t_n(\bar{x})$ which free variables are among $\bar{x} = x_1, \dots, x_m$: if $k \Vdash \varphi(I(t_1(\bar{d})), \dots, I(t_n(\bar{d})))$ and $k \preccurlyeq l$, then $l \Vdash \varphi(I(t_1(\bar{d})), \dots, I(t_n(\bar{d})))$.

The forcing relation $\Vdash$ is extended to all formulas in the usual way.

If no confusion is possible, the model $(K, \preccurlyeq, \mathcal{D}, \mathcal{I}, \Vdash)$ is denoted by K. The model has *constant domains* if all elements of $\mathcal{D}$ are equal. Note that the Kripke models are in general *not* required to have constant domains. Given a class of Kripke models $\mathcal{K}$, let $\mathcal{K}_{cd}$ denote the set of those models in $\mathcal{K}$ that have constant domains.

4 Skolemization

The most popular consequence of the Skolemization method is the statement that in classical predicate logic CQC, a prenex formula

$$\forall x_1 \exists y_1 \dots \forall x_n \exists y_n \varphi(\bar{x}, y_1, \dots, y_n)$$

is satisfiable if and only if its *Skolemization*

$$\forall x_1 \dots x_n \varphi\big(\bar{x}, f_1(x_1), f_2(x_1,x_2), \dots, f_n(x_1,\dots,x_n)\big)$$

is satisfiable, where f_i is a function symbol of arity i that does not occur in φ. This is equivalent to the statement that for such function symbols f_i:

$$\vdash_{\mathsf{CQC}} \exists x_1 \forall y_1 \dots \exists x_n \forall y_n \varphi(\bar{x}, y_1, \dots, y_n)$$
$$\Leftrightarrow$$
$$\vdash_{\mathsf{CQC}} \exists x_1 \dots x_n \varphi\big(\bar{x}, f_1(x_1), f_2(x_1,x_2), \dots, f_n(x_1,\dots,x_n)\big).$$

This formulation in terms of derivability rather than satisfiability is the one used in this paper.

Less well-known is the fact that Skolemization also applies to infix formulas, formulas that are not necessarily in prenex normal form. To state this result one needs to distinguish strong from weak quantifiers, defined in Section 2.

The *Skolemization*, φ^s, of a formula φ is the result of replacing every strong quantifier occurrence $Qx\psi(x,\bar{y})$ by $\psi(f(\bar{y}),\bar{y})$, where f is a fresh function symbol and the variables $\bar{y}$ are the variables of the weak quantifiers in the scope of which $Qx\psi(x,\bar{y})$ occurs. In formal terms: The *Skolemization*, φ^s, of a formula φ is such that φ^s does not contain strong quantifiers and there exist formulas $\varphi = \varphi_1, \dots, \varphi_n = \varphi^s$ such that every φ_{i+1} is the result of replacing the leftmost strong quantifier occurrence $Qx\psi(x,\bar{y})$ in φ_i by $\psi(f_i(\bar{y}),\bar{y})$, where the $f_1, \dots, f_{n-1}$ are distinct fresh function symbols that do not occur in φ and $\bar{y}$ are the variables of the weak quantifiers in the scope of which $Qx\psi(x,\bar{y})$ occurs.

The following is an example of Skolemization.

$$\big(\forall u \exists v \varphi(u,v) \to \forall x \exists y \forall z \psi(x,y)\big)^s = \forall u \varphi\big(u, f_1(u)\big) \to \exists y \psi\big(f_2, y, f_3(y)\big).$$

Note that f_2 is a constant, as the corresponding quantifier $\forall x$ is not in the scope of any weak quantifiers.

Classical logic admits Skolemization:

$$\vdash_{\mathsf{CQC}} \varphi \Leftrightarrow \vdash_{\mathsf{CQC}} \varphi^s.$$

Note that the result for prenex formulas given above is a special case of this theorem.

