STABLE CANONICAL RULES

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Abstract. We introduce stable canonical rules and prove that each normal modal rule system is axiomatizable by stable canonical rules. This solves an open problem of Jeřábek [15, p. 1204]. We apply these results to construct finite refutation patterns for each modal formula that is not derivable in the basic modal logic **K**, and prove that each normal modal logic is axiomatizable by stable canonical rules. This solves an open problem of Chagrov and Zakharyaschev [12, Ch. 9, p. 332, Prob. 9.5], but our solution is by means of multiple-conclusion rules rather than formulas.

1. Introduction

It is a well-known result of Zakharyaschev [32] that each normal extension of **K4** is axiomatizable by canonical formulas. This result was generalized in two directions. In [2] it was generalized to all normal extensions of **wK4**, and in [15] Zakharyaschev's canonical formulas were generalized to multiple-conclusion canonical rules and it was proved that each normal modal rule system over **K4** is axiomatizable by canonical rules.

The key ingredients of Zakharyaschev's technique include the concepts of subreduction, closed domain condition, and selective filtration. While selective filtration is very effective in the transitive case [12], and also generalizes to the weakly transitive case [4, 2], it is less effective for **K**. This is one of the reasons why canonical formulas and rules do not work well for **K** [12, 15]. In [3] a different approach to canonical formulas for intuitionistic logic was developed that uses the technique of filtration instead of selective filtration. The new canonical formulas were called stable canonical formulas, and it was shown that each superintuitionistic logic is axiomatizable by stable canonical formulas.

In this paper we generalize the technique of [3] to the modal setting. Since the technique of filtration works well for **K**, we show that this new technique is effective in the non-transitive case as well. We introduce stable canonical rules and show that each normal modal rule system is axiomatizable by stable canonical rules. This solves an open problem of Jeřábek [15, p. 1204]. It also allows us to construct finite refutation patterns for each modal formula that is not derivable in **K**, and to prove that each normal modal logic is axiomatizable by stable canonical rules. This yields a solution of an open problem of Chagrov and Zakharyaschev [12, Ch. 9, p. 332, Prob. 9.5], but our solution is by means of multiple-conclusion rules rather than formulas. For normal extensions of **K4** we show that stable canonical rules can be replaced by stable canonical formulas, thus providing an alternative to Zakharyaschev's axiomatization [32].

The paper is organized as follows. In Section 2 we recall basic facts about modal logics, modal algebras, modal spaces (descriptive Kripke frames), and modal rules systems. In Section 3 we introduce stable homomorphisms, their dual stable maps, and the closed domain condition (CDC) for stable maps. Section 4 provides an algebraic approach to filtrations and connects them with (CDC). The main results of the paper are proved in Section 5. We give finite refutation patterns

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for normal modal rule systems and normal modal logics. We introduce stable canonical rules and prove that every normal modal rule system and every normal modal logic is axiomatizable by stable canonical rules, thus solving open problems of Jeřábek and of Chagrov and Zakharyaschev. In Section 6 we provide an algebraic approach to transitive filtrations, introduce stable canonical formulas for **K4**, and prove that every normal extension of **K4** is axiomatizable by stable canonical formulas. This provides an alternative to Zakharyaschev's axiomatization. Finally, Section 7 defines stable rule systems and stable logics and proves that all stable rule systems and stable logics have the finite model property. It also gives a characterization of all splitting rule systems and splitting logics by means of Jankov rules and Jankov formulas, thus yielding alternative proofs of the results of Jeřábek and Blok, respectively.

2. Preliminaries

In this section we briefly discuss some of the basic facts that will be used throughout the paper. We use [12, 16, 9, 29] as our main references for the basic theory of normal modal logics, including their algebraic and relational semantics, and the dual equivalence between modal algebras and modal spaces (descriptive Kripke frames). We also use [11] for universal algebra, [25, 17] for modal rules, and [15, 8] for multiple-conclusion modal rules.

Modal logic. We recall that a *normal modal logic* is the set of formulas of the basic modal language containing classical tautologies and $\Box(p \to q) \to (\Box p \to \Box q)$, and closed under Modus Ponens $(\varphi, \varphi \to \psi/\psi)$, Necessitation $(\varphi/\Box\varphi)$, and Substitution $(\varphi(p_1,\dots,p_n)/\varphi(\psi_1,\dots,\psi_n))$. The least normal modal logic is denoted by $\mathbf{K}$, and $\Lambda(\mathbf{K})$ denotes the complete lattice of normal modal logics.

A *modal algebra* is a pair $\mathfrak{A} = (A, \Diamond)$, where A is a Boolean algebra and $\Diamond$ is a unary function on A that commutes with finite joins. As usual, the dual operator $\Box$ is defined as $\neg\Diamond\neg$. A *modal homomorphism* between two modal algebras is a Boolean homomorphism h satisfying $h(\Diamond a) = \Diamond h(a)$. Let MA be the category of modal algebras and modal homomorphisms.

A *modal space* (or *descriptive Kripke frame*) is a pair $\mathfrak{X} = (X, R)$, where X is a Stone space (zero-dimensional compact Hausdorff space) and R is a binary relation on X satisfying

$$R(x) := \{y \in X : xRy\}$$

is closed for each $x \in X$ and

$$R^{-1}(U) := \{x \in X : \exists y \in U \text{ with } xRy\}$$

is clopen (closed and open) for each clopen U of X. A *bounded morphism* (or *p-morphism*) between two modal spaces is a continuous map f such that $f(R(x)) = R(f(x))$. Let MS be the category of modal spaces and bounded morphisms.

It is a well-known theorem in modal logic that MA is dually equivalent to MS. The functors $(-)_* : \mathsf{MA} \to \mathsf{MS}$ and $(-)^* : \mathsf{MS} \to \mathsf{MA}$ that establish this dual equivalence are constructed as follows. For a modal algebra $\mathfrak{A} = (A, \Diamond)$, let $\mathfrak{A}_* = (X, R)$, where X is the Stone space of A (that is, the set of ultrafilters of A topologized by the basis $\{\varphi(a) : a \in A\}$, where $\varphi(a) = \{x \in X : a \in x\}$) and xRy iff $(\forall a \in A)(a \in y \Rightarrow \Diamond a \in x)$. We call R the *dual* of $\Diamond$. For a modal homomorphism h, let $h_* = h^{-1}$. For a modal space $\mathfrak{X} = (X, R)$, let $\mathfrak{X}^* = (A, \Diamond)$, where A is the Boolean algebra of clopens of X and $\Diamond(U) = R^{-1}(U)$. For a bounded morphism f, let $f^* = f^{-1}$.

Let $\mathfrak{A} = (A, \Diamond)$ be a modal algebra and let $\mathfrak{X} = (X, R)$ be its dual space. Then it is well known that R is reflexive iff $a \le \Diamond a$, and R is transitive iff $\Diamond\Diamond a \le \Diamond a$. A modal algebra $\mathfrak{A}$ is a **K4**-*algebra* if $\Diamond\Diamond a \le \Diamond a$ holds in $\mathfrak{A}$, and it is an **S4**-*algebra* if in addition $a \le \Diamond a$ holds in $\mathfrak{A}$. **S4**-algebras are also known as closure algebras, interior algebras, or topological Boolean algebras. Let $\mathsf{K4}$ be the full subcategory of MA consisting of **K4**-algebras, and let $\mathsf{S4}$ be the full subcategory of $\mathsf{K4}$ consisting of **S4**-algebras. A modal space $\mathfrak{X} = (X, R)$ is a *transitive space* if R is transitive, and it is a *quasi-ordered space* if R is reflexive and transitive. Let TS be the full subcategory of MS

consisting of transitive spaces, and let QS be the full subcategory of TS consisting of quasi-ordered spaces. Then the dual equivalence of MA and MS restricts to the dual equivalence of $\mathsf{K4}$ and TS, which restricts further to the dual equivalence of $\mathsf{S4}$ and QS.

For a modal algebra $\mathfrak{A} = (A, \Diamond)$ and $a \in A$, define $\Diamond^0 a = a$ and $\Diamond^{n+1} a = \Diamond\Diamond^n a$, and set $\blacklozenge_n a = \bigvee_{k \leq n} \Diamond^k a$. We define $\Box^n$ and $\blacksquare_n$ similarly. By Rautenberg's criterion [24], $\mathfrak{A}$ is subdirectly irreducible (s.i. for short) iff there is $c \neq 1$ such that for each $a \neq 1$ there is $n \in \omega$ with $\blacksquare_n a \leq c$. Such a c is called an *opremum* of $\mathfrak{A}$. It is not unique.

If $\mathfrak{A}$ is a **K4**-algebra, then for each $n \geq 1$, we have $\blacklozenge_n a = a \vee \Diamond a$ and $\blacksquare_n a = a \wedge \Box a$. Let $\Diamond^+ a := a \vee \Diamond a$ and $\Box^+ a := a \wedge \Box a$. Then $\mathfrak{A}^+ = (A, \Diamond^+)$ is an **S4**-algebra, the set H of fixed points of $\Box^+$ forms a Heyting algebra, and up to isomorphism, each Heyting algebra arises this way. Therefore, a **K4**-algebra $\mathfrak{A}$ is s.i. iff the **S4**-algebra $\mathfrak{A}^+$ is s.i., which happens iff the Heyting algebra H is s.i., which in turn means that there is a greatest element c in $H - \{1\}$.

A filter F of a modal algebra $\mathfrak{A} = (A, \Diamond)$ is called a *modal filter* if $a \in F$ implies $\Box a \in F$. It is well known that there is a 1-1 correspondence between congruences and modal filters of $\mathfrak{A}$, hence homomorphic images of $\mathfrak{A}$ are determined by modal filters. For a modal space $\mathfrak{X} = (X, R)$, a subset Y of X is called an *up-set* if from $x \in Y$ and xRy it follows that $y \in Y$. If $\mathfrak{A}$ is a modal algebra and $\mathfrak{X}$ is its dual space, then modal filters of $\mathfrak{A}$ correspond to closed up-sets of $\mathfrak{X}$. Thus, homomorphic images of $\mathfrak{A}$ are determined by closed up-sets of $\mathfrak{X}$.

For a modal space $\mathfrak{X} = (X, R)$ and $Y \subseteq X$, let $R^0(Y) = Y$, $R^{n+1}(Y) = R(R^n(Y))$, and $R^\omega(Y) = \bigcup_{n \in \omega} R^n(Y)$. If Y is a singleton $\{x\}$, then we write $R^n(x)$ and $R^\omega(x)$. We call $\mathfrak{X}$ *rooted* provided there is $x \in X$, called a *root* of $\mathfrak{X}$, such that $X = R^\omega(x)$. Note that if R is transitive, then $R^\omega(x) = \{x\} \cup R(x)$ and if R is reflexive and transitive, then $R^\omega(x) = R(x)$. By [26, Thm. 3.1], a finite modal algebra $\mathfrak{A}$ is s.i. iff its dual modal space $\mathfrak{X}$ is rooted. This result extends to the infinite case as follows [28]: Let $\mathfrak{X}$ be a modal space. Call $x \in X$ a *topo-root* of $\mathfrak{X}$ if X is the closure of $R^\omega(x)$. Then a modal algebra $\mathfrak{A}$ is s.i. iff in its dual modal space $\mathfrak{X}$ the set of topo-roots has a nonempty interior [28, Thm. 2]. We call such modal spaces *topo-rooted*.

Multiple-conclusion modal rules. A *multiple-conclusion modal rule* is an expression Γ/Δ, where Γ, Δ are finite sets of modal formulas. If $\Delta = \{\varphi\}$, then Γ/Δ is called a *single-conclusion modal rule* and is written Γ/φ. If $\Gamma = \varnothing$, then Γ/Δ is called an *assumption-free modal rule* and is written $/\Delta$. Assumption-free single-conclusion modal rules $/\varphi$ can be identified with modal formulas φ.

A *normal modal rule system* is a set $\mathcal{S}$ of modal rules such that

(1) $\varphi/\varphi \in \mathcal{S}$.
(2) $\varphi, \varphi \to \psi/\psi \in \mathcal{S}$.
(3) $\varphi/\Box\varphi \in \mathcal{S}$.
(4) $/\varphi \in \mathcal{S}$ for each theorem φ of **K**.
(5) If $\Gamma/\Delta \in \mathcal{S}$, then $\Gamma, \Gamma'/\Delta, \Delta' \in \mathcal{S}$.
(6) If $\Gamma/\Delta, \varphi \in \mathcal{S}$ and $\Gamma, \varphi/\Delta \in \mathcal{S}$, then $\Gamma/\Delta \in \mathcal{S}$.
(7) If $\Gamma/\Delta \in \mathcal{S}$ and s is a substitution, then $s(\Gamma)/s(\Delta) \in \mathcal{S}$.

We denote the least normal modal rule system by $\mathbf{S_K}$, and the complete lattice of normal modal rule systems by $\Sigma(\mathbf{S_K})$. For a set Ξ of multiple-conclusion modal rules, let $\mathbf{S_K} + \Xi$ be the least normal modal rule system containing Ξ. If $\mathcal{S} = \mathbf{S_K} + \Xi$, then we say that $\mathcal{S}$ is *axiomatizable* by Ξ, and if Ξ is finite, then we call $\mathcal{S}$ *finitely axiomatizable*. If $\rho \in \mathcal{S}$, then we say that the normal modal rule system $\mathcal{S}$ *entails* or *derives* the modal rule ρ, and write $\mathcal{S} \vdash \rho$.

Given a normal modal rule system $\mathcal{S}$, let $\Lambda(\mathcal{S}) = \{\varphi : /\varphi \in \mathcal{S}\}$ be the corresponding normal modal logic, and for a normal modal logic L, let $\Sigma(L) = \mathbf{S_K} + \{/\varphi : \varphi \in L\}$ be the corresponding normal modal rule system. Then $\Lambda : \Sigma(\mathbf{S_K}) \to \Lambda(\mathbf{K})$ and $\Sigma : \Lambda(\mathbf{K}) \to \Sigma(\mathbf{S_K})$ are order-preserving maps such that $\Lambda(\Sigma(L)) = L$ for each $L \in \Lambda(\mathbf{K})$ and $\mathcal{S} \supseteq \Sigma(\Lambda(\mathcal{S}))$ for each $\mathcal{S} \in \Sigma(\mathbf{S_K})$. We say

that a normal modal logic L is *axiomatized* (over $\mathbf{K}$) by a set Ξ of multiple-conclusion modal rules if $L = \Lambda(\mathbf{S_K} + \Xi)$.

A modal algebra $\mathfrak{A} = (A, \Diamond)$ *validates* a multiple-conclusion modal rule Γ/Δ provided for every valuation V on A, if $V(\gamma) = 1$ for all $\gamma \in \Gamma$, then $V(\delta) = 1$ for some $\delta \in \Delta$. Otherwise $\mathfrak{A}$ *refutes* Γ/Δ. If $\mathfrak{A}$ validates Γ/Δ, we write $\mathfrak{A} \models \Gamma/\Delta$, and if $\mathfrak{A}$ refutes Γ/Δ, we write $\mathfrak{A} \not\models \Gamma/\Delta$. If $\Gamma = \{\phi_1, \dots, \phi_n\}$, $\Delta = \{\psi_1, \dots, \psi_m\}$, and $\phi_i(\underline{x})$ and $\psi_j(\underline{x})$ are the terms in the first-order language of modal algebras corresponding to the ϕ_i and ψ_j, then $\mathfrak{A} \models \Gamma/\Delta$ iff $\mathfrak{A}$ is a model of the universal sentence $\forall \underline{x}\ (\bigwedge_{i=1}^n \phi_i(\underline{x}) = 1 \to \bigvee_{j=1}^m \psi_j(\underline{x}) = 1)$. Consequently, normal modal rule systems correspond to universal classes of modal algebras. It is well known (see, e.g., [11, Thm. V.2.20]) that a class of modal algebras is a universal class iff it is closed under isomorphisms, subalgebras, and ultraproducts.

