Intuitionistic modal logics and Dyckhoff's calculus

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Abstract

In 1992 Dyckhoff developed a sequent calculus for intuitionistic propositional logic in which proof search is terminating. In this paper this result is extended to several intuitionistic modal logics, including intuitionistic versions of the modal logics K and KD. Cut–free sequent calculi that are extensions of Dyckhoff's calculus are developed for these logics and are shown, by proof–theoretic means, to be equal to the more common sequent calculi for the intuitionistic modal logics considered here.

Keywords: intuitionistic modal logic, sequent calculus, intermediate logic
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1 Introduction

For several standard sequent calculi for classical propositional logic CPC, a corresponding calculus for intuitionistic propositional logic IPC can be obtained by restricting the right side of the sequents to one formula or, depending on the calculus, at most one formula. For the well–known calculus G3 in (Troelstra and Schwichtenberg, 1996) this is no longer the case. This calculus does not contain structural rules, which are admissible in it, and restricting the right side of the sequents to exactly one formula does not result in a calculus for intuitionistic propositional logic, as some of the structural rules are no longer admissible. This defect can be remedied by slightly reformulating some of the rules. In this way a calculus iG3 for intuitionistic propositional logic without structural rules is obtained. But an important property of the classical system G3 is lost. Namely, proof search in iG3 is no longer terminating.

The lack of termination makes iG3 unfit for certain proof–theoretic arguments about IPC, such as the proof that IPC has uniform interpolation by Pitts (1992). For his intricate proof Pitts uses a terminating calculus for IPC developed by Dyckhoff

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(1992). This calculus, called LJT by Dyckhoff and DY in this paper[1], is the result of replacing the left implication rule in $\mathsf{iG3}$ by four implication rules. Proof search in this calculus always terminates because in all the rules in the calculus the premisses come before the conclusion in an order on sequents introduced by Dyckhoff in the same paper.

In a forthcoming paper (Iemhoff, 2017) the author establishes that several intuitionistic modal logics have uniform interpolation. For these results, we use the terminating sequent calculi without structural rules à la Dyckhoff developed in this paper. They are extensions of DY in which the structural rules are admissible. The reason to not simply include the results in this paper in (Iemhoff, 2017) is that the latter paper would become quite long, and, more importantly, that terminating sequent calculi for modal intuitionistic logics might be useful tools outside the realm of uniform interpolation. Whether these calculi will indeed find such applications elsewhere, only the future will tell.

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2 Logics and calculi

The logics we consider are intermediate propositional logics, formulated in a language $\mathcal{L}$ that contains *constant* $\bot$, *propositional variables* or *atoms* $p, q, r, \ldots$ and the *connectives* $\wedge, \vee, \rightarrow$, where $\neg\varphi$ is defined as $\varphi \rightarrow \bot$. $\bot$ is by definition not an atom. $\mathcal{F}$ denotes the set of formulas in $\mathcal{L}$. Given a set of atoms $\mathcal{P}$, $\mathcal{F}(\mathcal{P})$ denotes all formulas in $\mathcal{L}$ in which all atoms belong to $\mathcal{P}$.

All logics we consider are extensions of intuitionistic propositional logic IPC and satisfy the necessitation rule in case the logic contains modalities. We do not assume the modal logics to be normal. The logics are given as sets of formulas, and $\vdash_{\mathsf{L}} \varphi$ denotes that formula φ holds in logic L. In this paper the logics are all defined via sequent calculi.

We only consider (single-conclusion) sequents, which are expressions $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas and Δ contains at most one formula. Such sequents are interpreted as $I(\Gamma \Rightarrow \Delta) = (\bigwedge \Gamma \rightarrow \bigvee \Delta)$. We denote finite multisets by $\Gamma, \Pi, \Delta, \Sigma$. In a sequent, notation Π, Γ is short for $\Gamma \cup \Pi$. In writing an expression $\Gamma \Rightarrow \Delta$ it is understood that Δ contains at most one formula. We also define (a for antecedent, s for succedent):

$$(\Gamma \Rightarrow \Delta)^a \equiv_{df} \Gamma \qquad (\Gamma \Rightarrow \varphi)^s \equiv_{df} \Delta.$$

When sequents are used in the setting of formulas, we often write S for $I(S)$, such as in $\vdash \bigvee_i (S_i \Rightarrow S)$, which thus means $\vdash \bigvee_i \big(I(S_i) \rightarrow I(S)\big)$.

[1] Since LJ is the name that Gentzen used for a calculus *with* structural rules, we think LJT perhaps not the best name for Dyckhoff's calculus and take the opportunity to replace it by one that refers to its creator.