Interestingly, many of the standard nonclassical logics do not admit Skolemization. For example, in IQC and the predicate versions of LC and KC[1] there are various counterexamples, such as the following formulas, in which φ ranges over predicates, and which are not derivable in the logics, though their Skolemization (at the right) is.[2]

DNS	$\forall x \neg\neg\varphi(x) \to \neg\neg\forall x \varphi(x)$	$\forall x \neg\neg\varphi(x) \to \neg\neg\varphi(c)$
EDNS	$\neg\neg\exists x \varphi(x) \to \exists x \neg\neg\varphi(x)$	$\neg\neg\varphi(c) \to \exists x \neg\neg\varphi(x)$
CD	$\forall x(\varphi(x) \vee \psi) \to \forall x \varphi(x) \vee \psi$	$\forall x(\varphi(x) \vee \psi) \to \varphi(c) \vee \psi$

[1] By this we mean the predicate logics QLC and QAJ from (Gabbay et al., 2009), axiomatized by $\forall \bar{x}\big((\varphi(x) \to \psi(\bar{x})) \vee (\psi(\bar{x}) \to \varphi(\bar{x}))\big)$ and $\forall \bar{x}\big(\neg\varphi(\bar{x}) \vee \neg\neg\varphi(\bar{x})\big)$, respectively.

[2] These principles can be found in (Gabbay et al., 2009): DNS is shown to be equivalent, over IQC, to the principle KF, which is $\neg\neg\forall x(\varphi(x) \vee \neg\varphi(x))$; the strong Markov principle SMP appears under the name Ma.

As mentioned above, in this paper we are not concerned with developing alternative methods but rather with proving that certain alternatives cannot obtain for certain logics. The question then is what one requires of such an alternative method, and the answer to that question clearly depends on the application one has in mind. Starting point in this paper is the idea that an alternative method $(\cdot)^a$ should produce a formula without strong quantifiers and that a logic L *admits* this method if

$$\vdash_{\mathsf{L}} \varphi \Leftrightarrow \vdash_{\mathsf{L}} \varphi^a. \tag{1}$$

In this way, the alternative method provides a connection between the propositional fragment of L and L itself, at least in case the logic admits some form of a Herbrand Theorem, by which we mean a translation $(\cdot)^h$ such that $\varphi^h \in \mathcal{F}_{nw}$ and for all $\varphi \in \mathcal{F}_{ns}$:

$$\vdash_{\mathsf{L}} \varphi \Leftrightarrow \vdash_{\mathsf{L}} \varphi^h.$$

Therefore, requirement (1) seems a reasonable one. However, if no further requirements are made, then the notion trivializes in the sense that every logic with $\top$ and $\bot$ admits at least one alternative Skolemization method:

$$\varphi^a \equiv_{def} \begin{cases} \top & \text{if } \vdash_{\mathsf{L}} \varphi \\ \bot & \text{if } \nvdash_{\mathsf{L}} \varphi. \end{cases}$$

This is the reason that alternative methods are required to be *computable* as well.

We show in this paper that under the mild condition of strictness, to be defined in Section 5, there is no intermediate logic except CQC that is sound and complete with respect to a class of frames and that admits a strict, alternative Skolemization method. Thus implying that the logic IQC, the predicate versions of the Gabbay-deJongh logics, the predicate version of DeMorgan logic and Gödel–Dummett logic, as given in (Gabbay et al., 2009), and all tabular logics, which are the logics of a single frame, do not admit any strict, alternative Skolemization method.

5 Alternative Skolemization methods

An *alternative Skolemization method* is a computable total translation $(\cdot)^a$ from formulas to formulas such that for all formulas φ, φ^a does not contain strong quantifiers. A logic L *admits* the alternative Skolemization method if

$$\vdash \varphi \Leftrightarrow \vdash \varphi^a. \tag{2}$$

The method is *strict* if for every Kripke model K of L and all formulas φ:

$$K \not\Vdash \varphi^a \Rightarrow K \not\Vdash \varphi. \tag{3}$$

Clearly, standard Skolemization is an alternative Skolemization method, and CQC admits that method since $\varphi \to \varphi^s$ holds in CQC. An example of a different alternative Skolemization method is the one where occurrences of strong quantifiers $Qx\psi(x, \bar{y})$ are replaced by $\psi(f(\bar{y})) \vee \psi(g(\bar{y}))$ for fresh distinct f and g. Note that this method, a special case of the *parallel Skolemization method* introduced in (Baaz and Iemhoff, 2016), is strict, as is parallel Skolemization. On the other hand, the *existence Skolemization method* from (Baaz and Iemhoff, 2006, 2011) is not strict.