On the other hand, normal modal logics correspond to equationally definable classes of modal algebras; that is, models of the sentences $\forall \underline{x}\ \phi(\underline{x}) = 1$ in the first-order language of modal algebras. It is well known (see, e.g., [11, Thm. II.11.9]) that a class of modal algebras is an equational class iff it is a variety (that is, it is closed under homomorphic images, subalgebras, and products).

We also point out that a modal algebra $\mathfrak{A}$ validates a single-conclusion modal rule Γ/ψ iff $\mathfrak{A}$ is a model of the sentence $\forall \underline{x}\ (\bigwedge_{i=1}^n \phi_i(\underline{x}) = 1 \to \psi(\underline{x}) = 1)$, where $\Gamma = \{\phi_1, \dots, \phi_n\}$ and $\phi_i(\underline{x})$ and $\psi(\underline{x})$ are the terms in the first-order language of modal algebras corresponding to the ϕ_i and ψ. Consequently, single-conclusion normal modal rule systems, which are also known as *normal modal consequence relations*, correspond to universal Horn classes of modal algebras. It is well known (see, e.g., [11, Thm. V.2.25]) that a class of modal algebras is a universal Horn class iff it is a quasi-variety (that is, it is closed under isomorphisms, subalgebras, products, and ultraproducts).

For a normal modal rule system $\mathcal{S}$, we denote by $\mathcal{U}(\mathcal{S})$ the universal class of modal algebras corresponding to $\mathcal{S}$, and for a universal class of modal algebras $\mathcal{U}$, we denote by $\mathcal{S}(\mathcal{U})$ the normal modal rule system corresponding to $\mathcal{U}$. Then $\mathcal{S}(\mathcal{U}(\mathcal{S})) = \mathcal{S}$ and $\mathcal{U}(\mathcal{S}(\mathcal{U})) = \mathcal{U}$. This yields an isomorphism between $\Sigma(\mathbf{S_K})$ and the complete lattice $\mathsf{U}(\mathsf{MA})$ of universal classes of modal algebras (ordered by reverse inclusion).

Similarly, for a normal modal logic L, let $\mathcal{V}(L)$ denote the variety of modal algebras corresponding to L, and for a variety $\mathcal{V}$, let $L(\mathcal{V})$ denote the normal modal logic corresponding to $\mathcal{V}$. Then $L(\mathcal{V}(L)) = L$ and $\mathcal{V}(L(\mathcal{V})) = \mathcal{V}$, yielding an isomorphism between $\Sigma(\mathbf{K})$ and the complete lattice $\mathsf{V}(\mathsf{MA})$ of varieties of modal algebras (ordered by reverse inclusion).

Under this correspondence, for a modal rule system $\mathcal{S}$, the variety $\mathcal{V}(\Lambda(\mathcal{S}))$ corresponding to the modal logic $\Lambda(\mathcal{S})$ is the variety generated by the universal class $\mathcal{U}(\mathcal{S})$. We will utilize this fact later on in the paper.

3. Stable homomorphisms and the closed domain condition

In this section we introduce the key concepts of stable homomorphisms and the closed domain condition, and show how the two relate to each other.

Definition 3.1. *Let $\mathfrak{A} = (A, \Diamond)$ and $\mathfrak{B} = (B, \Diamond)$ be modal algebras and let $h : A \to B$ be a Boolean homomorphism. We call h a* stable homomorphism *provided $\Diamond h(a) \leq h(\Diamond a)$ for each $a \in A$.*

It is easy to see that $h : A \to B$ is stable iff $h(\Box a) \leq \Box h(a)$ for each $a \in A$. Stable homomorphisms were considered in [5] under the name of semi-homomorphisms and in [14] under the name of continuous morphisms.

Definition 3.2. *Let $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, R)$ be modal spaces and let $f : X \to Y$ be a continuous map. We call f* stable *if xRy implies $f(x)Rf(y)$.*

Lemma 3.3. *Let $\mathfrak{A} = (A, \Diamond)$ and $\mathfrak{B} = (B, \Diamond)$ be modal algebras, $\mathfrak{X} = (X, R)$ be the dual of $\mathfrak{A}$, $\mathfrak{Y} = (Y, R)$ be the dual of $\mathfrak{B}$, and $h : A \to B$ be a Boolean homomorphism. Then $h : A \to B$ is stable iff $h_* : Y \to X$ is stable.*

Proof. By Stone duality, it is sufficient to show that $\Diamond h(a) \leq h(\Diamond a)$ for each $a \in A$ iff xRy implies $h_*(x)Rh_*(y)$ for each $x, y \in Y$. First suppose that $\Diamond h(a) \leq h(\Diamond a)$ for each $a \in A$. Let $x, y \in Y$ with xRy, and let $a \in h_*(y)$. Then $h(a) \in y$. From xRy it follows that $\Diamond h(a) \in x$. Since $\Diamond h(a) \leq h(\Diamond a)$ and x is a filter, $h(\Diamond a) \in x$, hence $\Diamond a \in h_*(x)$. Therefore, $h_*(x)Rh_*(y)$.

Conversely, suppose xRy implies $h_*(x)Rh_*(y)$ for each $x, y \in Y$. Let $a \in A$ and let $x \in R^{-1}h_*^{-1}\varphi(a)$. Then there is $y \in Y$ such that xRy and $h_*(y) \in \varphi(a)$. Therefore, $h_*(x)Rh_*(y)$ and $h_*(y) \in \varphi(a)$. Thus, $h_*(x) \in R^{-1}\varphi(a)$, and so $x \in h_*^{-1}R^{-1}\varphi(a)$. This implies $R^{-1}h_*^{-1}\varphi(a) \subseteq h_*^{-1}R^{-1}\varphi(a)$. But $R^{-1}h_*^{-1}\varphi(a) = \varphi(\Diamond h(a))$ and $h_*^{-1}R^{-1}\varphi(a) = \varphi(h(\Diamond a))$. This yields $\varphi(\Diamond h(a)) \subseteq \varphi(h(\Diamond a))$, and since φ is an isomorphism, we conclude that $\Diamond h(a) \leq h(\Diamond a)$ for each $a \in A$. □

Definition 3.4. *Let $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, R)$ be modal spaces, $f : X \to Y$ be a map, and U be a clopen subset of Y. We say that f satisfies the* closed domain condition (CDC) for U *if*

$$R(f(x)) \cap U \neq \varnothing \Rightarrow f(R(x)) \cap U \neq \varnothing.$$

Let $\mathfrak{D}$ be a collection of clopen subsets of Y. We say that $f : X \to Y$ satisfies the closed domain condition (CDC) for $\mathfrak{D}$ *if f satisfies* (CDC) *for each $U \in \mathfrak{D}$.*

Lemma 3.5. *Let $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, R)$ be modal spaces, $f : X \to Y$ be a map, and U be a clopen subset of Y. Then the following two conditions are equivalent:*

1. *f satisfies* (CDC) *for U.*
2. *$f^{-1}R^{-1}U \subseteq R^{-1}f^{-1}U$.*

Proof. (1)⇒(2): Suppose that f satisfies (CDC) for U and $x \in f^{-1}R^{-1}U$. Then $R(f(x)) \cap U \neq \varnothing$. By (CDC), $f(R(x)) \cap U \neq \varnothing$. Thus, $x \in R^{-1}f^{-1}U$.

(2)⇒(1): Suppose that $f^{-1}R^{-1}U \subseteq R^{-1}f^{-1}U$ and $R(f(x)) \cap U \neq \varnothing$. Then $x \in f^{-1}R^{-1}U$. By (2), $x \in R^{-1}f^{-1}U$, which means that $f(R(x)) \cap U \neq \varnothing$. Thus, (CDC) is satisfied. □

Theorem 3.6. *Let $\mathfrak{A} = (A, \Diamond)$ and $\mathfrak{B} = (B, \Diamond)$ be modal algebras, $h : A \to B$ be a stable homomorphism, and $a \in A$. The following two conditions are equivalent:*

1. *$h(\Diamond a) = \Diamond h(a)$.*
2. *$h_* : B_* \to A_*$ satisfies* (CDC) *for $\varphi(a)$.*

Proof. Since $h : A \to B$ is a stable homomorphism, $\Diamond h(a) \leq h(\Diamond a)$. Therefore, $h(\Diamond a) = \Diamond h(a)$ iff $h(\Diamond a) \leq \Diamond h(a)$, which happens iff $h_*^{-1}R^{-1}\varphi(a) \subseteq R^{-1}h_*^{-1}\varphi(a)$. By Lemma 3.5, the last condition is equivalent to $h_* : B_* \to A_*$ satisfying (CDC) for $\varphi(a)$. □

Theorem 3.6 motivates the following definition.

Definition 3.7. *Let $\mathfrak{A} = (A, \Diamond)$ and $\mathfrak{B} = (B, \Diamond)$ be modal algebras and let $h : A \to B$ be a stable homomorphism.*

1. *We say that h satisfies* the closed domain condition (CDC) for $a \in A$ *if $h(\Diamond a) = \Diamond h(a)$.*
2. *We say that h satisfies* the closed domain condition (CDC) for $D \subseteq A$ *if h satisfies* (CDC) *for each $a \in D$.*

4. Filtrations and the closed domain condition

The filtration method is the main tool for establishing the finite model property in modal logic. The method can be developed either algebraically [22, 23] or frame-theoretically [20, 27], and the two are connected via duality [18, 19]. For a recent account of filtrations we refer to [14, 13]. In this section we give a slightly different account which is more suited for our purposes, and also discuss the connection with stable homomorphisms and the closed domain condition.

We start by recalling the frame-theoretic approach to filtrations (see, e.g., [9, Def. 2.36] or [12, Sec. 5.3]). Let $\mathfrak{M} = (X, R, V)$ be a Kripke model and let Σ be a set of formulas closed under

subformulas. For our purposes, Σ will always be assumed to be finite. Define an equivalence relation $\sim_\Sigma$ on X by

$$x \sim_\Sigma y \text{ iff } (\forall \varphi \in \Sigma)(x \models \varphi \Leftrightarrow y \models \varphi).$$

Let $X' = X/{\sim_\Sigma}$ and let $V'(p) = \{[x] : x \in V(p)\}$, where $[x]$ is the equivalence class of x with respect to $\sim_\Sigma$.

Definition 4.1. *For a binary relation R' on X', we say that the triple $\mathfrak{M}' = (X', R', V')$ is a* filtration of $\mathfrak{M}$ through Σ *if the following two conditions are satisfied:*

(F1) $xRy \Rightarrow [x]R'[y]$.
(F2) $[x]R'[y] \Rightarrow (\forall \Diamond\varphi \in \Sigma)(y \models \varphi \Rightarrow x \models \Diamond\varphi)$.

Note that if Σ is finite, then X' is finite. In fact, if Σ consists of n elements, then X' consists of no more than 2^n elements.

Let $\mathfrak{A} = (A, \Diamond)$ be a modal algebra and let $\mathfrak{X} = (X, R)$ be the dual of $\mathfrak{A}$. If V is a valuation on A, then by identifying A with the clopen subsets of X, we can view V as a valuation on X.

Theorem 4.2. *Let $\mathfrak{A} = (A, \Diamond)$ be a modal algebra and let $\mathfrak{X} = (X, R)$ be the dual of $\mathfrak{A}$. For a valuation V on A and a set of formulas Σ closed under subformulas, let A' be the Boolean subalgebra of A generated by $V(\Sigma) \subseteq A$ and let $D = \{V(\varphi) : \Diamond\varphi \in \Sigma\}$. For a modal operator $\Diamond'$ on A', the following two conditions are equivalent:*

(1) *The inclusion $(A', \Diamond') \rightarrowtail (A, \Diamond)$ is a stable homomorphism satisfying* (CDC) *for D.*
(2) *Viewing V as a valuation on X, there is a filtration $\mathfrak{M}' = (X', R', V')$ of $\mathfrak{M} = (X, R, V)$ through Σ such that R' is the dual of $\Diamond'$.*

Proof. Since A' is a Boolean subalgebra of A, it follows from Stone duality that the dual of A' can be described as the quotient of X by the equivalence relation given by $x \sim y$ iff $x \cap A' = y \cap A'$. As A' is generated by $V(\Sigma)$, we have $x \sim y$ iff $x \sim_\Sigma y$, so we identify the dual of A' with X'. Define V' on X' by $V'(p) = \{[x] : x \in V(p)\}$. Let $\Diamond'$ be a modal operator on A', and let $R' \subseteq X' \times X'$ be the dual of $\Diamond'$. By Lemma 3.3, $\mathfrak{M}' = (X', R', V')$ satisfies (F1) iff the inclusion $(A', \Diamond') \rightarrowtail (A, \Diamond)$ is a stable homomorphism. Therefore, it remains to see that $\mathfrak{M}'$ satisfies (F2) iff the inclusion $(A', \Diamond') \rightarrowtail (A, \Diamond)$ satisfies (CDC) for D. The former means that $[x]R'[y] \Rightarrow (\forall a \in D)(a \in y \Rightarrow \Diamond a \in x)$, and the latter means that $\Diamond' a = \Diamond a$ for each $a \in D$. First suppose that the inclusion satisfies (CDC) for D. Let $[x]R'[y]$, $a \in D$, and $a \in y$. Since $[x]R'[y]$, we have that $(\forall b \in A')(b \in y \Rightarrow \Diamond' b \in x)$. As $a \in D \subseteq A'$, from $a \in y$ it follows that $\Diamond' a \in x$. By (CDC) for D we see that $\Diamond' a = \Diamond a$, so $\Diamond a \in x$, and hence $\mathfrak{M}'$ satisfies (F2). Conversely, suppose that $\mathfrak{M}'$ satisfies (F2). Let $a \in D$. Since the inclusion $(A', \Diamond') \rightarrowtail (A, \Diamond)$ is stable, we have $\Diamond a \leq \Diamond' a$. Let $[x] \in \varphi(\Diamond' a)$. Then $[x] \in (R')^{-1}\varphi(a)$, so there is $[y] \in X'$ with $[x]R'[y]$ and $[y] \in \varphi(a)$. As $a \in D \subseteq A'$, from $[y] \in \varphi(a)$ it follows that $a \in y$. By (F2), this yields $\Diamond a \in x$. Therefore, since $\Diamond a \in A'$, we have $[x] \in \varphi(\Diamond a)$. Thus, $\varphi(\Diamond' a) \subseteq \varphi(\Diamond a)$, yielding $\Diamond' a \leq \Diamond a$. Consequently, $\Diamond' a = \Diamond a$ for each $a \in D$, and hence the embedding satisfies (CDC) for D. □

Theorem 4.2 motivates the following definition.

