A note on linear Kripke models

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Abstract

Gödel logics correspond to linear models with constant domains. In this paper other truth value logics, Scott logics, are defined, that correspond to linear models with possibly non-constant domains. An extension of intuitionistic logic by an existence predicate is discussed, and it is shown that this provides a natural translation of Scott logics into Gödel logics extended by this predicate.

Keywords: Kripke models, Gödel logics, Scott logics, linear frames, cones, existence predicate, Kripke models, truth value sets.

1 Introduction

Logics are often defined either via a proof system or via a class of models. For example most well-known modal and intermediate logics are presented via their class of frames or via their axiomatization over respectively K or IQC. A different approach to defining logics occurs in the context of truth value sets, i.e. subsets of the unit interval $[0,1]$. One can, for a given truth value set V, interpret formulas by mapping them to elements of V. The logical symbols receive a meaning via restrictions on these interpretations, e.g. by stipulating that the interpretation of $\wedge$ is the infimum of the interpretations of the respective conjuncts. Given these interpretations, one can associate a logic with such a truth value set V: the logic of all sentences that are mapped to 1 under any interpretation on V.

Given these three possible ways of defining a logic (there are of course a lot more, but here we will only consider these three), one might wish to know what is the connection between them, and whether a given logic can be represented in all three ways. Of course, this is not always possible, as e.g. the logic of finite models shows: this logic is defined via its class of models, but is known not to have a r.e. axiomatization. On the other hand, e.g. classical predicate logic CQC can be defined as the logic of the truth value set $\{0,1\}$ (under the appropriate interpretations of the language), as the logic axiomatized by e.g. the Gentzen calculus LK, and as the logic of classical models.

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For some logics it is not so clear whether they can be represented in these different ways. Interesting examples are the many valued logics given via truth value sets. Here the correspondence between a truth value representation, an axiomatization, and a representation via frames is not always available, and even if so, often difficult to prove. In this paper, except for the last section, we will concentrate on the first and last representation: the one in terms of truth value sets and the one in terms of frames. The logics we consider are initially defined via truth value sets and are proved to have a representation via frames, i.e. are proved to be the logic of a frame, at least in case the frame is countable.

An interesting and famous example of this phenomenon is given by the Gödel logics. Here a truth value set V always is a Gödel set: a closed set of $[0,1]$ containing 0 and 1. The Gödel logic G_V consists of those sentences that receive value 1 under all interpretations I on V. An interpretation interprets terms and predicate symbols on a given domain D, which in the case of an n-ary predicate P means that $I(P)$ is a map from D^n to V, and extends to all formulas via one of the standard interpretations of the logical symbols in many-valued logics, the *Gödel logic interpretation* (not standard terminology):

$$\begin{array}{lcl}
I(P\bar{t}) & = & I(P)(I(\bar{t})) \\
I(A \wedge B) & = & \inf(I(A), I(B)) \\
I(A \vee B) & = & \sup(I(A), I(B)) \\
I(A \to B) & = & \begin{cases} 1 & \text{if } I(A) \leq I(B) \\ I(B) & \text{otherwise,} \end{cases} \\
I(\exists x Ax) & = & \sup\{I(Aa) \mid a \in D\} \\
I(\forall x Ax) & = & \inf\{I(Aa) \mid a \in D\}.
\end{array}$$

As to the frame models for this logic, they are restricted to the ones with constant domains. That is, in this context the (constant domain) logic of a frame F, denoted by L_F^{cd}, consists of all formulas that hold in all Kripke models based on F with constant domains. As has been proved by A. Beckmann and N. Preining in [3], we have the following two representations of Gödel logics, one via truth value sets and one via frames:

Theorem 1 (A. Beckmann and N. Preining [3]) For any countable linear frame F there exists a Gödel set V such that

$$G_V = L_F^{cd}, \tag{1}$$

and vice versa: for every Gödel set V there exists a countable linear frame F such that (1).

As to the third way of representing a logic, via axiom systems, the situation for first-order Gödel logics is complicated. We will mention some of the results by N. Preining on this topic in Section 6.