Note that the form of Skolemization that we consider here does not take into account the identity axioms for Skolem functions as is usually done in the setting of model theory. This strengthens our results in the sense that if the problematic direction from right to left in (2) fails to hold, it does so too if we allow the identity axioms for Skolem functions on the right.

The requirement of computability alone does not suffice to prove that intuitionistic logic does not admit alternative Skolemization methods, as the following translation satisfies (2): $\varphi^a = (\psi_1 \to \psi_2)$, where ψ_1 consists of a conjunction of defining axioms for suitable primitive recursive functions that imply ψ_2, which is a coded statement that φ is provable in IQC, exactly whenever φ is provable in IQC. Since ψ_1 and ψ_2 can be defined in such a way that the first is a universal and the second an existential formula, the translation thus defined is an alternative Skolemization method. It is, however, not strict.

6 The strong and the weak quantifier fragments

Given a Kripke model K (recalling that they are assumed to be rooted), $K^{\downarrow}$ denotes the Kripke model that is the result of replacing every domain in K by the domain at the root of K and $K^{\uparrow}$ denotes the Kripke model that is the result of replacing every domain in K by the union of all domains in K. For predicates $P(\bar{x})$ and nodes k, we put $K^{\downarrow}, k \Vdash P(\bar{d})$ precisely if $\bar{d}$ consists of elements in D and $K, k \Vdash P(\bar{d})$, and we put $K^{\uparrow}, k \Vdash P(\bar{d})$ precisely if $\bar{d}$ consists of elements in D_k and $K, k \Vdash P(\bar{d})$.

Lemma 6.1 Let K be a rooted Kripke model, which root has domain D. Then the following holds for all k in K. Recall that $\bar{d}$ is short for $d_1, \ldots, d_n$, and $\bar{d} \in D$ means that $d_i \in D$ for all $i \leq n$.

1. For all formulas $\varphi(\bar{x}) \in \mathcal{F}_{nw}$, for all $\bar{d} \in D$: $K, k \Vdash \varphi(\bar{d}) \Rightarrow\; K^{\downarrow}, k \Vdash \varphi(\bar{d})$.
2. For all formulas $\varphi(\bar{x}) \in \mathcal{F}_{nw}$, for all $\bar{d} \in D_k$: $K, k \nVdash \varphi(\bar{d}) \Rightarrow\; K^{\uparrow}, k \nVdash \varphi(\bar{d})$.
3. For all formulas $\varphi(\bar{x}) \in \mathcal{F}_{ns}$, for all $\bar{d} \in D_k$: $K, k \Vdash \varphi(\bar{d}) \Rightarrow\; K^{\uparrow}, k \Vdash \varphi(\bar{d})$.
4. For all formulas $\varphi(\bar{x}) \in \mathcal{F}_{ns}$, for all $\bar{d} \in D$: $K, k \nVdash \varphi(\bar{d}) \Rightarrow\; K^{\downarrow}, k \nVdash \varphi(\bar{d})$.

Proof The four properties are proved simultaneously, by formula induction. For atomic formulas $\varphi(\bar{x})$ the lemma follows by definition. The case where φ is a conjunction or disjunction follows immediately from the induction hypothesis.

Suppose $\varphi(\bar{x}) = \varphi_1(\bar{x}) \to \varphi_2(\bar{x})$. For 1., assume $\varphi \in \mathcal{F}_{nw}$ and $K, k \Vdash \varphi(\bar{d})$ for some $\bar{d} \in D$, and consider $l \succcurlyeq k$ such that $K^{\downarrow}, l \Vdash \varphi_1(\bar{d})$. Because φ_1 does not contain strong quantifiers, it follows from 4. that $K, l \Vdash \varphi_1(\bar{d})$. Hence $K, l \Vdash \varphi_2(\bar{d})$, and thus $K^{\downarrow}, l \Vdash \varphi_2(\bar{d})$ by 1. and the fact that φ_2 does not contain weak quantifiers.