Definition 4.3. *Let $\mathfrak{A} = (A, \Diamond)$ be a modal algebra, V be a valuation on A, and Σ be a set of formulas closed under subformulas. Let A' be the Boolean subalgebra of A generated by $V(\Sigma) \subseteq A$ and let $D = \{V(\varphi) : \Diamond\varphi \in \Sigma\}$. Suppose that $\Diamond'$ is a modal operator on A' such that the inclusion $(A', \Diamond') \rightarrowtail (A, \Diamond)$ is a stable homomorphism satisfying* (CDC) *for D. Then we call $\mathfrak{A}' = (A', \Diamond')$ a* filtration *of $\mathfrak{A}$ through Σ.*

Lemma 4.4. *Let $\mathfrak{A}' = (A', \Diamond')$ be a filtration of $\mathfrak{A}$ through Σ and let V' be a restriction of V to A'. If $V(\varphi) \in D$, then $V(\varphi) = V'(\varphi)$.*

Proof. Easy induction on the complexity of φ. Since A' is a Boolean subalgebra of A, the proof for Boolean connectives is obvious, and since $(A', \Diamond') \rightarrowtail (A, \Diamond)$ is a stable embedding satisfying (CDC) for D, the proof for $\Diamond$ follows. □

Let $D^\vee$ denote the sub-join-semilattice of A' generated by D. Then $a \in D^\vee$ iff $a = \bigvee F$ for some finite subset F of D. In particular, $0 \in D^\vee$. It is easy to see that if $a \in D^\vee$, then $\Diamond a \in A'$.

Lemma 4.5. *Let $\mathfrak{A} = (A, \Diamond)$, V, Σ, A', and D be as above with Σ and hence A' finite. Define $\Diamond^l$ and $\Diamond^g$ on A' by*

$$\Diamond^l a = \bigwedge\{b \in A' : \Diamond a \le b\} \quad \text{and} \quad \Diamond^g a = \bigwedge\{\Diamond b : a \le b \ \&\ b \in D^\vee\}.$$

Then

(1) $\Diamond a \le \Diamond^l a \le \Diamond^g a$.
(2) *If $a \in D^\vee$, then $\Diamond a = \Diamond^l a = \Diamond^g a$.*
(3) *$(A', \Diamond^l)$ and $(A', \Diamond^g)$ are modal algebras.*
(4) *The inclusions of $(A', \Diamond^l)$ and $(A', \Diamond^g)$ into $\mathfrak{A}$ are stable.*
(5) *The inclusions of $(A', \Diamond^l)$ and $(A', \Diamond^g)$ into $\mathfrak{A}$ satisfy* (CDC) *for D.*
(6) *$(A', \Diamond^l)$ and $(A', \Diamond^g)$ are filtrations of $\mathfrak{A}$ through Σ.*
(7) *If $\mathfrak{A}' = (A', \Diamond')$ is a filtration of $\mathfrak{A}$ through Σ, then $\Diamond^l a \le \Diamond' a \le \Diamond^g a$ for each $a \in A'$.*

Proof. (1) It follows from the definition that $\Diamond a \le \Diamond^l a$. As $a \le b \Rightarrow \Diamond a \le \Diamond b$, we have $\{\Diamond b : a \le b \ \&\ b \in D^\vee\} \subseteq \{b \in A' : \Diamond a \le b\}$, so $\Diamond^l a \le \Diamond^g a$.

(2) If $a \in D^\vee$, then $\Diamond^g a \le \Diamond a$. This by (1) yields $\Diamond a = \Diamond^l a = \Diamond^g a$.

(3) Since $\Diamond 0 = 0$ and $0 \in A'$, it is clear that $\Diamond^l 0 = 0$. Moreover,

$$\begin{aligned}
\Diamond^l a \vee \Diamond^l b &= \textstyle\bigwedge\{x \in A' : \Diamond a \le x\} \vee \bigwedge\{y \in A' : \Diamond b \le y\} \\
&= \textstyle\bigwedge\{x \vee y : x, y \in A' \ \&\ \Diamond a \le x \ \&\ \Diamond b \le y\} \\
&= \textstyle\bigwedge\{z \in A' : \Diamond a \vee \Diamond b \le z\} \\
&= \textstyle\bigwedge\{z \in A' : \Diamond(a \vee b) \le z\} \\
&= \Diamond^l(a \vee b).
\end{aligned}$$

Therefore, $(B, \Diamond^l)$ is a modal algebra. As $\Diamond 0 = 0$ and $0 \in D^\vee$, by (2), $\Diamond^g 0 = 0$. Because $D^\vee$ is closed under finite joins,

$$\begin{aligned}
\Diamond^g a \vee \Diamond^g b &= \textstyle\bigwedge\{\Diamond x : a \le x \ \&\ x \in D^\vee\} \vee \bigwedge\{\Diamond y : b \le y \ \&\ y \in D^\vee\} \\
&= \textstyle\bigwedge\{\Diamond x \vee \Diamond y : a \le x \ \&\ b \le y \ \&\ x, y \in D^\vee\} \\
&= \textstyle\bigwedge\{\Diamond(x \vee y) : a \le x \ \&\ b \le y \ \&\ x, y \in D^\vee\} \\
&= \textstyle\bigwedge\{\Diamond z : a \vee b \le z \ \&\ z \in D^\vee\} \\
&= \Diamond^g(a \vee b).
\end{aligned}$$

Thus, $(B, \Diamond^g)$ is a modal algebra.

(4) follows from (1), (5) follows from (2), and (6) follows from (4) and (5).

(7) Suppose $\mathfrak{A}' = (A', \Diamond')$ is a filtration of $\mathfrak{A}$ through Σ. Let $a \in A'$. Since the inclusion $\mathfrak{A}' \rightarrowtail \mathfrak{A}$ is a stable homomorphism, we have $\Diamond a \le \Diamond' a$. Therefore, $\Diamond' a \in \{b \in A' : \Diamond a \le b\}$, which yields $\Diamond^l a \le \Diamond' a$. Let $b \in D^\vee$ with $a \le b$. Then $b = \bigvee F$ for some finite $F \subseteq D$. Since $\mathfrak{A}'$ is a modal algebra, $a \le b$ implies $\Diamond' a \le \Diamond' b = \Diamond' \bigvee F = \bigvee\{\Diamond' x : x \in F\}$. As the inclusion $\mathfrak{A}' \rightarrowtail \mathfrak{A}$ satisfies (CDC) for D, from $x \in F \subseteq D$ it follows that $\Diamond' x = \Diamond x$. Thus, $\Diamond' a \le \bigvee\{\Diamond x : x \in F\} = \Diamond \bigvee F = \Diamond b$, yielding $\Diamond' a \le \Diamond^g a$. □

As a consequence, we obtain that $(A', \Diamond^l)$ is the least filtration and $(A', \Diamond^g)$ is the greatest filtration of $\mathfrak{A}$ through the finite set of formulas Σ. We next show that these correspond to the least and greatest filtrations of the dual of $\mathfrak{A}$. We recall (see, e.g., [9, Sec. 2.3] or [12, Sec. 5.3]) that the least filtration of $\mathfrak{M} = (X, R, V)$ through Σ is $\mathfrak{M}^l = (X', R^l, V')$ and the greatest filtration is $\mathfrak{M}^g = (X, R^g, V')$, where

$[x]R^l[y]$ iff $(\exists x', y' \in X)(x \sim_\Sigma x' \ \&\ y \sim_\Sigma y' \ \&\ x'Ry')$.
$[x]R^g[y]$ iff $(\forall \Diamond\varphi \in \Sigma)(y \models \varphi \Rightarrow x \models \Diamond\varphi)$.

Lemma 4.6. *Let $\mathfrak{A} = (A, \Diamond)$, $\mathfrak{X} = (X, R)$, A', and X' be as in Theorem 4.2, with A' and X' finite. Then R^l on X' is the dual of $\Diamond^l$ on A' and R^g on X' is the dual of $\Diamond^g$ on A'.*

Proof. Let $R_{\Diamond^l}$ be the dual of $\Diamond^l$. Then $[x]R_{\Diamond^l}[y]$ iff $(\forall a \in A')(a \in y \Rightarrow \Diamond^l a \in x)$. On the other hand, $[x]R^l[y]$ iff $(\exists x', y' \in X)(x \sim_\Sigma x' \ \&\ y \sim_\Sigma y' \ \&\ x'Ry')$. First suppose that $[x]R^l[y]$. Let $a \in A'$ and $a \in y$. Since $a \in A'$ and $y \sim_\Sigma y'$, from $a \in y$ it follows that $a \in y'$, so $x'Ry'$ implies $\Diamond a \in x'$. By Lemma 4.5(1), $\Diamond a \le \Diamond^l a$, so $\Diamond^l a \in x'$, and as $\Diamond^l a \in A'$, we conclude that $\Diamond^l a \in x$. Thus, $[x]R_{\Diamond^l}[y]$. Conversely, suppose that $[x]\not R^l[y]$. Since A' is finite, there is $a \in A'$ such that $[y] = \varphi(a)$. As $a \in A'$, we have $a \in y$. Also, since $[x]\not R^l[y]$, we have $[x] \notin (R_l)^{-1}\varphi(a) = \varphi(\Diamond^l a)$. Because $\Diamond^l a \in A'$, this yields $\Diamond^l a \notin x$. Therefore, we found $a \in A'$ such that $a \in y$ but $\Diamond^l a \notin x$. Thus, $[x]\not R_{\Diamond^l}[y]$, and so R^l is the dual of $\Diamond^l$.

Let $R_{\Diamond^g}$ be the dual of $\Diamond^g$. Then $[x]R_{\Diamond^g}[y]$ iff $(\forall a \in A')(a \in y \Rightarrow \Diamond^g a \in x)$. On the other hand, $[x]R^g[y]$ iff $(\forall a \in D)(a \in y \Rightarrow \Diamond a \in x)$. By Lemma 4.5, $\Diamond a \le \Diamond^g a$ for each $a \in A'$ and $\Diamond a = \Diamond^g a$ for each $a \in D$. Therefore, $[x]R_{\Diamond^g}[y]$ implies $[x]R^g[y]$. Conversely, if $[x]\not R_{\Diamond^g}[y]$, then there is $a \in A'$ such that $a \in y$ and $\Diamond^g a \notin x$. Thus, there is $b \in D^\vee$ such that $a \le b$ and $\Diamond b \notin x$. As $a \le b$ and $a \in y$, we see that $b \in y$, yielding that $[x]\not R^g[y]$. Consequently, R^g is the dual of $\Diamond^g$. □

5. Finite refutation patterns and stable canonical rules

In this section we show how to construct finite refutation patterns for each multiple-conclusion modal rule not derivable in $\mathbf{S_K}$. Moreover, we introduce stable canonical rules, develop their basic properties, and prove that each normal modal rule system is axiomatizable by stable canonical rules. This solves an open problem of Jeřábek [15, p. 1201]. Furthermore, we apply these results to construct finite refutation patterns for each modal formula that is not derivable in $\mathbf{K}$, and prove that each normal modal logic is axiomatizable by stable canonical rules. This solves an open problem of Chagrov and Zakharyaschev [12, Ch. 9, p. 332, Prob. 9.5], but our solution is by means of multiple-conclusion rules rather than formulas.

Theorem 5.1.

(1) *If $\mathbf{S_K} \not\vdash \Gamma/\Delta$, then there exist $(\mathfrak{A}_1, D_1), \dots, (\mathfrak{A}_n, D_n)$ such that each $\mathfrak{A}_i = (A_i, \Diamond_i)$ is a finite modal algebra, $D_i \subseteq A_i$, and for each modal algebra $\mathfrak{B} = (B, \Diamond)$, we have $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is $i \le n$ and a stable embedding $h : A_i \rightarrowtail B$ satisfying* (CDC) *for D_i.*

(2) *If $\mathbf{K} \not\vdash \varphi$, then there exist $(\mathfrak{A}_1, D_1), \dots, (\mathfrak{A}_n, D_n)$ such that each $\mathfrak{A}_i = (A_i, \Diamond_i)$ is a finite modal algebra, $D_i \subseteq A_i$, and for each modal algebra $\mathfrak{B} = (B, \Diamond)$, we have $\mathfrak{B} \not\models \varphi$ iff there is $i \le n$ and a stable embedding $h : A_i \rightarrowtail B$ satisfying* (CDC) *for D_i.*

Proof. (1). If $\mathbf{S_K} \not\vdash \Gamma/\Delta$, then there is a modal algebra $\mathfrak{A} = (A, \Diamond)$ refuting Γ/Δ. Therefore, there is a valuation V on A such that $V(\gamma) = 1_A$ for each $\gamma \in \Gamma$ and $V(\delta) \ne 1_A$ for each $\delta \in \Delta$. Let Σ be the set of subformulas of $\Gamma \cup \Delta$, A' be the Boolean subalgebra of A generated by $V(\Sigma)$, and $\mathfrak{A}' = (A', \Diamond')$ be a filtration of $\mathfrak{A}$ through Σ. By Lemmas 4.4 and 4.5, $\mathfrak{A}'$ is a finite modal algebra refuting Γ/Δ. In fact, $|A'| \le m$, where $m = 2^{2^{|\Sigma|}}$ is the size of the free Boolean algebra on $|\Sigma|$-generators.

Let $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ be the list of all finite modal algebras $\mathfrak{A}_i = (A_i, \Diamond_i)$ of size $\le m$ refuting Γ/Δ. Let V_i be a valuation on A_i refuting Γ/Δ; that is, $V_i(\gamma) = 1_{A_i}$ for each $\gamma \in \Gamma$ and $V_i(\delta) \ne 1_{A_i}$ for each $\delta \in \Delta$. Set $D_i = \{V_i(\psi) : \Diamond\psi \in \Sigma\}$.

Given a modal algebra $\mathfrak{B} = (B, \Diamond)$, we must show that $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is $i \le n$ and a stable embedding $h : A_i \rightarrowtail B$ satisfying (CDC) for D_i.

($\Leftarrow$): First suppose that there is $i \le n$ and a stable embedding $h_i : A_i \rightarrowtail B$ satisfying (CDC) for D_i. Define a valuation V_B on B by $V_B(p) = h_i \circ V_i(p)$ for each propositional letter p. For each

$\Diamond\psi \in \Sigma$, we have $V_i(\psi) \in D_i$. Therefore, $V_B(\Diamond\psi) = \Diamond V_B(\psi)$. Thus, since $V_i(\gamma) = 1_{A_i}$ for each $\gamma \in \Gamma$ and $V_i(\delta) \neq 1_{A_i}$ for each $\delta \in \Delta$, we see that $V_B(\gamma) = 1_B$ for each $\gamma \in \Gamma$ and $V_B(\delta) \neq 1_B$ for each $\delta \in \Delta$. Consequently, $\mathfrak{B} \not\models \Gamma/\Delta$.

($\Rightarrow$): Next suppose that $\mathfrak{B} \not\models \Gamma/\Delta$. We show that there is $i \leq n$ and a stable embedding $h : A_i \rightarrowtail B$ satisfying (CDC) for D_i. Since $\mathfrak{B} \not\models \Gamma/\Delta$, there is a valuation V_B on B such that $V_B(\gamma) = 1_B$ for each $\gamma \in \Gamma$ and $V_B(\delta) \neq 1_B$ for each $\delta \in \Delta$. Let B' be the Boolean subalgebra of B generated by $V_B(\Sigma)$. Since $|V_B(\Sigma)| \leq |\Sigma|$, we see that $|B'| \leq m$. Let V' be the restriction of V_B to B' and set $D = \{V'(\psi) : \Diamond\psi \in \Sigma\}$. Let $\mathfrak{B}' = (B', \Diamond')$ be a filtration of $\mathfrak{B}$ through Σ. By Lemma 4.5, the embedding $B' \rightarrowtail B$ is a stable embedding satisfying (CDC) for D. By Lemma 4.4, V' refutes Γ/Δ on $\mathfrak{B}'$. As $|B'| \leq m$, there is $i \leq n$ such that $\mathfrak{B}' = \mathfrak{A}_i$ and $D = D_i$. Thus, the embedding $A_i \rightarrowtail B$ is a stable embedding satisfying (CDC) for D_i.