1.1 Scott logics

As the above shows, Gödel logics correspond to frames when the models are restricted to the ones with constant domains. And indeed, the Gödel logics all contain the constant domain formula

$$\forall x(A \vee B(x)) \leftrightarrow (A \vee \forall x B(x)).$$

Here we ask ourselves what happens when we do not require that the domains be constant, but wish to keep the close relation between frames and truth value sets. We introduce logics based on Gödel sets that correspond to linear frames for which it is not assumed that the domains of the models be constant. For these logics, the quantifiers are interpreted in a different way than in the case of Gödel logics. Here a domain assignment is a pair (D, e) where D is a nonempty set and e is a function $e : D \to V$ satisfying

$$\exists a \in D \; e(a) = 1.$$

Given a domain assignment (D, e), a *Scott logic interpretation* I interprets terms and predicate symbols on D, satisfies

$$\inf_i e(a_i) \leq e(I(f)(\bar{a}))$$

for all n-ary function symbols f in the language and all sequences $\bar{a} = a_1, \ldots, a_n$ in D^n, and extends to all formulas as follows:

$$\begin{array}{lcl}
I(P\bar{t}) & = & I(P)(I(\bar{t})) \\
I(A \wedge B) & = & \inf(I(A), I(B)) \\
I(A \vee B) & = & \sup(I(A), I(B)) \\
I(A \to B) & = & \begin{cases} 1 & \text{if } I(A) \leq I(B) \\ I(B) & \text{otherwise,} \end{cases} \\
I(\exists x A x) & = & \sup\{e(a) \wedge I(Aa) \mid a \in D\} \\
I(\forall x A x) & = & \inf\{e(a) \to I(Aa) \mid a \in D\}.
\end{array}$$

The *Scott logic* S_V *of* V consists of those sentences A that receive the value 1 for any domain assignment and any Scott interpretation on V.

In this paper (Corollary 10) we prove the following correspondence.

> For every countable Gödel set V there exists a countable linear frame F such that $L_F = S_V$.
>
> For every countable linear frame F there exists a Gödel set V such that $L_F = S_V$.

Here L_F denotes the logic of a frame F, i.e. the set of formulas that hold in all Kripke models (also the ones with non-constant domains) based on F. Except for the quantifier cases, the methods used in the proof of this correspondence are quite similar to the ones used by Beckmann and Preining in [3]. The difference lies in the introduction of the Scott interpretation, i.e. in a new interpretation of the quantifiers, and its relation to the existence predicate, to be discussed below.

1.2 An existence predicate

The distinction between Scott logic interpretations and Gödel logic interpretations lies in the treatment of the quantifiers. Gödel logics can be viewed as a special kind of Scott logics, namely the ones in which e is the constant function mapping all elements to 1.

There is a kind of intuitive explanation of the interpretation of $\forall$ and $\exists$ in this setting. In [1] it has been observed that there is some sort of friction between quantifiers and terms in intuitionistic logic, which shows itself particularly clearly in the case of Skolemization. The Skolemization of an infix formula, i.e. a formula not necessarily in prenex normal form, is the replacement of strong quantifiers (occurrences of positive universal and negative existential quantifiers) by fresh function symbols. Classically, a formula is equiconsistent to its Skolemized version, intuitionistically it is not. One of the reasons for this is that quantifiers might range over increasing domains, while terms have to be interpreted at the root of a Kripke model. Therefore, a term cannot denote an object that only comes into existence at a later stage in a model.

In [1] an alternative Skolemization method is introduced that has the desired properties of standard Skolemization for a large class of formulas, including the formulas in which all strong quantifiers are existential. To this end the language of intuitionistic logic is extended by an *existence predicate* E, where Et has the meaning that t *exists*. This conservative extension of intuitionistic predicate logic, IQCE, was first studied by D. Scott in [9]. A Gentzen calculus for the system was first introduced in [2], where it is also shown that the system has cut-elimination. Terms may range over partial objects, variables and quantifiers are assumed to range over existing objects. Whence in a natural deduction formulation, the introduction rule for $\exists$ and the elimination rule for $\forall$ become:

$$\frac{A(t) \qquad Et}{\exists x A(x)} \qquad\qquad \frac{\forall x A(x) \qquad Et \qquad \begin{array}{c}[A(t)]\\ \vdots\\ C\end{array}}{C}$$