Given a sequent calculus G and a sequent S, $\vdash_{\mathsf{G}} S$ denotes that S is derivable in G. The logic $\mathsf{L_G}$ corresponding to G is defined as

$$\vdash_{\mathsf{L_G}} I(S) \ \equiv_{df} \vdash_{\mathsf{G}} S.$$

We use an order on sequents based on the *weight function* $w(\cdot)$ on formulas from (Dyckhoff, 1992), which is inductively defined as: the weight of atoms is 1, $w(\Box\varphi) = w(\varphi) + 1$, and $w(\varphi \circ \psi) = w(\varphi) + w(\psi) + i$, where $i = 1$ in case $\circ \in \{\vee, \to\}$ and $i = 2$ otherwise. We use the following ordering on sequents: $S_0 \preccurlyeq S_1$ if and only if $S_0^a \cup S_0^s \preccurlyeq S_1^a \cup S_1^s$, where $\preccurlyeq$ is the order on multisets determined by weight as in (Dershowitz and Manna, 1979): for multisets Γ, Δ we have $\Delta \preccurlyeq \Gamma$ if Δ is the result of replacing one or more formulas in Γ by zero or more formulas of lower weight.

2.1 Intuitionistic modal logic

The calculus DY in Figure 2.1 is Dyckhoff's calculus LJT (Dyckhoff, 1992) but then for the language of propositional modal logic instead of pure propositional logic, and with slightly different names for the rules. Recall that in this paper sequents are assumed to have at most one formula in the succedent, which is a slight but inessential deviation from Dyckhoff's approach, where sequents are assumed to have exactly one formula at the right. The calculus is terminating with respect to the above well-ordering $\preccurlyeq$ on sequents. The calculus has no structural rules, but they are admissible in it, as is shown below.

$$\Gamma, p \Rightarrow p \quad At \quad (p \text{ an atom}) \qquad\qquad \Gamma, \bot \Rightarrow \Delta \quad L\bot$$

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}\ R\wedge \qquad\qquad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ L\wedge$$

$$\frac{\Gamma \Rightarrow \varphi, \psi}{\Gamma \Rightarrow \varphi \vee \psi}\ R\vee \qquad\qquad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ L\vee$$

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\ R{\to} \qquad\qquad \frac{\Gamma, p, \varphi \Rightarrow \Delta}{\Gamma, p, p \to \varphi \Rightarrow \Delta}\ Lp{\to} \quad (p \text{ an atom})$$

$$\frac{\Gamma, \varphi \to (\psi \to \gamma) \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \to \gamma \Rightarrow \Delta}\ L{\wedge}{\to} \qquad\qquad \frac{\Gamma, \varphi \to \gamma, \psi \to \gamma \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \to \gamma \Rightarrow \Delta}\ L{\vee}{\to}$$

$$\frac{\Gamma, \psi \to \gamma \Rightarrow \varphi \to \psi \quad \gamma, \Gamma \Rightarrow \Delta}{\Gamma, (\varphi \to \psi) \to \gamma \Rightarrow \Delta}\ L{\to}{\to}$$

Figure 2.1: The Gentzen calculus DY

The calculus $\mathsf{iG3M}$ consists of the rules of DY where the four left implication rules are replaced by the following rule.

$$\frac{\Gamma, \varphi \to \psi \Rightarrow \varphi \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta} L\!\to$$

Note that $\mathsf{iG3M}$ is the propositional part of $\mathsf{iG3}$ from (Troelstra and Schwichtenberg, 1996) but then for the language of modal instead of propositional logic.

Modal and additional left implication rules, *where* Π *ranges over multisets that do not contain boxed formulas*:

$$\frac{\Gamma \Rightarrow \varphi}{\Pi, \Box\Gamma \Rightarrow \Box\varphi} \mathcal{R}_K \qquad \frac{\Gamma \Rightarrow \varphi \quad \Pi, \Box\Gamma, \psi \Rightarrow \Delta}{\Pi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \Delta} L\Box\!\to$$

$$\frac{\boxdot\Gamma \Rightarrow \varphi}{\Pi, \Box\Gamma \Rightarrow \Box\varphi} \mathcal{R}_{K4} \qquad \frac{\boxdot\Gamma \Rightarrow \varphi \quad \Pi, \Box\Gamma, \psi \Rightarrow \Delta}{\Pi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \Delta} L_{K4}\Box\!\to$$

$$\frac{\Gamma, \varphi \Rightarrow}{\Pi, \Box\Gamma, \Box\varphi \Rightarrow \Delta} \mathcal{R}_D$$

DYM is the calculus DY plus the rule $L\Box\!\to$. For $\mathsf{X} \in \{\mathsf{K}, \mathsf{D}, \mathsf{K4}\}$, the calculus $\mathsf{iG3X}$ consists of $\mathsf{iG3M}$ plus the rule $\mathcal{R}_X$, and $\mathsf{iG3KD}$ is $\mathsf{iG3K}$ plus the rule $\mathcal{R}_D$. The calculus DYK consists of DYM plus the rule $\mathcal{R}_K$, and $\mathsf{DYK4}$ consists of DYM plus the rules $\mathcal{R}_{K4}$ and $L_{K4}\Box\!\to$. The calculi DYD and DYKD consist of $\mathcal{R}_D$, plus DYM and DYK, respectively.