(2). Since for a modal algebra $\mathfrak{A}$, we have $\mathfrak{A} \models \varphi$ iff $\mathfrak{A} \models /\varphi$, we see that $\mathbf{K} \not\vdash \varphi$ iff $\mathbf{S_K} \not\vdash /\varphi$. Now apply (1). □

Definition 5.2. *Let $\mathfrak{A} = (A, \Diamond)$ be a finite modal algebra and let D be a subset of A. For each $a \in A$ we introduce a new propositional letter p_a and define the* stable canonical rule $\rho(\mathfrak{A}, D)$ *associated with $\mathfrak{A}$ and D as Γ/Δ, where:*

$$\begin{aligned}\Gamma \;=\; & \{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \cup \\ & \{p_{\neg a} \leftrightarrow \neg p_a : a \in A\} \cup \\ & \{\Diamond p_a \to p_{\Diamond a} : a \in A\} \cup \\ & \{p_{\Diamond a} \to \Diamond p_a : a \in D\},\end{aligned}$$

and

$$\Delta \;=\; \{p_a \leftrightarrow p_b : a, b \in A, a \neq b\}.$$

Lemma 5.3. *Let $\mathfrak{A} = (A, \Diamond)$ be a finite modal algebra and let $D \subseteq A$. Then $\mathfrak{A} \not\models \rho(\mathfrak{A}, D)$.*

Proof. Define a valuation V on A by $V(p_a) = a$ for each $a \in A$. Then $V(\gamma) = 1$ for each $\gamma \in \Gamma$ and $V(\delta) \neq 1$ for each $\delta \in \Delta$. Therefore, $\mathfrak{A} \not\models \rho(\mathfrak{A}, D)$. □

Theorem 5.4. *Let $\mathfrak{A} = (A, \Diamond)$ be a finite modal algebra, $D \subseteq A$, and $\mathfrak{B} = (B, \Diamond)$ be a modal algebra. Then $\mathfrak{B} \not\models \rho(\mathfrak{A}, D)$ iff there is a stable embedding $h : A \rightarrowtail B$ satisfying* (CDC) *for D.*

Proof. First suppose that there is a stable embedding $h : A \rightarrowtail B$ satisfying (CDC) for D. By Lemma 5.3, there is a valuation V on A refuting $\rho(\mathfrak{A}, D)$. We define a valuation V_B on B by $V_B(p_a) = h(V(p_a)) = h(a)$ for each $a \in A$. Since h is a stable homomorphism, $h(a\vee b) = h(a)\vee h(b)$, $h(\neg a) = \neg h(a)$, and $\Diamond h(a) \leq h(\Diamond a)$ for each $a, b \in A$. Therefore,

$$\begin{aligned} V_B(p_{a\vee b} \leftrightarrow p_a \vee p_b) &= V_B(p_{a\vee b}) \leftrightarrow V_B(p_a) \vee V_B(p_b) = h(a \vee b) \leftrightarrow h(a) \vee h(b) = 1, \\ V_B(p_{\neg a} \leftrightarrow \neg p_a) &= V_B(p_{\neg a}) \leftrightarrow \neg V_B(p_a) = h(\neg a) \leftrightarrow \neg h(a) = 1, \\ V_B(\Diamond p_a \to p_{\Diamond a}) &= V_B(\Diamond p_a) \to V_B(p_{\Diamond a}) = \Diamond h(a) \to h(\Diamond a) = 1. \end{aligned}$$

Since h satisfies (CDC) for D,

$$V_B(p_{\Diamond a} \to \Diamond p_a) = V_B(p_{\Diamond a}) \to \Diamond V_B(p_a) = h(\Diamond a) \to \Diamond h(a) = 1$$

for each $a \in D$. Thus, $V_B(\gamma) = 1$ for each $\gamma \in \Gamma$. On the other hand, since h is an embedding, from $a \neq b$ it follows that $V_B(p_a) = h(a) \neq h(b) = V_B(p_b)$. This yields $V_B(\delta) \neq 1$ for each $\delta \in \Delta$. Consequently, $\rho(\mathfrak{A}, D)$ is refuted on $\mathfrak{B}$.

Conversely, let $\mathfrak{B} \not\models \rho(\mathfrak{A}, D)$. Then there is a valuation V on B such that $V(\gamma) = 1$ for each $\gamma \in \Gamma$ and $V(\delta) \neq 1$ for each $\delta \in \Delta$. Define a map $h : A \to B$ by $h(a) = V(p_a)$ for each $a \in A$. We show that $h : A \to B$ is a stable embedding satisfying (CDC) for D.

Let $a, b \in A$. Since $V(\gamma) = 1$ for each $\gamma \in \Gamma$, we have $V(p_{a\vee b}) \leftrightarrow V(p_a) \vee V(p_b) = 1$. Therefore, $V(p_{a\vee b}) = V(p_a) \vee V(p_b)$. By a similar argument,

$$\begin{array}{rcl} V(p_{\neg a}) & = & \neg V(p_a), \\ \Diamond V(p_a) & \leq & V(p_{\Diamond a}), \text{ and} \\ V(p_{\Diamond a}) & = & \Diamond V(p_a) \text{ for } a \in D. \end{array}$$

Since $h(a) = V(p_a)$ for each $a \in A$, we have:

$$\begin{array}{rcl} h(a \vee b) & = & h(a) \vee h(b), \\ h(\neg a) & = & \neg h(a), \\ \Diamond h(a) & \leq & h(\Diamond a), \text{ and} \\ h(\Diamond a) & = & \Diamond h(a) \text{ for } a \in D. \end{array}$$

Thus, h is a stable homomorphism satisfying (CDC) for D. To see that h is an embedding, let $a, b \in A$ with $a \neq b$. Since $V(\delta) \neq 1$ for each $\delta \in \Delta$, we have $V(p_a \leftrightarrow p_b) \neq 1$, so $V(p_a) \neq V(p_b)$. This implies $h(a) \neq h(b)$, which yields that h is an embedding. □

As a consequence of Theorems 5.1 and 5.4, we obtain:

Theorem 5.5.

(1) *If* $\mathbf{S_K} \nvdash \Gamma/\Delta$, *then there exist* $(\mathfrak{A}_1, D_1), \dots, (\mathfrak{A}_n, D_n)$ *such that each* $\mathfrak{A}_i = (A_i, \Diamond_i)$ *is a finite modal algebra,* $D_i \subseteq A_i$, *and for each modal algebra* $\mathfrak{B} = (B, \Diamond)$, *we have:*

$$\mathfrak{B} \models \Gamma/\Delta \text{ iff } \mathfrak{B} \models \rho(\mathfrak{A}_1, D_1), \dots, \rho(\mathfrak{A}_n, D_n).$$

(2) *If* $\mathbf{K} \nvdash \varphi$, *then there exist* $(\mathfrak{A}_1, D_1), \dots, (\mathfrak{A}_n, D_n)$ *such that each* $\mathfrak{A}_i = (A_i, \Diamond_i)$ *is a finite modal algebra,* $D_i \subseteq A_i$, *and for each modal algebra* $\mathfrak{B} = (B, \Diamond)$, *we have:*

$$\mathfrak{B} \models \varphi \text{ iff } \mathfrak{B} \models \rho(\mathfrak{A}_1, D_1), \dots, \rho(\mathfrak{A}_n, D_n).$$

Proof. (1). Suppose $\mathbf{S_K} \nvdash \Gamma/\Delta$. By Theorem 5.1(1), there exist $(\mathfrak{A}_1, D_1), \dots, (\mathfrak{A}_n, D_n)$ such that each $\mathfrak{A}_i = (A_i, \Diamond_i)$ is a finite modal algebra, $D_i \subseteq A_i$, and for each modal algebra $\mathfrak{B} = (B, \Diamond)$, we have $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is $i \leq n$ and a stable embedding $h : A_i \rightarrowtail B$ satisfying (CDC) for D_i. By Theorem 5.4, this is equivalent to the existence of $i \leq n$ such that $\mathfrak{B} \not\models \rho(\mathfrak{A}_i, D_i)$. Thus, $\mathfrak{B} \models \Gamma/\Delta$ iff $\mathfrak{B} \models \rho(\mathfrak{A}_1, D_1), \dots, \rho(\mathfrak{A}_n, D_n)$.

(2). This is proved similarly but uses Theorem 5.1(2). □

We are ready to prove the main result of the paper.

Theorem 5.6.

(1) *Each normal rule system* $\mathcal{S}$ *over* $\mathbf{S_K}$ *is axiomatizable by stable canonical rules. Moreover, if* $\mathcal{S}$ *is finitely axiomatizable, then* $\mathcal{S}$ *is axiomatizable by finitely many stable canonical rules.*

(2) *Each normal modal logic* L *is axiomatizable by stable canonical rules. Moreover, if* L *is finitely axiomatizable, then* L *is axiomatizable by finitely many stable canonical rules.*

Proof. (1). Let $\mathcal{S}$ be a normal rule system. Then there is a family $\{\rho_i : i \in I\}$ of modal rules such that $\mathcal{S} = \mathbf{S_K} + \{\rho_i : i \in I\}$. Therefore, $\mathbf{S_K} \nvdash \rho_i$ for each $i \in I$. By Theorem 5.5(1), for each $i \in I$, there exist $(\mathfrak{A}_{i1}, D_{i1}), \dots, (\mathfrak{A}_{in_i}, D_{in_i})$ such that $\mathfrak{A}_{ij} = (A_{ij}, \Diamond_{ij})$ is a finite modal algebra, $D_{ij} \subseteq A_{ij}$, and for each modal algebra $\mathfrak{B} = (B, \Diamond)$, we have $\mathfrak{B} \models \rho_i$ iff $\mathfrak{B} \models \rho(\mathfrak{A}_{i1}, D_{i1}), \dots, \rho(\mathfrak{A}_{in_i}, D_{in_i})$. Thus, $\mathfrak{B} \models \mathcal{S}$ iff $\mathfrak{B} \models \{\rho_i : i \in I\}$, which happens iff $\mathfrak{B} \models \rho(\mathfrak{A}_{i1}, D_{i1}), \dots, \rho(\mathfrak{A}_{in_i}, D_{in_i})$ for each $i \in I$. Consequently, $\mathcal{S} = \mathbf{S_K} + \bigcup_{i\in I}\{\rho(\mathfrak{A}_{i1}, D_{i1}), \dots, \rho(\mathfrak{A}_{in_i}, D_{in_i})\}$, and so $\mathcal{S}$ is axiomatizable by stable canonical rules. In particular, if $\mathcal{S}$ is finitely axiomatizable, then $\mathcal{S}$ is axiomatizable by finitely many stable canonical rules.

(2). Let L be a normal modal logic. Then L is obtained by adding $\{\varphi_i : i \in I\}$ to $\mathbf{K}$ as new axioms. Therefore, $\mathbf{K} \nvdash \varphi_i$ for each $i \in I$. By Theorem 5.5(2), for each $i \in I$, there exist

$(\mathfrak{A}_{i1}, D_{i1}), \dots, (\mathfrak{A}_{in_i}, D_{in_i})$ such that $\mathfrak{A}_{ij} = (A_{ij}, \Diamond_{ij})$ is a finite modal algebra, $D_{ij} \subseteq A_{ij}$, and for each modal algebra $\mathfrak{B} = (B, \Diamond)$, we have $\mathfrak{B} \models \varphi_i$ iff $\mathfrak{B} \models \rho(\mathfrak{A}_{i1}, D_{i1}), \dots, \rho(\mathfrak{A}_{in_i}, D_{in_i})$.

Let $\mathfrak{B} = (B, \Diamond)$ be a modal algebra. Then $\mathfrak{B} \not\models L$ iff $\mathfrak{B} \not\models \varphi_i$ for some $i \in I$. By Theorem 5.5(2), this is equivalent to the existence of $j \leq n_i$ such that $\mathfrak{B} \not\models \rho(\mathfrak{A}_{ij}, D_{ij})$. Therefore, $\mathfrak{B} \models L$ iff $\mathfrak{B} \models \bigcup_{i\in I}\{\rho(\mathfrak{A}_{i1}, D_{i1}), \dots, \rho(\mathfrak{A}_{in_i}, D_{in_i})\}$. Thus, $\mathcal{V}(L) = \mathcal{U}\left(\mathbf{S_K} + \bigcup_{i\in I}\{\rho(\mathfrak{A}_{i1}, D_{i1}), \dots, \rho(\mathfrak{A}_{in_i}, D_{in_i})\}\right)$. This implies that $\mathcal{U}\left(\mathbf{S_K} + \bigcup_{i\in I}\{\rho(\mathfrak{A}_{i1}, D_{i1}), \dots, \rho(\mathfrak{A}_{in_i}, D_{in_i})\}\right)$ is a variety. So the variety corresponding to $\Lambda\left(\mathbf{S_K} + \bigcup_{i\in I}\{\rho(\mathfrak{A}_{i1}, D_{i1}), \dots, \rho(\mathfrak{A}_{in_i}, D_{in_i})\}\right)$ is equal to $\mathcal{U}(\mathbf{S_K} + \bigcup_{i\in I}\{\rho(\mathfrak{A}_{i1}, D_{i1}), \dots, \rho(\mathfrak{A}_{in_i}, D_{in_i})\})$. On the other hand, the variety corresponding to L is $\mathcal{V}(L)$. Since the varieties corresponding to L and $\Lambda\left(\mathbf{S_K} + \bigcup_{i\in I}\{\rho(\mathfrak{A}_{i1}, D_{i1}), \dots, \rho(\mathfrak{A}_{in_i}, D_{in_i})\}\right)$ are equal, we conclude that $L = \Lambda\left(\mathbf{S_K} + \bigcup_{i\in I}\{\rho(\mathfrak{A}_{i1}, D_{i1}), \dots, \rho(\mathfrak{A}_{in_i}, D_{in_i})\}\right)$. In particular, if L is finitely axiomatizable, then L is axiomatizable by finitely many stable canonical rules. □

Theorem 5.6(1) yields a solution of an open problem of Jeřábek [15, p. 1201], and Theorem 5.6(2) that of Chagrov and Zakharyaschev [12, Ch. 9, p. 332, Prob. 9.5]. However, our solution is by means of multiple-conclusion rules rather than formulas. It is also worth pointing out that our axiomatization requires to work with all finite modal algebras. It is not sufficient to work with only finite s.i. modal algebras. As we will see in the next section, the situation improves for **K4**, where stable canonical rules can be replaced by stable canonical formulas and it is sufficient to work with only finite s.i. **K4**-algebras.

Remark 5.7. Using duality between modal algebras and modal spaces, one can rephrase all the results in this and forthcoming sections in dual terms. In fact, stable canonical rules can be defined directly for finite modal spaces (finite Kripke frames) without using modal algebras.