In accordance with this the quantifiers are interpreted in Kripke models in the following way (to stress the difference between this way of forcing and the normal forcing $\Vdash$, we denote it by $\Vdash^e$):

$$k \Vdash^e \exists x A(x) \text{ iff } \exists d \in D_k \; k \Vdash^e Ed \wedge A(d)$$

$$k \Vdash^e \forall x A(x) \text{ iff } \forall k' \succcurlyeq k \forall d \in D_{k'} : \; k' \Vdash^e (Ed \to A(d)).$$

Observe the connection between the interpretation of the quantifiers in Scott logic and in IQCE ($I_e(A)$ denotes $\{w \mid w \Vdash^e A\}$):

$$\begin{array}{rl} \text{Scott logic} & I(\exists x Ax) = \sup\{e(a) \wedge I(Aa) \mid a \in D\} \\ \text{Existence logic} & I_e(\exists x Ax) \equiv \bigcup_{a \in D}\{w \mid a \in D_w,\ w \Vdash^e Ea \wedge Aa\} \\ \text{Scott logic} & I(\forall x Ax) = \inf\{e(a) \to I(Aa) \mid a \in D\} \\ \text{Existence logic} & I_e(\forall x Ax) \equiv \bigcap_{a \in D}\{w \mid \forall v \succcurlyeq w(a \in D_v \Rightarrow v \Vdash^e Ea \to Aa)\}. \end{array}$$

Now the meaning behind the treatment of quantifiers in Scott logics becomes more clear: the $e(a)$ can be considered as a measure of termination, or approximation: $e(a) < e(b)$, means that the value we give to the existence of b is higher than the one for a, or that we have a better approximation of b than of a. For the evaluation of a quantified statement $QxA(x)$, we wish to take into account the values of $I(Aa)$ only for those a that we know well enough compared to Aa, that is, those a for which $I(Aa) < e(a)$. And indeed, if there is an a such that $I(Aa) < e(a)$, then

$$\begin{aligned} I(\exists x Ax) &= \sup\big(\sup\{I(Aa) \mid a \in D,\ I(Aa) < e(a)\}, \\ &\qquad \sup\{e(a) \mid a \in D,\ e(a) \leq I(Aa)\}\big) \\ I(\forall x Ax) &= \inf\{I(Aa) \mid a \in D,\ I(Aa) < e(a)\}. \end{aligned}$$

On the other hand, if no such a such that $I(Aa) < e(a)$ exists, then we give $QxA(x)$ the benefit of the doubt. For the universal quantifier this means $I(\forall x A(x)) = 1$, and for the existential quantifier this means $I(\exists x Ax) = \sup\{e(a) \mid a \in D\}$.

We will not discuss IQCE here in more detail, but we will encounter the existence predicate later in Section 4, where we present a faithful interpretation, first suggested by an anonymous referee, of Scott logics S_V into so-called existence Gödel logics G_V^e. This interpretation, that uses the existence predicate, enables us to give another proof of the main result on the correspondence between Scott logics and linear frames.

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2 Preliminaries

$\mathcal{L}$ denotes a language of first-order predicate logic, without equality. For a set D, $\mathcal{L}_D$ denotes the language $\mathcal{L}$ extended by the elements of D, which are considered as constants of the language. An $\mathcal{L}$-*term* is a term in $\mathcal{L}$ and an $\mathcal{L}_D$-*term* is a term in $\mathcal{L}_D$. Similarly for $\mathcal{L}$-sentences and $\mathcal{L}_D$-sentences. A *closed* term is a term in which no free variables occur. Given a set D, t, s range over closed $\mathcal{L}_D$-terms, and a, b range over elements in D. $A, B, \ldots$ range over formulas in $\mathcal{L}_D$ unless explicitly stated that they are formulas in $\mathcal{L}$. For a sequence $\bar{a} = a_1, \ldots, a_n$, $\bar{a} \in D$ means that all a_i belong to D. The a_i are called the *elements* of $\bar{a}$. $\bar{a}\bar{b}$ denotes the concatenation of $\bar{a}$ and $\bar{b}$. When we write a formula A like $A(\bar{t}, \bar{a})$ this means that every $\mathcal{L}_D$-term in A is an element of $\bar{t}$ or $\bar{a}$, and all elements in D that occur in A are elements of $\bar{a}$. When we write A like $A(\bar{t})$, this means that all $\mathcal{L}_D$-terms in A are elements of $\bar{t}$. If we write A like $A(\bar{a})$, this means that all constants in A that belong to D are elements of $\bar{a}$. Thus $A(\bar{a})$ may contain other terms than the ones in D but they are just not indicated. Notation $A(\bar{x})$ means that all free variables in A are elements of $\bar{x}$.