In the rules of $\mathsf{iG3X}$, the *principal* formula of an inference is defined as usual for the connectives, and in the modal rules above all formulas in $\Box\Gamma$ as well as $\Box\varphi$ are principal.

The *depth* of a proof is the length of its longest branch. If $\vdash$ stands for derivation in a given calculus, the we write $\vdash_d S$ if S has a proof of depth at most d in that calculus.

3 Structural rules in $\mathsf{iG3X}$

Lemma 3.1 (*Weakening, Contraction, Inversion Lemma*)
For $\mathsf{X} \in \{\mathsf{M}, \mathsf{K}, \mathsf{D}, \mathsf{KD}, \mathsf{K4}\}$ and $\vdash$ denoting $\vdash_{\mathsf{iG3X}}$, the following statements hold.

Weakening $\vdash_d \Gamma \Rightarrow \Delta$ implies $\vdash_d \Gamma, \varphi \Rightarrow \Delta$.

Contraction $\vdash_d \Gamma, \varphi, \varphi \Rightarrow \Delta$ implies $\vdash_d \Gamma, \varphi \Rightarrow \Delta$.

Inversion $\wedge$ $\vdash_d \Gamma, \varphi \wedge \psi \Rightarrow \Delta$ implies $\vdash_d \Gamma, \varphi, \psi \Rightarrow \Delta$.

Inversion $\vee$ $\vdash_d \Gamma, \varphi_1 \vee \varphi_2 \Rightarrow \Delta$ implies $\vdash_d \Gamma, \varphi_i \Rightarrow \Delta$ for $i = 1, 2$.

Inversion $R\!\to$ $\vdash_d \Gamma \Rightarrow \varphi \to \psi$ implies $\vdash_d \Gamma, \varphi \Rightarrow \psi$.

Inversion $L\!\to$ $\vdash_d \Gamma, \varphi \to \psi \Rightarrow \Delta$ implies $\vdash_d \Gamma, \psi \Rightarrow \Delta$.

Proof The proofs are standard and therefore omitted. For details, see page 66–67 in (Troelstra and Schwichtenberg, 1996). $\dashv$

Given a proof in $\mathsf{iG3X}$ the *level* of a cut is the sum of the depths of the deductions of the premisses, and the *rank* of a cut with cutformula φ is $|\varphi|$, where $|\varphi|$ denotes the maximum length of a branch in its construction tree. The *cutrank* of a proof is the maximum of the cutranks of the cuts occurring in the proof.

Theorem 3.2 (*Cut Admissibility Lemma*)
For $\mathsf{X} \in \{\mathsf{M}, \mathsf{K}, \mathsf{D}, \mathsf{KD}, \mathsf{K4}\}$, the Cut Rule is admissible in $\mathsf{iG3X}$.

Proof We have to show that the rule

$$\frac{\Gamma_l \Rightarrow \varphi \quad \Gamma_r, \varphi \Rightarrow \Delta}{\Gamma_l, \Gamma_r \Rightarrow \Delta}$$

is admissible in $\mathsf{iG3X}$. Let $\vdash$ denote $\vdash_{\mathsf{iG3X}}$. Following (Troelstra and Schwichtenberg, 1996), we use induction to the cutrank and a subinduction to the maximum level of the cuts of maximal cutrank in the proof of $\Gamma \Rightarrow \Delta$. We use the fact that $\mathsf{iG3X}$ is closed under weakening and contraction implicitly all the time.

Thus it suffices to show that in a proof $\mathcal{D}$ that ends with a cut

$$\frac{\begin{array}{c}\mathcal{D}_l \\ \Gamma_l \Rightarrow \varphi\end{array} \quad \begin{array}{c}\mathcal{D}_r \\ \Gamma_r, \varphi \Rightarrow \Delta\end{array}}{\Gamma_l, \Gamma_r \Rightarrow \Delta}$$

where the cutranks of $\mathcal{D}_l$ and $\mathcal{D}_r$ are lower than $|\varphi|$, by a proof of the same endsequent of cutrank lower than $|\varphi|$. By the induction hypothesis we can assume that $\mathcal{D}_l$ and $\mathcal{D}_r$ are cutfree.

There are three possibilities:

1. at least one of the premisses is an axiom;
2. both premisses are not axioms and the cutformula is not principal in at least one of the premisses;
3. the cutformula is principal in both premisses, which are not axioms.

1. As in (Troelstra and Schwichtenberg, 1996), straightforward, by checking all possible cases.

2. First, the case that φ is not pricipal in $\mathcal{D}_l$. Thus the last inference in $\mathcal{D}_l$ is $\mathcal{R}_D$, if present, or one of the nonmodal rules of $\mathsf{iG3X}$. In the first case, $\Gamma_l = \Pi, \Box\Gamma, \Box\psi$ and the last inference is of the form

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}' \\ \Gamma', \psi \Rightarrow\end{array}}{\Pi, \Box\Gamma', \Box\psi \Rightarrow \varphi} \quad \Gamma_r, \varphi \Rightarrow \Delta}{\Gamma_l, \Gamma_r \Rightarrow \Delta}$$

Clearly, the following is a derivation of the same endsequent.

$$\dfrac{\begin{matrix}\mathcal{D}'\\ \Gamma', \psi \Rightarrow\end{matrix}}{\Gamma_r, \Pi, \Box\Gamma', \Box\psi \Rightarrow \Delta}\ \mathcal{R}_D$$

In case the last inference in $\mathcal{D}_l$ is not a modal rule, the lower part of $\mathcal{D}$ looks as follows, in case $\mathcal{R}$ is a two premiss rule.