Let $\mathfrak{X} = (X, R)$ be a finite modal space and $\mathfrak{D} \subseteq \mathcal{P}(X)$. For each $x \in X$ we introduce a new propositional letter p_x and define the *stable canonical rule* $\sigma(\mathfrak{X}, \mathfrak{D})$ as the rule Γ/Δ, where

$$\begin{aligned}\Gamma \;=\; & \{\bigvee\{p_x : x \in X\}\} \cup \\ & \{p_x \to \neg p_y : x, y \in X, x \neq y\} \cup \\ & \{p_x \to \neg\Diamond p_y : x, y \in X, x \not R y\} \cup \\ & \{p_x \to \Diamond p_y : x \in X, y \in U \in \mathfrak{D}, xRy\},\end{aligned}$$

and

$$\Delta \;=\; \{\neg p_x : x \in X\}.$$

Then a modal space $\mathfrak{Y} = (Y, R)$ refutes $\sigma(\mathfrak{X}, \mathfrak{D})$ iff there is an onto stable map $f : Y \to X$ satisfying (CDC) for $\mathfrak{D}$. This provides an alternative way of defining stable canonical rules by avoiding algebraic terminology. Indeed, let $\mathfrak{A} = (A, \Diamond)$ be a finite modal algebra, $\mathfrak{X} = (X, R)$ be its dual modal space, $D \subseteq A$, and $\mathfrak{D} = \{\varphi(a) : a \in D\}$. Then for each modal algebra $\mathfrak{B} = (B, \Diamond)$ with the dual space $\mathfrak{Y} = (Y, R)$, we have $\mathfrak{B} \models \rho(\mathfrak{A}, D)$ iff $\mathfrak{Y} \models \sigma(\mathfrak{X}, \mathfrak{D})$.

6. Stable canonical formulas for **K4**

As we have seen, all normal modal logics are axiomatizable by stable canonical rules. In general, these rules are not equivalent to formulas. In this section we show that for transitive normal modal logics we can replace stable canonical rules by stable canonical formulas. This provides an axiomatization of transitive normal modal logics, which is an alternative to Zakharyaschev's axiomatization [32].

Transitive filtrations. We start by developing the transitive analogues of the least and greatest filtrations.

Definition 6.1. *Let $\mathfrak{A} = (A, \Diamond)$ be a* **K4***-algebra, V be a valuation on A, Σ be a set of formulas closed under subformulas, and $\mathfrak{A}' = (A', \Diamond')$ be a filtration of $\mathfrak{A}$ through Σ. We call $\mathfrak{A}'$ a* transitive filtration *if $\mathfrak{A}'$ is also a* **K4***-algebra.*

For a **K4**-algebra $\mathfrak{A} = (A, \Diamond)$ and $a \in A$, we recall that $\Diamond^+ a = a \vee \Diamond a$.

Lemma 6.2. *Let $\mathfrak{A} = (A, \Diamond)$ be a* **K4***-algebra, and let V, Σ, A', D, and $D^\vee$ be as in Lemma 4.5. Define $\Diamond^t$ and $\Diamond^L$ on A' by*

$$\Diamond^t a = \bigwedge\{\Diamond b : \Diamond a \le \Diamond b \;\&\; b, \Diamond b \in A'\} \quad \text{and} \quad \Diamond^L a = \bigwedge\{\Diamond b : \Diamond a \le \Diamond b \;\&\; \Diamond^+ a \le \Diamond^+ b \;\&\; b \in D^\vee\}.$$

Then both $(A', \Diamond^t)$ and $(A', \Diamond^L)$ are transitive filtrations of $\mathfrak{A}$ through Σ.

Proof. Since $\Diamond 0 = 0$ and $0 \in A'$, it is obvious that $\Diamond^t 0 = 0$. As A' is closed under finite joins,

$$\begin{array}{rcl} \Diamond^t a \vee \Diamond^t b & = & \bigwedge\{\Diamond x : \Diamond a \le \Diamond x \;\&\; x, \Diamond x \in A'\} \vee \bigwedge\{\Diamond y : \Diamond b \le \Diamond y \;\&\; y, \Diamond y \in A'\} \\ & = & \bigwedge\{\Diamond x \vee \Diamond y : \Diamond a \le \Diamond x \;\&\; \Diamond b \le \Diamond y \;\&\; x, \Diamond x, y, \Diamond y \in A'\} \\ & = & \bigwedge\{\Diamond (x \vee y) : \Diamond a \le \Diamond x \;\&\; \Diamond b \le \Diamond y \;\&\; x, \Diamond x, y, \Diamond y \in A'\} \\ & = & \bigwedge\{\Diamond z : \Diamond (a \vee b) \le \Diamond z \;\&\; z, \Diamond z \in A'\} \\ & = & \Diamond^t (a \vee b). \end{array}$$

Since $\Diamond 0 = 0$ and $0 \in D^\vee$, it is obvious that $\Diamond^L 0 = 0$. As $D^\vee$ is closed under finite joins,

$$\begin{array}{rcl} \Diamond^L a \vee \Diamond^L b & = & \bigwedge\{\Diamond x : \Diamond a \le \Diamond x \;\&\; \Diamond^+ a \le \Diamond^+ x \;\&\; x \in D^\vee\} \vee \\ & & \bigwedge\{\Diamond y : \Diamond b \le \Diamond y \;\&\; \Diamond^+ b \le \Diamond^+ y \;\&\; y \in D^\vee\} \\ & = & \bigwedge\{\Diamond x \vee \Diamond y : \Diamond a \le \Diamond x \;\&\; \Diamond^+ a \le \Diamond^+ x \;\&\; \Diamond b \le \Diamond y \;\&\; \Diamond^+ b \le \Diamond^+ y \;\&\; x, y \in D^\vee\} \\ & = & \bigwedge\{\Diamond (x \vee y) : \Diamond a \le \Diamond x \;\&\; \Diamond^+ a \le \Diamond^+ x \;\&\; \Diamond b \le \Diamond y \;\&\; \Diamond^+ b \le \Diamond^+ y \;\&\; x, y \in D^\vee\} \\ & = & \bigwedge\{\Diamond z : \Diamond (a \vee b) \le \Diamond z \;\&\; \Diamond^+ (a \vee b) \le \Diamond^+ z \;\&\; z \in D^\vee\} \\ & = & \Diamond^L (a \vee b). \end{array}$$

Therefore, both $(A', \Diamond^t)$ and $(A', \Diamond^L)$ are modal algebras. It is obvious that $\Diamond^l a \le \Diamond^t a \le \Diamond^L a \le \Diamond^g a$ for each $a \in A'$. Thus, both $(A', \Diamond^t)$ and $(A', \Diamond^L)$ are filtrations of $\mathfrak{A}$ through Σ. It remains to show that both $(A', \Diamond^t)$ and $(A', \Diamond^L)$ are **K4**-algebras. We have

$$\Diamond^t a = \bigwedge\{\Diamond x : \Diamond a \le \Diamond x \;\&\; x, \Diamond x \in A'\}$$

and

$$\Diamond^t \Diamond^t a = \bigwedge\{\Diamond y : \Diamond \Diamond^t a \le \Diamond y \;\&\; y, \Diamond y \in A'\}.$$

Let $x \in A'$ with $\Diamond a \le \Diamond x$ and $\Diamond x \in A'$. Then

$$\begin{array}{rcl} \Diamond \Diamond^t a & = & \Diamond \bigwedge\{\Diamond y : \Diamond a \le \Diamond y \;\&\; y, \Diamond y \in A'\} \\ & \le & \bigwedge\{\Diamond \Diamond y : \Diamond a \le \Diamond y \;\&\; y, \Diamond y \in A'\} \\ & \le & \bigwedge\{\Diamond y : \Diamond a \le \Diamond y \;\&\; y, \Diamond y \in A'\} \;\le\; \Diamond x, \end{array}$$

so $\Diamond^t \Diamond^t a \le \Diamond^t a$. Also,

$$\Diamond^L a = \bigwedge\{\Diamond x : \Diamond a \le \Diamond x \;\&\; \Diamond^+ a \le \Diamond^+ x \;\&\; x \in D^\vee\}$$

and

$$\Diamond^L \Diamond^L a = \bigwedge\{\Diamond y : \Diamond \Diamond^L a \le \Diamond y \;\&\; \Diamond^+ \Diamond^L a \le \Diamond^+ y \;\&\; y \in D^\vee\}.$$

Let $x \in D^\vee$ with $\Diamond a \le \Diamond x$ and $\Diamond^+ a \le \Diamond^+ x$. Then

$$\begin{array}{rcl} \Diamond \Diamond^L a & = & \Diamond \bigwedge\{\Diamond y : \Diamond a \le \Diamond y \;\&\; \Diamond^+ a \le \Diamond^+ y \;\&\; y \in D^\vee\} \\ & \le & \bigwedge\{\Diamond \Diamond y : \Diamond a \le \Diamond y \;\&\; \Diamond^+ a \le \Diamond^+ y \;\&\; y \in D^\vee\} \\ & \le & \bigwedge\{\Diamond y : \Diamond a \le \Diamond y \;\&\; \Diamond^+ a \le \Diamond^+ y \;\&\; y \in D^\vee\} \;\le\; \Diamond x \end{array}$$

and

$$\begin{aligned}\Diamond^+\Diamond^L a &= \Diamond^+ \bigwedge\{\Diamond y : \Diamond a \le \Diamond y \ \&\ \Diamond^+ a \le \Diamond^+ y \ \&\ y \in D^\vee\}\\ &\le \bigwedge\{\Diamond^+\Diamond y : \Diamond a \le \Diamond y \ \&\ \Diamond^+ a \le \Diamond^+ y \ \&\ y \in D^\vee\}\\ &= \bigwedge\{\Diamond y : \Diamond a \le \Diamond y \ \&\ \Diamond^+ a \le \Diamond^+ y \ \&\ y \in D^\vee\}\\ &\le \bigwedge\{\Diamond^+ y : \Diamond a \le \Diamond y \ \&\ \Diamond^+ a \le \Diamond^+ y \ \&\ y \in D^\vee\} \ \le\ \Diamond^+ x.\end{aligned}$$

This implies $\Diamond^L\Diamond^L a \le \Diamond^L a$. Thus, both $(A', \Diamond^t)$ and $(A', \Diamond^L)$ are **K4**-algebras. □

Lemma 6.3. *Suppose that $\mathfrak{A} = (A, \Diamond)$ is a* **K4***-algebra and $\mathfrak{X} = (X, R)$ is its dual. Let A' and X' be as in Theorem 4.2, with A' and X' finite, R^l be as in Lemma 4.5, and $\Diamond^t$ and $\Diamond^L$ be as in Lemma 6.2. The dual of $\Diamond^t$ is the transitive closure of R^l and the dual of $\Diamond^L$ is the Lemmon filtration.*

Proof. Let R^t denote the transitive closure of R^l. Then $[x]R^t[y]$ iff there exist $z_1, \dots, z_n \in X$ such that $[x] = [z_1]R^l \dots R^l[z_n] = [y]$. Also, $[x]R_{\Diamond^t}[y]$ iff $(\forall a \in A')(a \in y \Rightarrow \Diamond^t a \in x)$. We show $R^t = R_{\Diamond^t}$. Since $\Diamond^l a \le \Diamond^t a$ for each $a \in A'$, we have $R^l \subseteq R_{\Diamond^t}$. Also, since $(A', \Diamond^t)$ is a **K4**-algebra, $R_{\Diamond^t}$ is transitive. Thus, $R^t \subseteq R_{\Diamond^t}$. Conversely, suppose that $[x]\not R^t[y]$. To see that $[x]\not R_{\Diamond^t}[y]$, it is sufficient to find $a \in A'$ such that $a \in y$ and $\Diamond^t a \notin x$. Let $a \in A'$ be such that $\varphi(a) = [y]$. Then $a \in y$. Since $[x]\not R^t[y]$, we have $[x] \cap R^{-1}[y] = \varnothing$. If $R^{-1}[y]$ is saturated (that is, $R^{-1}[y]$ is a union of equivalence classes), then $\varphi(\Diamond a) = R^{-1}[y]$ is saturated, so $\Diamond a \in A'$. This yields $\Diamond^t a = \Diamond a$. As $x \notin R^{-1}[y]$, we have $x \notin \varphi(\Diamond a)$, so $\Diamond a \notin x$. Thus, $a \in y$ and $\Diamond^t a \notin x$. If $R^{-1}[y]$ is not saturated, then we consider the saturation $[R^{-1}[y]]$ of $R^{-1}[y]$. Since $[x]\not R^t[y]$, we have $[x] \cap (R^{-1}[R^{-1}[y]] \cup R^{-1}[y]) = \varnothing$. If $R^{-1}[R^{-1}[y]]$ is saturated, then let $b \in A'$ be such that $\varphi(b) = [R^{-1}[y]] \cup [y]$. So $\varphi(\Diamond b) = R^{-1}[R^{-1}[y]] \cup R^{-1}[y] \supseteq \varphi(\Diamond a)$ is saturated. Therefore, $\Diamond b \in A'$, $\Diamond a \le \Diamond b$, and $x \notin \varphi(\Diamond b)$. Thus, $a \in y$ and $\Diamond^t a \le \Diamond b \notin x$. If $[R^{-1}[y]]$ is not saturated, then we continue the process by taking its saturation. Since there are only finitely many saturated subsets of X, the process will end after finitely many steps, which will produce $b \in A'$ such that $\Diamond a \le \Diamond b$, $\Diamond b \in A'$, and $x \notin \varphi(\Diamond b)$. Thus, $a \in y$ and $\Diamond^t a \le \Diamond b \notin x$, and hence $[x]\not R_{\Diamond^t}[y]$.

Let R^L be the Lemmon filtration. Then $[x]R^L[y]$ iff $(\forall \Diamond\varphi \in \Sigma)(y \models \Diamond^+\varphi \Rightarrow x \models \Diamond\varphi)$, which is equivalent to $(\forall a \in D)(\Diamond^+ a \in y \Rightarrow \Diamond a \in x)$. Also, $[x]R_{\Diamond^L}[y]$ iff $(\forall a \in A')(a \in y \Rightarrow \Diamond^L a \in x)$. We show $R^L = R_{\Diamond^L}$. First suppose that $[x]\not R^L[y]$. Then there exists $a \in D$ such that $\Diamond^+ a \in y$ but $\Diamond a \notin x$. From $\Diamond^+ a \in y$ it follows that $a \in y$ or $\Diamond a \in y$. As $a \in D$, we have $\Diamond^L a = \Diamond a$. So if $a \in y$, then $\Diamond^L a \notin x$. On the other hand, if $\Diamond a \in y$, then letting $b = \Diamond a$, we have $b \in A'$, $b \in y$, and $\Diamond^L b = \Diamond^L\Diamond a \le \Diamond^L\Diamond^L a \le \Diamond^L a = \Diamond a \notin x$. Therefore, in both cases we have $[x]\not R_{\Diamond^L}[y]$. Next suppose that $[x]\not R_{\Diamond^L}[y]$. Then there exists $a \in A'$ such that $a \in y$ but $\Diamond^L a \notin x$. The latter implies that there exists $b \in D^\vee$ such that $\Diamond a \le \Diamond b$, $\Diamond^+ a \le \Diamond^+ b$, and $\Diamond b \notin x$. As $a \le \Diamond^+ a \le \Diamond^+ b$, the former implies that $\Diamond^+ b \in y$. Thus, $[x]\not R^L[y]$. □

Refutation patterns and stable canonical formulas for K4. Next we apply the results of Section 5 to obtain refutation patterns for **K4**. We will utilize the following corollary of Venema's characterization [28] of s.i. modal algebras.