2.1 Kripke models

A *classical* $\mathcal{L}_{D_w}$*-structure* is a pair (D_w, I_w) such that D_w is a set and I_w is a map from $\mathcal{L}_{D_w}$ such that

> for every n-ary predicate P in $\mathcal{L}$, $I_w(P)$ is an n-ary predicate on D_w,
>
> for every n-ary function f in $\mathcal{L}_{D_w}$, $I_w(f)$ is an n-ary function on D_w (constants are considered as 0-ary functions),
>
> $I_w(a) = a$ for every constant $a \in D_w$.

For any closed $\mathcal{L}_{D_w}$-term t, $I_w(t)$ denotes the interpretation of t under I_w in D_w, which is defined as usual. $I_w(t_1, \ldots, t_n)$ is short for $I_w(t_1), \ldots, I_w(t_n)$. For $\mathcal{L}_{D_w}$-sentences A, let $(D_w, I_w) \models A$ denote that A holds in the structure (D_w, I_w), which is defined as usual for classical structures.

A *frame* is a pair $(W, \preccurlyeq)$ where W is a nonempty set and $\preccurlyeq$ is a partial order on W. A *Kripke model* on a frame $F = (W, \preccurlyeq)$ is a triple $K = (F, D, I)$, where $D = \{D_w \mid w \in W\}$ is a collection of nonempty sets satisfying

$$\bigcap_w D_w \neq \emptyset \qquad (w \preccurlyeq v \;\Rightarrow\; D_w \subseteq D_v),$$

and I is a collection $\{I_w \mid w \in W\}$, such that the (D_w, I_w) are classical $\mathcal{L}_{D_w}$-structures that satisfy the persistency requirement: for all $w \in W$, for all predicates $P(\bar{x})$ in $\mathcal{L}$ and for all closed $\mathcal{L}_{D_w}$-terms $\bar{t}$,

$$w \preccurlyeq v \;\Rightarrow\; \big((D_w, I_w) \models P(\bar{t}) \;\Rightarrow\; (D_v, I_v) \models P(\bar{t})\big),$$

$$w \preccurlyeq v \;\Rightarrow\; I_w(\bar{t}) = I_v(\bar{t}).$$

Note the point that was made in Section 1.2: closed $\mathcal{L}$-terms are interpreted as elements that belong to every domain D_w in the model. Moreover, the persistency requirement implies that this interpretation is the same at every node. We sometimes call D a *domain assignment* to F. $w\!\uparrow$ denotes $\{v \in W \mid$

$w \preccurlyeq v\}$. A model is said to have *constant domains* when all nodes have the same domain: $D_w = D_v$, for all w and v. For a frame $F = (W, \preccurlyeq)$ we define

$$Cones(F) \ \equiv_{def} \ \{U \subseteq W \mid \forall x \in U \forall y \in W \, (x \preccurlyeq y \ \Rightarrow \ y \in U)\}.$$