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow \varphi_l \quad \Gamma_2 \Rightarrow \varphi}{\Gamma_l \Rightarrow \varphi}\ \mathcal{R} \qquad \begin{matrix}\mathcal{D}_r\\ \Gamma_r, \varphi \Rightarrow \Delta\end{matrix}}{\Gamma_l, \Gamma_r \Rightarrow \Delta}$$

For all possible rules except L$\rightarrow$, $\varphi_l = \varphi$. The cut can be pushed upwards:

$$\dfrac{\Gamma_r, \Gamma_1 \Rightarrow \varphi_l \qquad \dfrac{\Gamma_2 \Rightarrow \varphi \quad \Gamma_r, \varphi \Rightarrow \Delta}{\Gamma_r, \Gamma_2 \Rightarrow \Delta}}{\Gamma_l, \Gamma_r \Rightarrow \Delta}\ \mathcal{R}$$

Thus we obtain a proof of $\Gamma_r, \Gamma_2 \Rightarrow \Delta$ of the same cutrank, but where the cut of maximal cutrank is of lower level, and the induction hypothesis can be applied. The case of a one premiss rule is similar.

Second, the case that φ is not pricipal in $\mathcal{D}_r$. The nonmodal rules are treated as in the previous case. We treat the case that it is an application of $\mathcal{R}_K$, the other modal rules can be treated in the same way. Thus the lower part of $\mathcal{D}$ looks as follows.

$$\dfrac{\Gamma_l \Rightarrow \varphi \qquad \dfrac{\begin{matrix}\mathcal{D}'\\ \Gamma \Rightarrow \psi\end{matrix}}{\Pi, \Box\Gamma, \varphi \Rightarrow \Box\psi}\ \mathcal{R}_K}{\Gamma_l, \Pi, \Box\Gamma \Rightarrow \Box\psi}$$

Clearly, the following is a proof of the same endsequent but of lower cutrank.

$$\dfrac{\begin{matrix}\mathcal{D}'\\ \Gamma \Rightarrow \psi\end{matrix}}{\Gamma_l, \Pi, \Box\Gamma \Rightarrow \Box\psi}\ R_K$$

3. The cutformula is principal in both premisses, which are not axioms. We distinguish by cases according to the form of the cutformula, and treat implications and boxed formulas.

If the cutformula is an implication, the last inference looks as follows.

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma_l, \varphi \Rightarrow \psi\end{matrix}}{\Gamma_l \Rightarrow \varphi \rightarrow \psi}\ R\rightarrow \qquad \dfrac{\begin{matrix}\mathcal{D}_2\\ \Gamma_r, \varphi \rightarrow \psi \Rightarrow \varphi\end{matrix} \quad \begin{matrix}\mathcal{D}_3\\ \Gamma_r, \psi \Rightarrow \Delta\end{matrix}}{\Gamma_r, \varphi \rightarrow \psi \Rightarrow \Delta}\ L\rightarrow}{\Gamma_l, \Gamma_r \Rightarrow \Delta}$$

This is replaced by the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma_l, \varphi \Rightarrow \psi}}{\Gamma_l \Rightarrow \varphi \to \psi} \quad \dfrac{\mathcal{D}_2}{\Gamma_r, \varphi \to \psi \Rightarrow \varphi}}{\Gamma_l, \Gamma_r \Rightarrow \varphi}\, Cut \quad \dfrac{\mathcal{D}_1}{\Gamma_l, \varphi \Rightarrow \psi}}{\Gamma_l, \Gamma_l, \Gamma_r \Rightarrow \psi}\, Cut \quad \dfrac{\mathcal{D}_3}{\Gamma_r, \psi \Rightarrow \Delta}}{\Gamma_l, \Gamma_l, \Gamma_r, \Gamma_r \Rightarrow \Delta}\, Cut$$

in which each cut either is of lower rank or of the same rank but of lower level. The other connectives can be treated in the same way.

If the cutformula is a boxed formula, then the form of the last inference depends on which modal rules are present in the calculus. We treat one case and leave the others to the reader. Suppose the last inference has the following form.