For 2., assume $\varphi \in \mathcal{F}_{nw}$ and $K, k \nVdash \varphi(\bar{d})$ for some $\bar{d} \in D_k$ and consider $l \succcurlyeq k$ such that $K, l \Vdash \varphi_1(\bar{d})$ and $K, l \nVdash \varphi_2(\bar{d})$. Because φ_1 does not contain strong quantifiers, it follows from 3. that $K^{\uparrow}, l \Vdash \varphi_1(\bar{d})$. Because φ_2 does not contain weak quantifiers, $K^{\uparrow}, l \nVdash \varphi_2(\bar{d})$ follows from 2. Thus $K^{\uparrow}, k \nVdash \varphi_1(\bar{d}) \to \varphi_2(\bar{d})$. The proofs of 3. and 4. are analogous.

Suppose $\varphi(\bar{y}) = \forall x\psi(x,\bar{y})$. For 1., assume $\varphi \in \mathcal{F}_{nw}$ and $K, k \Vdash \forall x\psi(x,\bar{e})$ for some $\bar{e} \in D$ and consider $l \succcurlyeq k$ and $d \in D$. By the induction hypothesis and the fact that ψ does not contain weak quantifiers and D is the domain at the root of K, it follows that $K^{\downarrow}, l \Vdash \psi(d,\bar{e})$. Hence $K^{\downarrow}, k \Vdash \forall x\psi(x,\bar{e})$. For 2., assume $\varphi \in \mathcal{F}_{nw}$ and $K, k \nVdash \forall x\psi(x,\bar{e})$ for some $\bar{e} \in D_k$ and consider $l \succcurlyeq k$ and $d \in D_l$ such that $K, l \nVdash \psi(d,\bar{e})$. By the induction hypothesis and the fact that ψ does not contain weak quantifiers, it follows that $K^{\uparrow}, l \nVdash \psi(d,\bar{e})$. Hence $K^{\uparrow}, k \nVdash \forall x\psi(x,\bar{e})$. Cases 3. and 4. do not apply, as φ contains a strong quantifier.

Suppose $\varphi = \exists x\psi(x)$. Cases 1. and 2. do not apply, as φ contains a weak quantifier. For 3., assume $\varphi \in \mathcal{F}_{ns}$ and $K, k \Vdash \exists x\psi(x,\bar{e})$ for some $\bar{e} \in D_k$ and consider $d \in D_k$ such that $K, k \Vdash \psi(d)$. By the induction hypothesis and the fact that ψ does not contain strong quantifiers, it follows that $K^{\uparrow}, k \Vdash \psi(d)$. Hence $K^{\uparrow}, k \Vdash \exists x\psi(x,\bar{e})$. For 4., assume $\varphi \in \mathcal{F}_{ns}$ and $K, k \nVdash \exists x\psi(x,\bar{e})$ for some $\bar{e} \in D$. Thus for all $d \in D$, $K, k \nVdash \psi(d,\bar{e})$. Since ψ does not contain strong quantifiers the induction hypothesis gives $K^{\downarrow}, k \nVdash \psi(d,\bar{e})$ for all $d \in D$. Hence $K^{\downarrow}, k \nVdash \exists x\psi(x,\bar{e})$. ⊣

Theorem 6.2 Let L be a logic that is sound and complete with respect to a class of Kripke models $\mathcal{K}$ which is closed under $\downarrow$ and $\uparrow$, then the strong quantifier free fragment of L is sound and complete with respect to $\mathcal{K}_{cd}$. And so is the weak quantifier free fragment of L.