Given a frame $F = (W, \preccurlyeq)$ and a Kripke model $K = (F, D, I)$, the forcing relation $\Vdash$ is defined as follows. For our purposes it suffices to define the forcing relation $K, w \Vdash A$ at node w inductively only for *sentences* in $\mathcal{L}_{D_w}$. For predicates $P(\bar{x})$ in $\mathcal{L}$ and closed $\mathcal{L}_{D_w}$-terms t, we put

$$K, w \Vdash P(\bar{t}) \equiv_{def} (D_w, I_w) \models P(\bar{t}),$$

and extend $K, w \Vdash A$ to all sentences in $\mathcal{L}_{D_w}$ in the usual way. Note that $K, w \Vdash A$ is defined only for $\mathcal{L}_{D_w}$-sentences A, which suffices for our purposes. It follows that $K, w \Vdash A$ is defined for all $\mathcal{L}$-sentences A. When K is clear from the context we write $w \Vdash A$ instead of $K, w \Vdash A$. A $\mathcal{L}$-sentence A is *valid in* K, denoted $K \Vdash A$, if for all $w \in W$, $K, w \Vdash A$. A $\mathcal{L}$-sentence A is *valid on a frame* F, denoted $F \Vdash A$, when A is valid in all Kripke models on F. The *logic of* F consists of all $\mathcal{L}$-sentences that hold in all Kripke models on F, and is denoted by L_F. The set of all $\mathcal{L}$-sentences that hold in all Kripke models based on F with constant domains is denoted by L_F^{cd}.

2.2 Truth value logics

A *Gödel set* is a closed set $V \subseteq [0, 1]$ that contains 0 and 1. A *domain assignment* to V consist of a pair (D, e) such that D is a nonempty set and e is a function from D to V satisfying

$$\exists a \in D \ e(a) = 1.$$

An *interpretation* on (V, D, e) is a map I such that for every n-ary predicate symbol P in $\mathcal{L}$, $I(P) : D^n \to V$, for every n-ary function h in $\mathcal{L}$, $I(h) : D^n \to D$, and $I(a) = a$ for all $a \in D$, and such that for all n-ary function symbols f in $\mathcal{L}$, for all sequences $\bar{a} = a_1, \ldots, a_n \in D$

$$\inf_i e(a_i) \leq e(I(f)(\bar{a})).$$

I can be extended to interpret all terms in $\mathcal{L}_D$ in the usual way. I can be made into a map from sentences in $\mathcal{L}_D$ to V in different ways. We saw two possibilities in the introduction. In this paper all truth value logics interpret the connectives in the same way:

$$\begin{array}{lcl} I(P\bar{t}) & = & I(P)(\bar{I(t)}) \\ I(A \wedge B) & = & \inf(I(A), I(B)) \\ I(A \vee B) & = & \sup(I(A), I(B)) \\ I(A \to B) & = & \left\{ \begin{array}{ll} 1 & \text{if } I(A) \leq I(B) \\ I(B) & \text{otherwise,} \end{array} \right. \end{array}$$

(The P ranges over predicates in $\mathcal{L}$, the $\bar{t}$ over closed $\mathcal{L}_D$-terms.) The interpretation of the quantifiers differs from case to case, as we saw in the case of the

Gödel and Scott logics in the introduction. A *Scott logic interpretation* I on (V, D, e) is an interpretation for which the quantifiers are interpreted as

$$\begin{array}{rcl} I(\exists x A x) & = & \sup\{e(a) \wedge I(Aa) \mid a \in D\} \\ I(\forall x A x) & = & \inf\{e(a) \to I(Aa) \mid a \in D\}. \end{array}$$

Note that we can define I for $\exists$ and $\forall$ without referring to $\wedge$ and $\to$, by replacing them by their respective definitions.

A *Gödel logic interpretation* I on (V, D, e) is an interpretation for which the quantifiers are interpreted as

$$\begin{array}{rcl} I(\exists x A x) & = & \sup\{I(Aa) \mid a \in D\} \\ I(\forall x A x) & = & \inf\{I(Aa) \mid a \in D\}. \end{array}$$

As the function e does not play a role in the case of Gödel logics, a domain assignment in that case just consists of a nonempty set D. Note that Gödel logic interpretations on (V, D) can be considered as Scott logic interpretations on (V, D, e) where $e \equiv 1$. Given an interpretation, we say that a $\mathcal{L}_D$-sentence A is *valid in* (V, D, e, I), denoted by $(V, D, e, I) \models A$, if $I(A) = 1$. The *Scott logic* (*Gödel logic*) of a Gödel set V is the class of $\mathcal{L}$-sentences A such that for all domain assignments (D, e) and all Scott logic (Gödel logic) interpretations I on (V, D, e), $(V, D, e, I) \models A$. The Scott logic (Gödel logic) of V is denoted by S_V (G_V).