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma_l \Rightarrow \varphi}}{\Pi_l, \Box\Gamma_l \Rightarrow \Box\varphi}\, \mathcal{R}_K \quad \dfrac{\dfrac{\mathcal{D}_2}{\Gamma_r, \varphi \Rightarrow \psi}}{\Pi_r, \Box\Gamma_r, \Box\varphi \Rightarrow \Box\psi}\, \mathcal{R}_K}{\Pi_l, \Pi_r, \Box\Gamma_l, \Box\Gamma_r \Rightarrow \Box\psi}$$

This can be replaced by a proof of lower cutrank, namely

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma_l \Rightarrow \varphi} \quad \dfrac{\mathcal{D}_2}{\Gamma_r, \varphi \Rightarrow \psi}}{\Gamma_l, \Gamma_r \Rightarrow \psi}\, Cut}{\Pi_l, \Pi_r, \Box\Gamma_l, \Box\Gamma_r \Rightarrow \Box\psi}$$

This completes the proof of cut admissibility. $\dashv$

4 Strict proofs in DYX

In this section we prove a normal form theorem for proofs in the Dyckhoff calculi which enables us to establish, in the next section, that iG3X and DYX are equal. This in turn implies that the structural rules are admissible in the Dyckhoff calculi, a fact that is used in the application of these calculi in the context of uniform interpolation in (Iemhoff, 2017).

A multiset is *irreducible* if it has no element that is a disjunction or a conjunction or falsum and for no atom p does it contain both $p \to \psi$ and p. A sequent S is *irreducible* if S^a is. An application of L$\to$ in a proof in iG3X is *awkward* if the principal formula is an implication $p \to \psi$ where p is an atom. A proof is *sensible* if the last inference is not awkward.

A proof in iG3X is *strict* if in any application of L$\to$ with an irreducible conclusion and principal formula of the form $\Box\varphi \to \psi$, the left premiss is an axiom or the conclusion of an application of one of the modal rules $\mathcal{R}_K$, $\mathcal{R}_{K4}$ or $\mathcal{R}_{GL}$.

Lemma 4.1 For $\mathsf{X} \in \{\mathsf{M}, \mathsf{K}, \mathsf{D}, \mathsf{KD}, \mathsf{K4}\}$: Every irreducible sequent that is provable in $\mathsf{iG3X}$ has a sensible proof.

Proof This is proved in the same way as the corresponding lemma (Lemma 1) in (Dyckhoff, 1992). Arguing by contradiction, assume that among all provable irreducible sequents that have no sensible proofs, S is such a sequent with the shortest proof, where shortness is measured along the leftmost branch of a proof. Thus the last inference in the proof is an application of L$\to$ and has a principal formula of the form $p \to \psi$:

$$\dfrac{\overset{\mathcal{D}_l}{\Gamma, p \to \psi \Rightarrow p} \quad \overset{\mathcal{D}_r}{\Gamma, \psi \Rightarrow \Delta}}{\Gamma, p \to \psi \Rightarrow \Delta}$$

Since S^a is irreducible, $p, \bot \notin S^a$. Therefore the left premiss cannot be an axiom and whence is the conclusion of a rule, say R. Since the succedent of the conclusion of R consists of an atom, this either is not a modal rule or it is $\mathcal{R}_D$.
In the last case, for some Π, Γ' such that $\Gamma = \Pi, \Box\Gamma'$ the last part of the proof looks as

$$\dfrac{\dfrac{\overset{\mathcal{D}'}{\Gamma' \Rightarrow}}{\Pi, \Box\Gamma', p \to \psi \Rightarrow p}\ \mathcal{R}_D \quad \overset{\mathcal{D}_r}{\Gamma, \psi \Rightarrow \Delta}}{\Pi, \Box\Gamma', p \to \psi \Rightarrow \Delta}$$

Hence the following is a sensible proof of the same endsequent.

$$\dfrac{\overset{\mathcal{D}'}{\Gamma' \Rightarrow}}{\Pi, \Box\Gamma', p \to \psi \Rightarrow \Delta}\ R_D$$

In the first case, we proceed as in (Dyckhoff, 1992). Sequent $(\Gamma, p \to \psi \Rightarrow p)$ has a sensible proof, and since the sequent is irreducible, the last inference of this proof is L$\to$ with a principal formula $\varphi \to \psi'$ such that φ is not an atom. Let $\mathcal{D}'$ be the proof of the left premiss $(\Gamma, p \to \psi \Rightarrow \varphi)$. Then we obtain the following sensible proof of S, where $\Pi, \varphi \to \psi' = \Gamma$.

$$\dfrac{\overset{\mathcal{D}'}{\Pi, p \to \psi, \varphi \to \psi' \Rightarrow \varphi} \quad \dfrac{\overset{\mathcal{D}''}{\Pi, p \to \psi, \psi' \Rightarrow p} \quad \overset{\mathcal{D}'''}{\Pi, \psi, \psi' \Rightarrow \Delta}}{\Pi, p \to \psi, \psi' \Rightarrow \Delta}}{\Pi, p \to \psi, \varphi \to \psi' \Rightarrow \Delta}$$

The existence of $\mathcal{D}''$ and $\mathcal{D}'''$ follows from the Inversion Lemma and the existence of $\mathcal{D}_l$ and $\mathcal{D}_r$, respectively.