Proposition 6.4. *Let $\mathfrak{A} = (A, \Diamond)$ be a finite modal algebra and let $\mathfrak{B} = (B, \Diamond)$ be a s.i. modal algebra. If there is a stable embedding $h : A \rightarrowtail B$, then $(A, \Diamond)$ is also s.i.*

Proof. Let $\mathfrak{X} = (X, R)$ be the dual of $\mathfrak{A}$, $\mathfrak{Y} = (Y, R)$ be the dual of $\mathfrak{B}$, and $f : Y \to X$ be the dual of h. Since h is 1-1, f is onto. As $\mathfrak{B}$ is s.i., by [28, Thm. 2], the set of topo-roots of $\mathfrak{Y}$ has nonempty interior. Let t belong to this interior. We show that $f(t)$ is a root of $\mathfrak{X}$. Because $\mathfrak{A}$ is finite, this will imply that $\mathfrak{A}$ is s.i. For $Y \subseteq X$, we denote by $\overline{Y}$ the topological closure of Y. Since t is a topo-root of $\mathfrak{Y}$, we have $\overline{R^\omega(t)} = Y$. Therefore, $f\left(\overline{R^\omega(t)}\right) = X$. As f is continuous and X is

finite, $f\left(\overline{R^\omega(t)}\right) \subseteq \overline{f\left(R^\omega(t)\right)} = f\left(R^\omega(t)\right)$. So $f\left(R^\omega(t)\right) = X$. Since h is stable, by Lemma 3.3, f is stable. So for each $y \in Y$, we have $f(R(y)) \subseteq R(f(y))$, and hence $f(R^\omega(y)) \subseteq R^\omega(f(y))$. Thus, $f\left(R^\omega(t)\right) \subseteq R^\omega(f(t))$, which yields that $R^\omega(f(t)) = X$. Consequently, $f(t)$ is a root of $\mathfrak{X}$, so $\mathfrak{X}$ is rooted, an hence $\mathfrak{A}$ is s.i. □

We next prove the following version of Theorem 5.1(2) for **K4**.

Theorem 6.5. *If* $\mathbf{K4} \nvdash \varphi$, *then there exist* $(\mathfrak{A}_1, D_1), \dots, (\mathfrak{A}_n, D_n)$ *such that each* $\mathfrak{A}_i = (A_i, \Diamond_i)$ *is a finite s.i.* **K4***-algebra,* $D_i \subseteq A_i$, *and for each s.i. modal algebra* $\mathfrak{B} = (B, \Diamond)$, *the following conditions are equivalent:*

(1) $\mathfrak{B} \not\models \varphi$.
(2) *There is* $i \leq n$ *and a stable embedding* $h : A_i \rightarrowtail B$ *satisfying* (CDC) *for* D_i.
(3) *There is a homomorphic image* $\mathfrak{C} = (C, \Diamond)$ *of* $\mathfrak{B}$, $i \leq n$, *and a stable embedding* $h : A_i \rightarrowtail C$ *satisfying* (CDC) *for* D_i.

Proof. Let $\mathbf{S_{K4}}$ be the least normal modal rule system containing $/\varphi$ for each $\varphi \in \mathbf{K4}$. Then $\mathbf{K4} \nvdash \varphi$ iff $\mathbf{S_{K4}} \nvdash /\varphi$. Suppose that $\mathbf{K4} \nvdash \varphi$. Then there is a s.i. **K4**-algebra $\mathfrak{A} = (A, \Diamond)$ refuting φ. Let Σ be the set of subformulas of φ and m be the size of the free Boolean algebra on $|\Sigma|$-generators. As in the proof of Theorem 5.1(2), but using a transitive filtration instead of an arbitrary filtration, we construct a finite **K4**-algebra $\mathfrak{A}' = (A', \Diamond')$ of size $\leq m$ refuting φ. Since $\mathfrak{A}$ is s.i., by Proposition 6.4, so is $\mathfrak{A}'$. Let $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ be the list of all finite s.i. **K4**-algebras $\mathfrak{A}_i = (A_i, \Diamond_i)$ of size $\leq m$ refuting φ. Let V_i be a valuation on A_i refuting φ. Set $D_i = \{V_i(\psi) : \Diamond\psi \in \Sigma\}$. Let $\mathfrak{B} = (B, \Diamond)$ be a s.i. **K4**-algebra.

(1)⇒(2): Suppose that $\mathfrak{B} \not\models \varphi$. As in the proof of Theorem 5.1(2), but using a transitive filtration instead of an arbitrary filtration, we construct a finite **K4**-algebra $\mathfrak{B}' = (B', \Diamond')$ of size $\leq m$, a valuation V' on B' refuting φ, and a stable embedding $B' \rightarrowtail B$ satisfying (CDC) for $D = \{V'(\psi) : \Diamond\psi \in \Sigma\}$. Since $\mathfrak{B}$ is s.i., by Proposition 6.4, so is $\mathfrak{B}'$. Therefore, there is $i \leq n$ such that $\mathfrak{B}' = \mathfrak{A}_i$ and $D = D_i$. Thus, there is $i \leq n$ and a stable embedding $h : A_i \rightarrowtail B$ satisfying (CDC) for D_i.

(2)⇒(3): This is obvious.

(3)⇒(1): Suppose that there is a homomorphic image $\mathfrak{C}$ of $\mathfrak{B}$, $i \leq n$, and a stable embedding $h : A_i \rightarrowtail C$ satisfying (CDC) for D_i. The same argument as in the proof of Theorem 5.1(2) yields that $\mathfrak{C} \not\models \varphi$. Since $\mathfrak{C}$ is a homomorphic image of $\mathfrak{B}$, we conclude that $\mathfrak{B} \not\models \varphi$. □

Remark 6.6. While Theorem 6.5 also holds for **K**, unlike **K4**, it does not yield any substantial gains because the next definition, producing stable canonical formulas for **K4**, does not work for **K**.

Definition 6.7. *Let* $\mathfrak{A} = (A, \Diamond)$ *be a finite s.i.* **K4***-algebra and* $D \subseteq A$. *For each* $a \in A$ *we introduce a new propositional letter* p_a *and define the* stable canonical formula $\gamma(\mathfrak{A}, D)$ *associated with* $\mathfrak{A}$ *and* D *as follows:*

$$\begin{aligned} \gamma(\mathfrak{A}, D) &= \bigwedge\{\Box^+\gamma : \gamma \in \Gamma\} \to \bigvee\{\Box^+\delta : \delta \in \Delta\} \\ &= \Box^+ \bigwedge \Gamma \to \bigvee\{\Box^+\delta : \delta \in \Delta\}, \end{aligned}$$

where Γ *and* Δ *are as in Definition 5.2.*

Theorem 6.8. *Let* $\mathfrak{A} = (A, \Diamond)$ *be a finite s.i.* **K4***-algebra,* $D \subseteq A$, *and* $\mathfrak{B} = (B, \Diamond)$ *be a* **K4***-algebra. Then* $\mathfrak{B} \not\models \gamma(\mathfrak{A}, D)$ *iff there is a s.i. homomorphic image* $\mathfrak{C} = (C, \Diamond)$ *of* $\mathfrak{B}$ *and a stable embedding* $h : A \rightarrowtail C$ *such that* $h(\Diamond a) = \Diamond h(a)$ *for each* $a \in D$.

Proof. First suppose that there is a s.i. homomorphic image $\mathfrak{C}$ of $\mathfrak{B}$ and a stable embedding $h : A \rightarrowtail C$ such that $h(\Diamond a) = \Diamond h(a)$ for each $a \in D$. Define a valuation V_A on A by $V_A(p_a) = a$

for each $a \in A$. Then $V_A(\gamma) = 1_A$ for each $\gamma \in \Gamma$ and $V_A(\delta) \neq 1_A$ for each $\delta \in \Delta$. Therefore, $V_A(\Box^+ \bigwedge \Gamma) = 1_A$ and $\Box^+\delta \neq 1_A$ for each $\delta \in \Delta$. Since $\mathfrak{A}$ is a s.i. **K4**-algebra, there is the second largest element c in H, where we recall that H is the Heyting algebra of the fixed points of $\Box^+$. Thus, $\bigvee\{\Box^+\delta : \delta \in \Delta\} \leq c$, and hence $\mathfrak{A} \not\models \gamma(\mathfrak{A}, D)$. Next define a valuation V_C on C by $V_C(p_a) = h(V_A(p_a)) = h(a)$ for each $a \in A$. The same argument as in the proof of Theorem 5.4 shows that $V_C(\gamma) = 1_C$ for each $\gamma \in \Gamma$ and $V_C(\delta) \neq 1_C$ for each $\delta \in \Delta$. Therefore, $V_C(\Box^+ \bigwedge \Gamma) = 1_C$ and $\Box^+\delta \neq 1_C$ for each $\delta \in \Delta$. Because $\mathfrak{C}$ is s.i., it has an opremum, hence $\bigvee\{\Box^+\delta : \delta \in \Delta\}$ is underneath the opremum, so $\mathfrak{C} \not\models \gamma(\mathfrak{A}, D)$. Since $\mathfrak{C}$ is a homomorphic image of $\mathfrak{B}$, we conclude that $\mathfrak{B} \not\models \gamma(\mathfrak{A}, D)$.

Conversely, suppose that $\mathfrak{B} \not\models \gamma(\mathfrak{A}, D)$. Since $\mathfrak{B}$ is a **K4**-algebra, by [1, Lem. 4.1] (which is a modal analogue of [31, Lem. 1]), there is a s.i. homomorphic image $\mathfrak{C}$ of $\mathfrak{B}$ and a valuation V_C on C such that $V_C(\Box^+ \bigwedge \Gamma) = 1_C$ and $V_C(\bigvee\{\Box^+\delta : \delta \in \Delta\}) \neq 1_C$. Next we define a map $h : A \to C$ by $h(a) = V_C(p_a)$ for each $a \in A$. The proof of Theorem 5.4 then shows that h is a stable embedding such that $h(\Diamond a) = \Diamond h(a)$ for each $a \in D$. □

Combining Theorems 6.5 and 6.8 yields.

Corollary 6.9. *If* $\mathbf{K4} \nvdash \varphi$*, then there exist* $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ *such that each* $\mathfrak{A}_i = (A_i, \Diamond_i)$ *is a finite s.i.* **K4***-algebra,* $D_i \subseteq A_i$*, and for each s.i.* **K4***-algebra* $\mathfrak{B} = (B, \Diamond)$*, we have:*

$$\mathfrak{B} \models \varphi \text{ iff } \mathfrak{B} \models \bigwedge_{i=1}^{n} \gamma(\mathfrak{A}_i, D_i).$$

Proof. Suppose $\mathbf{K4} \nvdash \varphi$. By Theorem 6.5, there exist $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ such that each $\mathfrak{A}_i = (A_i, \Diamond_i)$ is a finite s.i. modal algebra, $D_i \subseteq A_i$, and for each s.i. **K4**-algebra $\mathfrak{B} = (B, \Diamond)$, we have $\mathfrak{B} \not\models \varphi$ iff there is a homomorphic image $\mathfrak{C} = (C, \Diamond)$ of $\mathfrak{B}$, $i \leq n$, and a stable embedding $h : A_i \rightarrowtail C$ satisfying (CDC) for D_i. By Theorem 6.8, this is equivalent to the existence of $i \leq n$ such that $\mathfrak{B} \not\models \gamma(\mathfrak{A}_i, D_i)$. Thus, $\mathfrak{B} \models \varphi$ iff $\mathfrak{B} \models \bigwedge_{i=1}^{n} \gamma(\mathfrak{A}_i, D_i)$. □

Consequently, we arrive at a new axiomatization of modal logics above **K4**, which is an alternative to Zakharyaschev's axiomatization.

Theorem 6.10. *Each normal transitive logic* L *is axiomatizable over* **K4** *by stable canonical formulas. Moreover, if* L *is finitely axiomatizable, then* L *is axiomatizable by finitely many stable canonical formulas.*

Proof. Let L be a normal transitive logic. Then L is obtained by adding $\{\varphi_i : i \in I\}$ to **K4** as new axioms. Therefore, $\mathbf{K4} \nvdash \varphi_i$ for each $i \in I$. By Corollary 6.9, for each $i \in I$, there exist $(\mathfrak{A}_{i1}, D_{i1}), \ldots, (\mathfrak{A}_{in_i}, D_{in_i})$ such that $\mathfrak{A}_{ij} = (A_{ij}, \Diamond_{ij})$ is a finite s.i. **K4**-algebra, $D_{ij} \subseteq A_{ij}$, and for each s.i. **K4**-algebra $\mathfrak{B} = (B, \Diamond)$, we have $\mathfrak{B} \models \varphi_i$ iff $\mathfrak{B} \models \bigwedge_{j=1}^{n_i} \gamma(\mathfrak{A}_{ij}, D_{ij})$. Since every modal logic is determined by the class of its s.i. modal algebras, $L = \mathbf{K4} + \{\bigwedge_{j=1}^{m_i} \gamma(\mathfrak{A}_{ij}, D_{ij}) : i \in I\}$. In particular, if L is finitely axiomatizable, then L is axiomatizable by finitely many stable canonical formulas. □

Remark 6.11. Let $\mathfrak{A} = (A, \Diamond)$ be a finite s.i. **K4**-algebra and let $D \subseteq A$. In general, $\mathbf{K4} + \gamma(\mathfrak{A}, D)$ is not equal to $\Lambda(\mathbf{S_{K4}} + \rho(\mathfrak{A}, D))$. We do have that $\Lambda(\mathbf{S_{K4}} + \rho(\mathfrak{A}, D)) \subseteq \mathbf{K4} + \gamma(\mathfrak{A}, D)$. Indeed, for a s.i. modal algebra $\mathfrak{B} = (B, \Diamond)$, if $\mathfrak{B} \not\models \Lambda(\mathbf{S_{K4}} + \rho(\mathfrak{A}, D))$, then $\mathfrak{B} \not\models \rho(\mathfrak{A}, D)$. Therefore, by Theorems 5.4 and 6.8, $\mathfrak{B} \not\models \gamma(\mathfrak{A}, D)$. This yields $\Lambda(\mathbf{S_{K4}} + \rho(\mathfrak{A}, D)) \subseteq \mathbf{K4} + \gamma(\mathfrak{A}, D)$. The other inclusion, in general, may not be true. However, if $\mathcal{U}(\mathbf{S_{K4}} + \rho(\mathfrak{A}, D))$ is a variety, then $\Lambda(\mathbf{S_{K4}} + \rho(\mathfrak{A}, D)) = \mathbf{K4} + \gamma(\mathfrak{A}, D)$. To see this, let $\mathfrak{B} \not\models \mathbf{K4} + \gamma(\mathfrak{A}, D)$. Then $\mathfrak{B} \not\models \gamma(\mathfrak{A}, D)$. Therefore, by Theorem 6.8, there is a s.i. homomorphic image $\mathfrak{C} = (C, \Diamond)$ of $\mathfrak{B}$ and a stable embedding $h : A \rightarrowtail C$ satisfying (CDC) for D. By Theorem 5.4, $\mathfrak{C} \not\models \rho(\mathfrak{A}, D)$. If $\mathfrak{B} \models \rho(\mathfrak{A}, D)$, then $\mathfrak{B} \in \mathcal{U}(\mathbf{S_{K4}} + \rho(\mathfrak{A}, D))$, and since this class is a variety, it is closed under homomorphic images, so $\mathfrak{C} \in \mathcal{U}(\mathbf{S_{K4}} + \rho(\mathfrak{A}, D))$. But then $\mathfrak{C} \models \rho(\mathfrak{A}, D)$, a contradiction. Thus, $\mathfrak{B} \not\models \rho(\mathfrak{A}, D)$, and hence

$\Lambda(\mathbf{S_{K4}} + \rho(\mathfrak{A}, D)) = \mathbf{K4} + \gamma(\mathfrak{A}, D)$. We leave it as an interesting open question to determine when $\mathcal{U}(\mathbf{S_K} + \rho(\mathfrak{A}, D))$ is a variety.