2.3 An existence predicate

Let $\mathcal{L}^e$ denote the language $\mathcal{L}$ extended by a unary predicate E. Given this extended language $\mathcal{L}^e$ we define analogues of Gödel logics and frame logics for this language. These are not just the Gödel logics and frame logics for the language $\mathcal{L}^e$, in which the predicate E would not play a special role, but are slight variations because in this setting interpretations satisfy some extra requirements related to the existence predicate. To stress these requirements we put *existence* in the name.

The *Gödel existence logic* G_V^e is defined for the language $\mathcal{L}^e$. A *Gödel existence logic interpretation* I on (V, D), where D is a domain assignment to V, is a Gödel logic interpretation on (V, D) with satisfies the extra requirements that

$$\exists a \in D \; I(Ea) = 1,$$

and for all functions h in $\mathcal{L}$, for all $\bar{a} = a_1, \ldots, a_n \in D$,

$$I(\bigwedge_{i \leq n} Ea_i) \leq I(Eh(\bar{a})).$$

The Gödel existence logic G_V^e of a Gödel set V consists of all $\mathcal{L}^e$-sentences A such that for all domain assignments D and all Gödel existence logic interpretations I on (V, D), $(V, D, I) \models A$.

As we did for Gödel logics, we also define *Kripke existence models* for $\mathcal{L}^e$, which are not just Kripke models for the extended language, but Kripke models with two extra properties that correspond to the extra properties on Gödel existence logic interpretations above. We call a Kripke model K for $\mathcal{L}^e$ a *Kripke existence model* when $K \Vdash Ea$ for some a contained in all domains, and such that for all functions h in $\mathcal{L}$, for all nodes w and all $\bar{a} = a_1, \ldots, a_n \in D_w$,

$$K, w \Vdash \bigwedge_{i \leq n} Ea_i \;\Rightarrow K, w \Vdash Eh(\bar{a}).$$

The *existence (constant domain) logic* of a frame F, denoted by L^e_F (L^{cde}_F), consists of all $\mathcal{L}^e$-sentences A such that $K \Vdash A$ for all Kripke existence models K on F (with constant domains).

3 A correspondence

In this section we prove, Theorem 2, one of the two main ingredients of the proof, in Section 5, that to every Scott logic based on a countable Gödel set there corresponds a logic of a countable frame, and vice versa to every logic of a countable frame there corresponds a Scott logic, of a possibly uncountable Gödel set. The other main ingredient is a result by A. Beckmann and N. Preining in [3], stated in Section 5.

Theorem 2 states that isomorphisms between cones of frames and Gödel sets imply the equivalence of the corresponding logics. There are two ways to prove Theorem 2. One uses an interpretation of Scott logics into Gödel existence logics. This is the short proof, it is discussed in Section 4. The other proof is long but gives constructions that given a Kripke model provide its corresponding Scott logic interpretation and vice versa. In this and the next section we present these two proofs. In Section 5 it is shown how Theorem 2 implies the main theorem of the paper, i.e. the correspondence between frames and Scott logics (Corollary 10). The reader only interested in the short proof of Theorem 2 and the proof how it implies the main theorem could therefore skip this section and proceed with the next section on interpretations and Section 5.

We proceed with the long proof of Theorem 2. In [3] a similar observation (Lemma 13 and Proposition 18) is made, reading L^{cd}_F for L_F and G_V for S_V in the theorem below. In that case, as mentioned by the authors, the observation is straightforward. Here the situation is different, as the quantifier cases are more involved and the fact that the domains might not be constant makes it all more complicated.

3.1 First proof of Theorem 2

Theorem 2 When F is a linear frame and V is a Gödel set such that it holds that $\big(Cones(F), \subseteq, \bigcap, \bigcup\big) \cong (V, \leq, \inf, \sup)$, then $L_F = S_V$.