$\dashv$

Lemma 4.2 For $\mathsf{X} \in \{\mathsf{M}, \mathsf{K}, \mathsf{D}, \mathsf{KD}, \mathsf{K4}\}$: Every irreducible sequent that is provable in $\mathsf{iG3X}$ has a sensible strict proof.

Proof We treat the case iG3KD, the proofs for the other calculi are similar. We use induction on $\preccurlyeq$. Let S be an irreducible sequent. If S does not contain connectives or $\Box$, then if it is provable it has to be an axiom, which certainly is a sensible strict proof.

Suppose S contains connectives or $\Box$ and is provable, and consider a sensible proof $\mathcal{D}$ of S. With a subinduction to the depth of the left premiss of the last inference of the proof we show that S has a sensible strict proof. If $\mathcal{D}$ consists of an instance of an axiom, then S clearly has a sensible strict proof, namely $\mathcal{D}$. If $\mathcal{D}$ does not consist of an axiom, consider the last inference of the proof. By the induction hypothesis, the premiss(es) of the last inference have strict proofs. Thus if the last inference is not an application of L$\to$ with a principal formula of the form $\Box\varphi \to \psi$, it follows that S has a sensible strict proof.

Suppose the last inference of $\mathcal{D}$ is an application of L$\to$ and looks as follows.

$$\frac{\begin{array}{c}\mathcal{D}_l \\ \Gamma, \Box\varphi \to \psi \Rightarrow \Box\varphi\end{array} \quad \begin{array}{c}\mathcal{D}_r \\ \Gamma, \psi \Rightarrow \Delta\end{array}}{\Gamma, \Box\varphi \to \psi \Rightarrow \Delta}$$

If $\Gamma, \Box\varphi \to \psi \Rightarrow \Box\varphi$ is an axiom or the conclusion of $\mathcal{R}_K$, the proof is strict. If not, the irreducibility of S implies that it can only be the conclusion of L$\to$. We distinguish two cases.

If the principal formula is the same as in the last inference, $\mathcal{D}_l$ looks like

$$\frac{\begin{array}{c}\mathcal{D}' \\ \Gamma, \Box\varphi \to \psi \Rightarrow \Box\varphi\end{array} \quad \Gamma, \psi \Rightarrow \Box\varphi}{\Gamma, \Box\varphi \to \psi \Rightarrow \Box\varphi}$$

By replacing $\mathcal{D}_l$ in $\mathcal{D}$ by $\mathcal{D}'$ we obtain a proof of S where the depth of the left premiss of the last inference is lower than in the original proof, and the induction hypothesis applies.

If the principal formula is different from the one in the last inference, then consider the left most branch in $\mathcal{D}_l$ and call it b. It ends with $(\Gamma, \Box\varphi \to \psi \Rightarrow \Box\varphi)$. As it is not an axiom, there are sequents $S_1, \dots, S_n, S_{n+1}$ along b and atoms $p_1, \dots, p_n$ such that for all $i \leq n$: $S_i = (\Gamma, \Box\varphi \to \psi \Rightarrow p_i)$, S^s_{n+1} is not an atom, and S_i is the conclusion of an application of L$\to$ with left premiss S_{i+1}. Thus if the principal formula in the last inference of $\mathcal{D}_l$ is an implication $\varphi' \to \psi'$ such that φ' is not an atom, one can take $n = 0$ and for S_1 the left premiss of the last inference.

Let $\mathcal{D}_0$ be the proof of S_{n+1} and let $\varphi' \to \psi'$ be the principal formula in the application of L$\to$ with $S_{n+1} = (\Pi, \Box\varphi \to \psi, \varphi' \to \psi' \Rightarrow \varphi') = (\Gamma, \Box\varphi \to \psi \Rightarrow \varphi')$ as left premiss and conclusion S_n. Thus $S^s_{n+1} = \varphi'$ and φ' is not an atom. By the Inversion Lemma, the provability of the S implies the provability of $(\Pi, \Box\varphi \to \psi, \psi' \Rightarrow \Delta)$. Let $\mathcal{D}'$ be a strict proof of that latter sequent, which exists by the induction hypothesis. Then the following is a sensible proof S.

$$\frac{\begin{array}{c}\mathcal{D}_0 \\ \Pi, \Box\varphi \to \psi, \varphi' \to \psi' \Rightarrow \varphi'\end{array} \quad \begin{array}{c}\mathcal{D}' \\ \Pi, \Box\varphi \to \psi, \psi' \Rightarrow \Delta\end{array}}{\Pi, \Box\varphi \to \psi, \varphi' \to \psi' \Rightarrow \Delta}$$

If φ' is not boxed, its strictness follows immediately, and if φ' is boxed it follows by the induction hypothesis from the fact that the depth of the left premiss is lower than in the original proof. $\dashv$

5 Equivalence of $\mathsf{iG3X}$ and DYX

Theorem 5.1 For $\mathsf{X} \in \{\mathsf{M}, \mathsf{K}, \mathsf{D}, \mathsf{KD}, \mathsf{K4}\}$: $\vdash_{\mathsf{iG3X}} S$ if and only if $\vdash_{\mathsf{DYX}} S$.