Remark 6.12. As noted in Remark 5.7, our results can be phrased in dual terms. As with stable canonical rules, stable canonical formulas can also be defined directly for finite rooted transitive spaces (finite rooted transitive Kripke frames).

Let $\mathfrak{X} = (X, R)$ be a finite rooted transitive space and let $\mathfrak{D} \subseteq \mathcal{P}(X)$. For each $x \in X$ we introduce a new propositional letter p_x and define the *stable canonical formula* $\tau(\mathfrak{X}, \mathfrak{D})$ as follows:

$$\begin{aligned} \tau(\mathfrak{X}, \mathfrak{D}) &= \bigwedge\{\Box^+\gamma : \gamma \in \Gamma\} \to \bigvee\{\Box^+\delta : \delta \in \Delta\} \\ &= \Box^+ \bigwedge \Gamma \to \bigvee\{\Box^+\delta : \delta \in \Delta\}, \end{aligned}$$

where Γ and Δ are as in Remark 5.7. Then a transitive space $\mathfrak{Y} = (Y, R)$ refutes $\tau(\mathfrak{X}, \mathfrak{D})$ iff there is a closed up-set Z of Y and an onto stable map $f : Z \to X$ satisfying (CDC) for $\mathfrak{D}$. This provides an alternative way of defining stable canonical formulas for **K4** by avoiding algebraic terminology. Indeed, let $\mathfrak{A} = (A, \Diamond)$ be a finite s.i. **K4**-algebra, $\mathfrak{X} = (X, R)$ be its dual, $D \subseteq A$, and $\mathfrak{D} = \{\varphi(a) : a \in D\}$. Then for each **K4**-algebra $\mathfrak{B} = (B, \Diamond)$ with its dual $\mathfrak{Y} = (Y, R)$, we have $\mathfrak{B} \models \gamma(\mathfrak{A}, D)$ iff $\mathfrak{Y} \models \tau(\mathfrak{X}, \mathfrak{D})$.

7. Stable rules and Jankov rules

As we saw in Section 5, stable canonical rules $\rho(\mathfrak{A}, D)$ axiomatize all normal rule systems and all normal modal logics. In this section we consider two extreme cases, when $D = \varnothing$ and when $D = A$. In the first case we call the stable canonical rule $\rho(\mathfrak{A}, \varnothing)$ simply a *stable rule* and denote it by $\rho(\mathfrak{A})$. In the second case we denote the stable canonical rule $\rho(\mathfrak{A}, \mathfrak{A})$ by $\chi(\mathfrak{A})$ and call it a *Jankov rule*. We characterize normal rule systems and normal modal logics axiomatized by stable rules and prove that they all have the finite model property. On the other hand, as follows from [15] and [10], Jankov rules axiomatize splittings and join splittings in the lattices of normal rule systems and normal modal logics, respectively. We give alternate proofs of these results.

We start by an immediate consequence of Theorem 5.4.

Proposition 7.1. *Let $\mathfrak{A} = (A, \Diamond)$ and $\mathfrak{B} = (B, \Diamond)$ be modal algebras with A finite.*

1. *$\mathfrak{B} \not\models \rho(\mathfrak{A})$ iff there is a stable embedding $h : A \rightarrowtail B$.*
2. *$\mathfrak{B} \not\models \chi(\mathfrak{A})$ iff there is a 1-1 modal homomorphism $h : A \rightarrowtail B$.*

Definition 7.2.

1. *We call a class $\mathcal{K}$ of modal algebras* stable *provided for modal algebras $\mathfrak{A} = (A, \Diamond)$ and $\mathfrak{B} = (B, \Diamond)$, if $\mathfrak{B} \in \mathcal{K}$ and there is a stable embedding $h : A \rightarrowtail B$, then $\mathfrak{A} \in \mathcal{K}$.*
2. *We call a normal modal rule system $\mathcal{S}$* stable *provided the corresponding universal class $\mathcal{U}(\mathcal{S})$ is stable.*

Theorem 7.3. *A normal modal rule system $\mathcal{S}$ is stable iff $\mathcal{S}$ is axiomatizable by stable rules.*

Proof. First suppose that $\mathcal{S}$ is a stable modal rule system. Let $\mathcal{X}_\mathcal{S}$ be the set of all nonisomorphic finite modal algebras refuting $\mathcal{S}$. We show that $\mathcal{S} = \mathbf{S_K} + \{\rho(\mathfrak{A}) : \mathfrak{A} \in \mathcal{X}_\mathcal{S}\}$. Let $\mathfrak{B} = (B, \Diamond)$ be a modal algebra. If $\mathfrak{B} \not\models \mathcal{S}$, then there is $\rho \in \mathcal{S}$ such that $\mathfrak{B} \not\models \rho$. The construction in the proof of Theorem 5.1 yields a finite modal algebra $\mathfrak{A} = (A, \Diamond)$ such that $\mathfrak{A} \not\models \rho$ and the inclusion $A \rightarrowtail B$ is a stable embedding. Therefore, $\mathfrak{A} \in \mathcal{X}_\mathcal{S}$. By Proposition 7.1(1), $\mathfrak{B} \not\models \rho(\mathfrak{A})$. Thus, $\mathfrak{B} \not\models \mathbf{S_K} + \{\rho(\mathfrak{A}) : \mathfrak{A} \in \mathcal{X}_\mathcal{S}\}$. Conversely, if $\mathfrak{B} \not\models \mathbf{S_K} + \{\rho(\mathfrak{A}) : \mathfrak{A} \in \mathcal{X}_\mathcal{S}\}$, then there is $\mathfrak{A} \in \mathcal{X}_\mathcal{S}$ such that $\mathfrak{B} \not\models \rho(\mathfrak{A})$. By Proposition 7.1(1), there is a stable embedding $A \rightarrowtail B$. If $\mathfrak{B} \models \mathcal{S}$, then since $\mathcal{S}$ is stable, $\mathfrak{A} \models \mathcal{S}$, a contradiction. Therefore, $\mathfrak{B} \not\models \mathcal{S}$. Thus, $\mathcal{S} = \mathbf{S_K} + \{\rho(\mathfrak{A}) : \mathfrak{A} \in \mathcal{X}_\mathcal{S}\}$, and so $\mathcal{S}$ is axiomatizable by stable rules.

Next let $\mathcal{S}$ be axiomatizable by stable rules. Then $\mathcal{S} = \mathbf{S_K} + \{\rho(\mathfrak{C}_i) : i \in I\}$. Suppose that $\mathfrak{B} \models \mathcal{S}$ and $h : A \rightarrowtail B$ is a stable embedding. If $\mathfrak{A} \not\models \mathcal{S}$, then there is $i \in I$ such that $\mathfrak{A} \not\models \rho(\mathfrak{C}_i)$.

By Proposition 7.1(1), there is a stable embedding $C_i \rightarrowtail A$. Therefore, there is a stable embedding $C_i \rightarrowtail B$. Applying Proposition 7.1(1) again yields $\mathfrak{B} \not\models \rho(\mathfrak{C}_i)$. The obtained contradiction proves that $\mathfrak{A} \models \mathcal{S}$. Thus, $\mathcal{S}$ is a stable normal modal rule system. □

Definition 7.4. *We call a normal modal logic L* stable *provided the corresponding variety $\mathcal{V}(L)$ is stable.*

The proof of the next theorem is similar to that of Theorem 7.3, but there are some subtle differences due to the fact that if a modal logic L is axiomatizable by modal rules, then the variety corresponding to L is generated by the universal class corresponding to these rules. Thus, in general, this variety may contain algebras that do not validate (some of) these rules.

Theorem 7.5. *A normal modal logic L is stable iff L is axiomatizable by stable rules.*

Proof. First suppose that L is a stable modal logic. Let $\mathcal{X}_L$ be the set of all nonisomorphic finite modal algebras refuting L. We show that $L = \Lambda(\mathbf{S_K} + \{\rho(\mathfrak{A}) : \mathfrak{A} \in \mathcal{X}_L\})$. Let $\mathfrak{B} = (B, \Diamond)$ be a modal algebra. If $\mathfrak{B} \not\models L$, then there is $\varphi \in L$ such that $\mathfrak{B} \not\models \varphi$. The construction in the proof of Theorem 5.1 yields a finite modal algebra $\mathfrak{A} = (A, \Diamond)$ such that $\mathfrak{A} \not\models \varphi$ and the inclusion $A \rightarrowtail B$ is a stable embedding. Therefore, $\mathfrak{A} \in \mathcal{X}_L$. By Proposition 7.1(1), $\mathfrak{B} \not\models \rho(\mathfrak{A})$. Thus, $\mathfrak{B} \not\models \mathbf{S_K} + \{\rho(\mathfrak{A}) : \mathfrak{A} \in \mathcal{X}_L\}$. This yields that the universal class $\mathcal{U}(\mathbf{S_K} + \{\rho(\mathfrak{A}) : \mathfrak{A} \in \mathcal{X}_L\})$ is contained in the variety $\mathcal{V}_L$. Consequently, the variety generated by $\mathcal{U}(\mathbf{S_K} + \{\rho(\mathfrak{A}) : \mathfrak{A} \in \mathcal{X}_L\})$ is contained in $\mathcal{V}_L$, and we see that $L \subseteq \Lambda(\mathbf{S_K} + \{\rho(\mathfrak{A}) : \mathfrak{A} \in \mathcal{X}_L\})$. Conversely, suppose that $\mathfrak{B} \not\models \Lambda(\mathbf{S_K} + \{\rho(\mathfrak{A}) : \mathfrak{A} \in \mathcal{X}_L\})$. Then $\mathfrak{B} \not\models \mathbf{S_K} + \{\rho(\mathfrak{A}) : \mathfrak{A} \in \mathcal{X}_L\}$, so there is $\mathfrak{A} \in \mathcal{X}_L$ such that $\mathfrak{B} \not\models \rho(\mathfrak{A})$. By Proposition 7.1(1), there is a stable embedding $A \rightarrowtail B$. If $\mathfrak{B} \models L$, then since L is stable, $\mathfrak{A} \models L$, a contradiction. Therefore, $\mathfrak{B} \not\models L$, yielding $\Lambda(\mathbf{S_K} + \{\rho(\mathfrak{A}) : \mathfrak{A} \in \mathcal{X}_L\}) \subseteq L$. Thus, $L = \Lambda(\mathbf{S_K} + \{\rho(\mathfrak{A}) : \mathfrak{A} \in \mathcal{X}_L\})$, and hence L is axiomatizable by stable rules.

Next let L be axiomatizable by stable rules. Then $L = \Lambda(\mathbf{S_K} + \{\rho(\mathfrak{C}_i) : i \in I\})$. Suppose that $\mathfrak{B} \models L$ and $h : A \rightarrowtail B$ is a stable embedding. If $\mathfrak{A} \not\models L$, then $\mathfrak{A} \not\models \mathbf{S_K} + \{\rho(\mathfrak{C}_i) : i \in I\}$, so there is $i \in I$ such that $\mathfrak{A} \not\models \rho(\mathfrak{C}_i)$. By Proposition 7.1(1), there is a stable embedding $C_i \rightarrowtail A$. Therefore, there is a stable embedding $C_i \rightarrowtail B$. Applying Proposition 7.1(1) again yields $\mathfrak{B} \not\models \rho(\mathfrak{C}_i)$. The obtained contradiction proves that $\mathfrak{A} \models L$. Thus, L is a stable normal modal logic. □

Definition 7.6.

(1) *A normal modal rule system $\mathcal{S}$ has the* finite model property (fmp) *if for each rule ρ with $\mathcal{S} \not\vdash \rho$, there exists a finite modal algebra $\mathfrak{A} = (A, \Diamond)$ such that $\mathfrak{A} \models \mathcal{S}$ and $\mathfrak{A} \not\models \rho$.*

(2) *A normal modal logic L has the* finite model property (fmp) *if for each formula φ with $L \not\vdash \varphi$, there exists a finite modal algebra $\mathfrak{A} = (A, \Diamond)$ such that $\mathfrak{A} \models L$ and $\mathfrak{A} \not\models \varphi$.*

Theorem 7.7.

(1) *Every stable normal modal rule system has the finite model property.*

(2) *Every stable normal modal logic has the finite model property.*

Proof. (1). Let $\mathcal{S}$ be a stable normal modal rule system and let ρ be a modal rule such that $\mathcal{S} \not\vdash \rho$. Then there is a modal algebra $\mathfrak{A} = (A, \Diamond)$ such that $\mathfrak{A} \models \mathcal{S}$ and $\mathfrak{A} \not\models \rho$. The proof of Theorem 5.1 yields a finite modal algebra $\mathfrak{A}' = (A', \Diamond')$ such that $\mathfrak{A}' \not\models \rho$ and the embedding $A' \rightarrowtail A$ is stable. Since $\mathcal{S}$ is stable, $\mathfrak{A}' \models \mathcal{S}$. Thus, $\mathcal{S}$ has the fmp.

(2). Let L be a stable normal modal logic and let φ be a formula such that $L \not\vdash \varphi$. Then there is a modal algebra $\mathfrak{A} = (A, \Diamond)$ such that $\mathfrak{A} \models L$ and $\mathfrak{A} \not\models \varphi$. The proof of Theorem 5.1 yields a finite modal algebra $\mathfrak{A}' = (A', \Diamond')$ such that $\mathfrak{A}' \not\models \varphi$ and the embedding $A' \rightarrowtail A$ is stable. Since L is stable, $\mathfrak{A}' \models L$. Thus, L has the fmp. □

Remark 7.8. Transitive stable logics play the same role in the theory of stable canonical formulas for **K4** as transitive subframe logics for Zakharyaschev's canonical formulas. Since transitive subframe logics admit selective filtration, they all have the fmp (see e.g., [12, Ch. 11.3]). The

method of selective filtration does not work well for non-transitive logics. Indeed, as was shown by Wolter [30], there exist non-transitive subframe logics without the fmp. On the other hand, by Theorem 7.7(2), all stable logics have the fmp, transitive or not. The reason for this is that unlike selective filtration, the standard filtration works just as well for non-transitive logics.

Stable logics also have some nice proof-theoretic properties. It is shown in [8] that every stable logic has the bounded proof property (bpp). We refer to [8] for more details on this.

We next turn to Jankov rules. We call a normal modal rule system $\mathcal{S}$ a *splitting rule system* if there is a normal modal rule system $\mathcal{T}$ such that $\mathcal{S} \not\subseteq \mathcal{T}$ and for each normal modal rule system $\mathcal{U}$, we have $\mathcal{S} \subseteq \mathcal{U}$ or $\mathcal{U} \subseteq \mathcal{T}$. The pair $(\mathcal{S}, \mathcal{T})$ is called a *splitting pair*. We call a normal modal rule system a *join splitting rule system* if it is a join (in the lattice $\Sigma(\mathbf{S_K})$) of splitting rule systems. Splitting and join splitting normal modal logics are defined similarly.