Proof For the first case, suppose that φ is a formula without strong quantifiers that is not derivable. Thus there is a model K in $\mathcal{K}$ that refutes φ. Let $\mathrm{Sub}_{\mathrm{neg}}(\varphi)$ and $\mathrm{Sub}_{\mathrm{pos}}(\varphi)$ denote the formulas that occur in φ negatively and positively, respectively. It suffices to show that

1. For all $\psi(\bar{x}) \in \mathrm{Sub}_{\mathrm{pos}}(\varphi)$, for all $\bar{d}$ in D: $K, k \nVdash \psi(\bar{d}) \Rightarrow K^{\downarrow}, k \nVdash \psi(\bar{d})$.
2. For all $\psi(\bar{x}) \in \mathrm{Sub}_{\mathrm{neg}}(\varphi)$, for all $\bar{d}$ in D: $K, k \Vdash \psi(\bar{d}) \Rightarrow K^{\downarrow}, k \Vdash \psi(\bar{d})$.

This follows from the previous lemma, using the fact that $\mathrm{Sub}_{\mathrm{pos}}(\varphi) \subseteq \mathcal{F}_{ns}$ and $\mathrm{Sub}_{\mathrm{neg}}(\varphi) \subseteq \mathcal{F}_{nw}$.

The second case is similar, using $K^{\uparrow}$ instead of $K^{\downarrow}$. ⊣

Corollary 6.3 Except for CQC, there is no intermediate logic that is sound and complete with respect to a class of frames and that admits a strict, alternative Skolemization method.

Proof Consider an intermediate logic L that is sound and complete with respect to a class of frames, that is not equal to CQC, and that admits an alternative Skolemization method $(\cdot)^a$ that is strict. We show how this leads to a contradiction. Let $\mathcal{K}$ be the class of Kripke models based on the frames in the given class.

First, we show that L is sound and complete with respect to the class $\mathcal{K}_{cd}$ of models in $\mathcal{K}$ that have constant domains:

$$\vdash_{\mathsf{L}} \varphi \Leftrightarrow \ \forall K \in \mathcal{K}_{cd}(K \Vdash \varphi).$$

$$\begin{array}{lll} 1 & \Vdash \varphi(d) \;\; \nVdash \varphi(e) \;\; \Vdash \psi & D_1 = \{d, e\} \\ | & & \\ 0 & \Vdash \varphi(d) \;\; \nVdash \psi & D_0 = \{d\} \end{array}$$

Figure 1: Model that refutes CD

The direction from left to right is trivial. The other direction is easy too: If $\nvdash \varphi$, then $\nvdash \varphi^a$, and so $K \nVdash \varphi^a$ for some $K \in \mathcal{K}$. Therefore $K^{\downarrow} \nVdash \varphi^a$ by Lemma 6.1. Thus $K^{\downarrow} \nVdash \varphi$ by strictness, and since $K^{\downarrow} \in \mathcal{K}_{cd}$, this completes the argument.

Having proven that L is sound and complete with respect to $\mathcal{K}_{cd}$, it follows that the constant domain formula CD (Section 4) holds in L, as it holds in all models with constant domains. However, if $\mathsf{L} \neq \mathsf{CQC}$, then its class of frames contains at least one frame in which at least one node has a successor. Since on such a frame there exists a model that refutes CD, as in Figure 1, CD does not hold in L. The desired contradiction has been obtained. $\dashv$

Let $\mathsf{QD_n}$ be the intermediate predicate logic of the frames of branching at most n, let QKC be the logic of the frames with one maximal node, and QLC be the logic of linear frames.

Corollary 6.4 The logics IQC, $\mathsf{QD_n}$, QKC, QLC, and all tabular logics, do not admit any strict, alternative Skolemization method.

Note that the constant domain logics, such as the Gödel logics, are not covered by Corollary 6.4, as they are not complete with respect to a class of frames, but with respect to the constant domain models on a certain class of frames.

We close with a short observation about logics that do not admit any strict alternative Skolemization method. Suppose that for such a logic there is an alternative method $(\cdot)^a$ that it admits, and that the proof of this fact is semantical, showing that for every countermodel K to φ there is a countermodel K' to φ^a and vice versa. Then from the fact that the method cannot be strict, and thus cannot satisfy (3), it follows that not in all cases one can take K for K', as one could do in CQC.

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