Proof For $\mathsf{X}{=}\mathsf{M}$, the equivalence can be treated as in the proof of Theorem 1 in (Dyckhoff, 1992). We treat the other cases. Let $\mathsf{X} \in \{\mathsf{K}, \mathsf{D}, \mathsf{KD}, \mathsf{K4}\}$. Because $\mathsf{iG3X}$ is closed under the structural rules and *Cut*, the proof of the direction from right to left is easy and therefore omitted. The other direction is proved by induction on the well-ordering $\preccurlyeq$ defined in Section 2.

Consider a sequent S provable in $\mathsf{iG3X}$. If S^a contains $\bot$, we are done immediately. Therefore assume this is not the case. If S^a contains a conjunction, $S = (\Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow \Delta)$, then $\Gamma, \varphi_1, \varphi_2 \Rightarrow \Delta$ is provable in $\mathsf{iG3X}$ too by the Inversion Lemma. Hence it is provable in DYX by the induction hypothesis. Thus so is $\Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow \Delta$. A disjunction in S^a as well as the case that both p and $p \to \varphi$ belong to S^a, can be treated in the same way.

Thus only the case that S is irreducible remains, and by Lemma 4.2 we may assume its proof to be sensible and strict. Thus the last inference, R, has a principal formula that either is in the succedent, is an implication $\gamma \to \psi$ in the antecedent, where γ is not atomic, or, in the case of $\mathsf{iG3D}$ and $\mathsf{iG3KD}$, is a boxed formula $\Box\psi$ and $R = R_D$. In the first and the last case, R belongs to both calculi and the premiss of R is lower in the well-ordering $\preccurlyeq$ than S, and thus the induction hypothesis applies. In the second case we distinguish according to the form of γ. Suppose $S = (\Gamma, \gamma \to \psi \Rightarrow \Delta)$.

If $\gamma = \bot$, then $\Gamma \Rightarrow \Delta$ is derivable in $\mathsf{iG3X}$, and therefore in DYX. As DYX is closed under weakening, Lemma 3.1, S is derivable in DYX too.

If $\gamma = \varphi_1 \wedge \varphi_2$, then $\Gamma, \varphi_1 \to (\varphi_2 \to \psi) \Rightarrow \Delta$ is derivable in $\mathsf{iG3X}$. Thus the sequent is derivable in DYX by the induction hypothesis. Hence so is $\Gamma, \varphi_1 \wedge \varphi_2 \to \psi \Rightarrow \Delta$. The case that γ is a disjunction can be treated in the same way.

If $\gamma = \varphi_1 \vee \varphi_2$, then $\Gamma, \varphi_1 \to \psi, \varphi_2 \to \psi \Rightarrow \Delta$ is derivable in $\mathsf{iG3X}$. Thus the sequent is derivable in DYX by the induction hypothesis. Hence so is $\Gamma, \varphi_1 \vee \varphi_2 \to \psi \Rightarrow \Delta$.

If $\gamma = \varphi_1 \to \varphi_2$, then because $\gamma \to \psi$ is the principal formula, both sequents $\Gamma, \gamma \to \psi \Rightarrow \gamma$ and $\Gamma, \psi \Rightarrow \Delta$ are derivable in $\mathsf{iG3X}$. Thus so is $\Gamma, \varphi_2 \to \psi \Rightarrow \varphi_1 \to \varphi_2$. Since this sequent and $\Gamma, \psi \Rightarrow \Delta$ are lower in complexity than S, they are derivable in DYX by the induction hypothesis. Hence so is S.

If $\gamma = \Box\varphi$, then because the proof is strict, the left premiss, $(\Gamma, \Box\varphi \to \psi \Rightarrow \Box\varphi)$, is the conclusion of an application of R_K to a sequent $\Pi \Rightarrow \varphi$ such that $\Box\Pi \subseteq \Gamma$. By the induction hypothesis, $\Pi \Rightarrow \varphi$ and the right premiss $(\Gamma, \psi \Rightarrow \Delta)$ are derivable in DYX, say with proofs $\mathcal{D}$ and $\mathcal{D}'$, respectively. Hence the following is a proof of

$(\Gamma, \Box\varphi \to \psi \Rightarrow \Delta)$ in DYX.

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Pi \Rightarrow \varphi\end{array} \quad \begin{array}{c}\mathcal{D}'\\ \Gamma, \psi \Rightarrow \Delta\end{array}}{\Gamma, \Box\varphi \to \psi \Rightarrow \Delta}$$

$\dashv$

From the previous theorem and Lemma 3.1 the following can be obtained.

Corollary 5.2 For $\mathsf{X} \in \{\mathsf{M}, \mathsf{K}, \mathsf{D}, \mathsf{KD}, \mathsf{K4}\}$, the rules *Cut*, *Weakening* and *Contraction* are admissible in DYX.