For a modal algebra $\mathfrak{A} = (A, \Diamond)$, let $\mathcal{S}(\mathfrak{A}) = \{\rho : \mathfrak{A} \models \rho\}$ and $L(\mathfrak{A}) = \{\varphi : \mathfrak{A} \models \varphi\}$. Then it is straightforward to verify that $\mathcal{S}(\mathfrak{A})$ is a normal modal rule system and $L(\mathfrak{A})$ is a normal modal logic. The following theorem was first proved by Jeřábek [15, Thm. 6.5] using a different technique.

Theorem 7.9. *Let $\mathcal{S}$ be a normal modal rule system.*

(1) *S is a splitting rule system iff $\mathcal{S}$ is axiomatizable by a Jankov rule.*
(2) *S is a join splitting rule system iff $\mathcal{S}$ is axiomatizable by Jankov rules.*

Proof. (1). First suppose that $\mathcal{S}$ is axiomatizable by a Jankov rule $\chi(\mathfrak{A})$. It is sufficient to show that $(\mathcal{S}, \mathcal{S}(\mathfrak{A}))$ is a splitting pair in $\Sigma(\mathbf{S_K})$. Since $\mathfrak{A} \not\models \chi(\mathfrak{A})$, we have $\mathcal{S} \not\subseteq \mathcal{S}(\mathfrak{A})$. Let $\mathcal{T}$ be a rule system such that $\mathcal{S} \not\subseteq \mathcal{T}$. Then there is a modal algebra $\mathfrak{B} = (B, \Diamond)$ such that $\mathfrak{B} \models \mathcal{T}$ and $\mathfrak{B} \not\models \mathcal{S}$. Therefore, $\mathfrak{B} \not\models \chi(\mathfrak{A})$. By Proposition 7.1(2), there is a 1-1 modal homomorphism $A \rightarrowtail B$. Thus, $\mathcal{T} \subseteq \mathcal{S}(\mathfrak{B}) \subseteq \mathcal{S}(\mathfrak{A})$, and hence $(\mathcal{S}, \mathcal{S}(\mathfrak{A}))$ is a splitting pair in $\Sigma(\mathbf{S_K})$.

Next suppose that $\mathcal{S}$ is a splitting rule system. Then there is a normal modal rule system $\mathcal{T}$ such that $(\mathcal{S}, \mathcal{T})$ is a splitting pair in $\Sigma(\mathbf{S_K})$. Since $\mathbf{S_K}$ has the fmp, there is a finite modal algebra $\mathfrak{A} = (A, \Diamond)$ such that $\mathcal{T} = \mathcal{S}(\mathfrak{A})$ (see, e.g., [21, Sec. 4]). Therefore, $(\mathcal{S}, \mathcal{S}(\mathfrak{A}))$ is a splitting pair. We show that $\mathcal{S} = \mathbf{S_K} + \chi(\mathfrak{A})$. For this it is sufficient to see that for each modal algebra $\mathfrak{B} = (B, \Diamond)$, we have $\mathfrak{B} \not\models \mathcal{S}$ iff $\mathfrak{B} \not\models \chi(\mathfrak{A})$. If $\mathfrak{B} \not\models \chi(\mathfrak{A})$, then by Proposition 7.1(2), there is a 1-1 modal homomorphism $A \rightarrowtail B$. This gives $\mathcal{S}(\mathfrak{B}) \subseteq \mathcal{S}(\mathfrak{A})$. If $\mathfrak{B} \models \mathcal{S}$, then $\mathcal{S} \subseteq \mathcal{S}(\mathfrak{B}) \subseteq \mathcal{S}(\mathfrak{A})$. This is a contradiction because $\mathcal{S} \not\subseteq \mathcal{S}(\mathfrak{A})$. Therefore, $\mathfrak{B} \not\models \mathcal{S}$. Conversely, if $\mathfrak{B} \not\models \mathcal{S}$, then $\mathcal{S} \not\subseteq \mathcal{S}(\mathfrak{B})$. Since $(\mathcal{S}, \mathcal{S}(\mathfrak{A}))$ is a splitting pair, this yields $\mathcal{S}(\mathfrak{B}) \subseteq \mathcal{S}(\mathfrak{A})$. As $\chi(\mathfrak{A}) \notin \mathcal{S}(\mathfrak{A})$, this gives $\mathfrak{B} \not\models \chi(\mathfrak{A})$. Thus, $\mathcal{S} = \mathbf{S_K} + \chi(\mathfrak{A})$.

(2). This follows from (1). □

Remark 7.10. In [6, 7, 2, 3] the theory of algebra-based (or equivalently frame-based) formulas is developed and a general criterion when a logic is axiomatized by these formulas is established. Such well-known classes of formulas as Jankov formulas, stable formulas, subframe formulas and others are particular instances of algebra-based formulas. This theory has a natural generalization to the theory of algebra-based (or equivalently frame-based) rules. We will not pursue it here, and only note that stable rules and Jankov rules are particular instances of these algebra-based rules.

We call a modal algebra $\mathfrak{A} = (A, \Diamond)$ *cycle-free* if $\Box^n 0 = 1$ for some $n \in \omega$ (equivalently $\Diamond^n 1 = 0$ for some $n \in \omega$). Cycle-free modal algebras correspond to cycle-free modal spaces, where a modal space $\mathfrak{X} = (X, R)$ is *cycle-free* if $x \notin R^\omega(x)$ for each $x \in X$ [12, p. 357]. We call $\mathfrak{A}$ *n-cycle-free* if $\Box^{n+1} 0 = 1$.

Lemma 7.11. *Let $\mathfrak{A} = (A, \Diamond)$ be n-cycle-free and $a, b \in A$ with $\blacksquare_n a \not\leq b$. Then there is a s.i modal algebra $\mathfrak{B} = (B, \Diamond)$ and an onto modal homomorphism $h : A \twoheadrightarrow B$ such that $h(\blacksquare_n a) = 1$ and $h(b) \neq 1$.*

Proof. The proof is similar to that of [1, Lem. 4.1] and we only sketch it. Let F be the filter generated by $\blacksquare_n a$. Then $\blacksquare_n a \in F$ and $b \notin F$. If $x \in F$, then $\blacksquare_n a \leq x$, so $\Box \blacksquare_n a \leq \Box x$. Since

$\mathfrak{A}$ is n-cycle-free, $\blacksquare_n a \leq \Box\blacksquare_n a$. Therefore, $\blacksquare_n a \leq \Box x$, and hence F is a modal filter. By Zorn's lemma, there is a maximal modal filter G such that $\blacksquare_n a \in G$ and $b \notin G$. Since G is maximal with this property, the quotient algebra $\mathfrak{B} = \mathfrak{A}/G$ is s.i. Let $h : A \twoheadrightarrow B$ be the quotient map. Then $h(\blacksquare_n a) = 1$ and $h(b) \neq 1$. □

Definition 7.12. *Let $\mathfrak{A} = (A, \Diamond)$ be a finite s.i. n-cycle-free modal algebra, and let $D \subseteq A$. For each $a \in A$ we introduce a new propositional letter p_a and define the* stable canonical formula *$\varepsilon(\mathfrak{A}, D)$ associated with A and D as follows:*

$$\begin{aligned} \varepsilon(\mathfrak{A}, D) &= \bigwedge\{\blacksquare_n\gamma : \gamma \in \Gamma\} \to \bigvee\{\blacksquare_n\delta : \delta \in \Delta\} \\ &= \blacksquare_n \bigwedge \Gamma \to \bigvee\{\blacksquare_n\delta : \delta \in \Delta\}, \end{aligned}$$

where Γ and Δ are as in Definition 5.2.

Theorem 7.13. *Let $\mathfrak{A} = (A, \Diamond)$ be a finite s.i. n-cycle-free modal algebra, $D \subseteq A$, and $\mathfrak{B} = (B, \Diamond)$ be a modal algebra. Then $\mathfrak{B} \not\models \varepsilon(\mathfrak{A}, D)$ iff there is a s.i. homomorphic image $\mathfrak{C} = (C, \Diamond)$ of $\mathfrak{B}$ and a stable embedding $h : A \rightarrowtail C$ such that $h(\Diamond a) = \Diamond h(a)$ for each $a \in D$.*

Proof. The proof follows the same pattern as the proof of Theorem 6.8. First suppose that there is a s.i. homomorphic image $\mathfrak{C}$ of $\mathfrak{B}$ and a stable embedding $h : A \rightarrowtail C$ such that $h(\Diamond a) = \Diamond h(a)$ for each $a \in D$. Define a valuation V_A on A by $V_A(p_a) = a$ for each $a \in A$. Then $V_A(\gamma) = 1_A$ for each $\gamma \in \Gamma$ and $V_A(\delta) \neq 1_A$ for each $\delta \in \Delta$. Therefore, $V_A(\blacksquare_n \bigwedge \Gamma) = 1_A$ and $V_A(\blacksquare_n\delta) \neq 1_A$ for each $\delta \in \Delta$. Since $\mathfrak{A}$ is s.i., it has an opremum c. Let $a \neq 1$. Then there is $m \in \omega$ with $\blacksquare_m a \leq c$. As $\mathfrak{A}$ is n-cycle-free, $\blacksquare_n a \leq \blacksquare_m a$, so $\blacksquare_n a \leq c$. Thus, $\bigvee\{V_A(\blacksquare_n\delta) : \delta \in \Delta\} \leq c$, and hence $\mathfrak{A} \not\models \varepsilon(\mathfrak{A}, D)$. Next define a valuation V_C on C by $V_C(p_a) = h(V_A(p_a)) = h(a)$ for each $a \in A$. The same argument as in the proof of Theorem 5.4 shows that $V_C(\gamma) = 1_C$ for each $\gamma \in \Gamma$ and $V_C(\delta) \neq 1_C$ for each $\delta \in \Delta$. Therefore, $V_C(\blacksquare_n \bigwedge \Gamma) = 1_C$ and $V_C(\blacksquare_n\delta) \neq 1_C$ for each $\delta \in \Delta$. Because $\mathfrak{C}$ is s.i., it has an opremum. As $\mathfrak{A}$ is n-cycle-free, so is $\mathfrak{C}$, and the same argument as above yields that $\bigvee\{V_C(\blacksquare_n\delta) : \delta \in \Delta\}$ is underneath the opremum, hence $\mathfrak{C} \not\models \varepsilon(\mathfrak{A}, D)$. Since $\mathfrak{C}$ is a homomorphic image of $\mathfrak{B}$, we conclude that $\mathfrak{B} \not\models \varepsilon(\mathfrak{A}, D)$.

Conversely, suppose that $\mathfrak{B} \not\models \varepsilon(\mathfrak{A}, D)$. By Lemma 7.11, there is a s.i. homomorphic image $\mathfrak{C}$ of $\mathfrak{B}$ and a valuation V_C on C such that $V_C(\blacksquare_n \bigwedge \Gamma) = 1_C$ and $V_C(\bigvee\{\blacksquare_n\delta : \delta \in \Delta\}) \neq 1_C$. Next we define a map $h : A \to C$ by $h(a) = V_C(p_a)$ for each $a \in A$. The proof of Theorem 5.4 then shows that h is a stable embedding such that $h(\Diamond a) = \Diamond h(a)$ for each $a \in D$. □

For a finite s.i. n-cycle-free modal algebra $\mathfrak{A} = (A, \Diamond)$, we denote $\varepsilon(\mathfrak{A}, \mathfrak{A})$ by $\varepsilon(\mathfrak{A})$ and call it the *Jankov formula* of $\mathfrak{A}$. The next theorem provides a different proof of Blok's theorem [10].

Theorem 7.14. *Let L be a normal modal logic.*

1. *L is a splitting logic iff L is axiomatizable by the Jankov formula of a finite s.i. cycle-free modal algebra.*
2. *L is a join splitting logic iff L is axiomatizable by Jankov formulas of finite s.i. cycle-free modal algebras.*

Proof. (1). First suppose that $L = \mathbf{K}+\varepsilon(\mathfrak{A})$ for some finite s.i. cycle-free modal algebra $\mathfrak{A} = (A, \Diamond)$. Then $\varepsilon(\mathfrak{A}) \in L$. On the other hand, by Theorem 7.13, $\mathfrak{A} \not\models \varepsilon(\mathfrak{A})$. Therefore, $\varepsilon(\mathfrak{A}) \notin L(\mathfrak{A})$, and hence $L \not\subseteq L(\mathfrak{A})$. Let M be a normal modal logic such that $L \not\subseteq M$. Then there is a modal algebra $\mathfrak{B} = (B, \Diamond)$ such that $\mathfrak{B} \models M$ and $\mathfrak{B} \not\models L$. This gives $\mathfrak{B} \not\models \varepsilon(\mathfrak{A})$. By Theorem 7.13, $\mathfrak{A}$ is isomorphic to a subalgebra of a s.i. homomorphic image $\mathfrak{C}$ of $\mathfrak{B}$. Thus, $M \subseteq L(\mathfrak{B}) \subseteq L(\mathfrak{C}) \subseteq L(\mathfrak{A})$. Consequently, $(L, L(\mathfrak{A}))$ is a splitting pair.

Conversely, suppose that L is a splitting logic. Then there is a normal modal logic M such that (L, M) is a splitting pair. It is well known (see, e.g., [12, Cor. 3.29]) that $\mathbf{K}$ is the modal logic of all finite trees. Since (L, M) is a splitting pair, M is a completely meet-prime element

of $\Lambda(\mathbf{K})$. Therefore, there is the dual modal algebra $\mathfrak{E}$ of some finite tree such that $L(\mathfrak{E}) \subseteq M$. This, by Jónsson's lemma, means that $M = L(\mathfrak{A})$, where $\mathfrak{A}$ is a s.i. homomorphic image of a subalgebra of $\mathfrak{E}$. Because $\mathfrak{E}$ is cycle-free, so is $\mathfrak{A}$. We show that $L = \mathbf{K} + \varepsilon(\mathfrak{A})$. Let $\mathfrak{B}$ be a modal algebra. If $\mathfrak{B} \not\models \mathbf{K} + \varepsilon(\mathfrak{A})$, then $\mathfrak{B} \not\models \varepsilon(\mathfrak{A})$. By Theorem 7.13, $\mathfrak{A}$ is isomorphic to a subalgebra of a s.i. homomorphic image $\mathfrak{C}$ of $\mathfrak{B}$, so $L(\mathfrak{B}) \subseteq L(\mathfrak{C}) \subseteq L(\mathfrak{A})$. If $\mathfrak{B} \models L$, then $L \subseteq L(\mathfrak{B}) \subseteq L(\mathfrak{C}) \subseteq L(\mathfrak{A})$, which is a contradiction as $L \not\subseteq L(\mathfrak{A})$. So we have $\mathfrak{B} \not\models L$. Conversely, if $\mathfrak{B} \not\models L$, then $L \not\subseteq L(\mathfrak{B})$. Since $(L, L(\mathfrak{A}))$ is a splitting pair, $L(\mathfrak{B}) \subseteq L(\mathfrak{A})$. As $\mathfrak{A} \not\models \varepsilon(\mathfrak{A})$, this yields $\mathfrak{B} \not\models \mathbf{K} + \varepsilon(\mathfrak{A})$. Thus, $L = \mathbf{K} + \varepsilon(\mathfrak{A})$.

(2). This follows from (1). □

Remark 7.15. As follows from Remarks 5.7 and 6.12, stable canonical rules and stable canonical formulas for **K4** can be defined directly for finite modal spaces without using algebraic terminology. The same is true for Jankov formulas, see [12, p. 357].

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