6 Termination

A sequent calculus is *terminating* if backwards proof search always terminates, that is, if the process that starts with a given sequent and applies the rules of the calculus backwards, in whatever order as long as possible to the sequents obtained thus far, always in a finite number of steps reaches the point where no rules can be applied backwards to the obtained sequents anymore.
Clearly, for $\mathsf{X} \in \{\mathsf{M}, \mathsf{K}, \mathsf{D}, \mathsf{KD}\}$ the calculus DYX is terminating, since in every rule the conclusion is higher in the order $\preccurlyeq$ on sequents than the premisses.

Theorem 6.1 For $\mathsf{X} \in \{\mathsf{M}, \mathsf{K}, \mathsf{D}, \mathsf{KD}\}$, the calculus DYX is a terminating sequent calculus for the intuitionistic modal logic $\mathsf{L}_{\mathsf{DYX}}$, and the cut rule and the structural rules are admissible in it.

7 Conclusion

It has been shown that for $\mathsf{X} \in \{\mathsf{M}, \mathsf{K}, \mathsf{D}, \mathsf{KD}, \mathsf{K4}\}$ the calculus $\mathsf{iG3X}$ is sound and complete for the intuitionistic modal logic $\mathsf{L}_{\mathsf{iG3X}}$ and that $\mathsf{iG3X}$ is equivalent to the calculus DYX. Thus providing any logic of the form $\mathsf{L}_{\mathsf{iG3X}}$, except $\mathsf{L}_{\mathsf{iG3K4}}$, with a terminating sequent calculus in which the structural rules, including the Cut Rule, are admissible.

7.1 Other intuitionistic modal logics

In this section we compare the logics in this paper to the logics in the literature on intuitionistic modal logics.
In his thesis Simpson (1994) formulates six requirements that an intuitionistic modal logic should obey. All logics in this paper satisfy the first four requirements:

- $\mathsf{L}_{\mathsf{iG3X}}$ is conservative over IPC;
- $\mathsf{L}_{\mathsf{iG3X}}$ contains all substitution instances of theorems of IPC and is closed under modus ponens;

- the addition of $(\Rightarrow \varphi \vee \neg\varphi)$ to $\mathsf{L}_{\mathsf{iG3X}}$ yields a standard modal logic;
- if $\varphi \vee \psi$ holds in $\mathsf{L}_{\mathsf{iG3X}}$, then either φ or ψ holds in $\mathsf{L}_{\mathsf{iG3X}}$.

The last two requirements from (Simpson, 1994) do not apply as they are about the diamond operator and the semantics.

In (Simpson, 1994) the intuitionistic modal logic *IK without* $\Diamond$ is defined by the Hilbert system given by the rules Modus Ponens and Necessiation, the K-axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and all theorems of IPC. For the logics with diamonds *IK*, *IK4*, *IKD* from (Simpson, 1994), we have that they contain $\mathsf{L}_{\mathsf{iG3K}}$, $\mathsf{L}_{\mathsf{iG3K4}}$, and $\mathsf{L}_{\mathsf{iG3KD}}$, respectively. Whether the latter logics are actually equivalent to the diamond–free fragments of the former logics we have not (yet) investigated. For $\mathsf{L}_{\mathsf{iG3D}}$ the correspondence is less clear, as the D–axiom, $\Diamond\top$, is formulated in terms of $\Diamond$, while in our setting the rule R_D amounts to $\neg\Box\bot$.

Most other logics in the literature also contain the modal operator diamond or are about logics S4 and S5, and therefore different from the logics in this paper. However, our logics $\mathsf{L}_{\mathsf{iG3K}}$ and $\mathsf{L}_{\mathsf{iG3KD}}$ occur as the system $\mathbf{HK}_\Box$ in (Božić and Došen, 1984) and $\mathbf{HD}_\Box$ in (Došen, 1985), respectively. In these two papers the logics are proved to be sound and complete with respect to certain classes of Kripke models with two accesibility relations. $\mathsf{L}_{\mathsf{iG3K}}$, $\mathsf{L}_{\mathsf{iG3K4}}$ and $\mathsf{L}_{\mathsf{iG3KD}}$ occur as the systems K^{i}, $\mathsf{K4}^{\mathsf{i}}$ and NV^{i} in (Litak, 2014), and in (Wolter and Zakharyaschev, 1999) the first two are denoted by $\mathbf{IntK}_\Box$ and $\mathbf{IntK4}_\Box$.

7.2 Future work

In any endeavor that requires a terminating calculus for some intermediate or intuitionistic modal logic, it seems worthwhile to try extend Dyckhoff's calculus to a calculus for the logic in such a way that termination is preserved. A first step in the extension of the results in Iemhoff (2017) on uniform interpolation to other intuitionistic modal logics, such as iG3GL, would be the development of extensions of Dyckhoff's calculus to such logics. Furthermore, it would be interesting to establish whether the intuitionistic modal logics from the literature that contain the diamond operator, such as the ones from (Simpson, 1994), have a sequent calculus à la Dyckhoff's calculus as well.

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