Proof Let $f : (Cones(F), \subseteq, \bigcap, \bigcup) \to (V, \leq, \inf, \sup)$ be an isomorphism, let $g = f^{-1}$. We show that

(a): to any Kripke model (F, D, I) on F there corresponds a domain assignment (D_F, e_F) and Scott interpretation I_F such that for all $\mathfrak{L}$-sentences A

$$(F, D, I) \Vdash A \Leftrightarrow (V, D_F, e_F, I_F) \models A. \tag{2}$$

And we show that

(b): for any domain assignment (D, e) on V and for any Scott interpretation I on (V, D, e) there is a Kripke model (F, D_V, I_V) on F such that for all $\mathfrak{L}$-sentences A

$$(F, D_V, I_V) \Vdash A \Leftrightarrow (V, D, e, I) \models A. \tag{3}$$

(a) and (b) together prove the theorem.

We start with the proof of (a). Let (F, D, I) be a Kripke model. We will define a domain assignment (D_F, e_F) to V and a Scott logic interpretation I_F as follows. Since no confusion is possible, we write e for e_F. We define e as follows:

$$C_a \equiv_{def} \{w \in W \mid a \in D_w\} \qquad e(a) \equiv_{def} f(C_a).$$

Put $D_F = \bigcup D$. To define the interpretation I_F on (V, D_F, e) we let I_F mimic the I_w in the following way. For predicates P in $\mathfrak{L}$ and $\bar{a} \in D$, define

$$I_F(P)(\bar{a}) \equiv_{def} f(\{w \in W \mid \bar{a} \in D_w,\ K, w \Vdash P\bar{a}\}).$$

To define $I_F(h) : D^n \to D$ for n-ary function symbols h in $\mathfrak{L}$, pick for every $\bar{a} \in D$ a node $w_{\bar{a}}$ such that $\bar{a} \in D_{w_{\bar{a}}}$. Note that the fact that K is linear implies that such a node always exists. Then define

$$I_F(h)(\bar{a}) \equiv_{def} I_{w_{\bar{a}}}(h)(\bar{a}).$$

Note that $I_F(h)$ is well-defined by the persistency requirements on I, see Section 2.1. Then we extend I_F to other sentences as required for Scott logic interpretations. To prove (a) it suffices to show that (D_F, e) is a domain assignment to V, I_V is a Scott logic interpretation on (V, D_F, e) and that (2) holds.

First we show that (D_F, e) indeed is a domain assignment to V. That D_F is nonempty follows from the fact that the D_w are nonempty. By assumption on the Kripke models $\bigcap_w D_w$ is nonempty. Let a be in the intersection. Then $e(a) = f(W) = 1$. This proves that (D_F, e) indeed is a domain assignment to V.

Next we show that I_F is a Scott logic interpretation on (V, D_F, e), for which it remains to show that $I_F(P)$ is a function from D^n to V, that $I_F(a) = a$ for all $a \in D$, and that for n-ary function symbols h, and sequences $\bar{a} = a_1, \ldots, a_n$ in D_F, $\inf_i e(a_i) \leq e(I_F(h)(\bar{a}))$, that is,

$$\inf_i f(\{w \in W \mid a_i \in D_w\}) \leq f(\{w \in W \mid I_F(h)(\bar{a}) \in D_w\}). \tag{4}$$

The first two conditions are clear, for the latter, note that if $a_i \in D_w$ for all i, then $I_F(h)(\bar{a}) \in D_w$, which implies (4).

Finally, to show (a) we have to show that (2), for which it suffices to show that for all $\mathcal{L}$-formulas $A(\bar{x})$ for all $\bar{a} \in D_F$:

$$f\big(\{w \in W \mid \bar{a} \in D_w,\ w \Vdash A\bar{a}\}\big) = \inf(\inf_i e(a_i), I_F(A\bar{a})). \tag{5}$$

To prove (5) we use formula induction. For $\bar{a} \in D$ define

$$C_{\bar{a}} \equiv_{def} \bigcap_{b \in \bar{a}} C_b.$$

For the atomic case, consider a predicate symbol $P(\bar{x})$ in $\mathcal{L}$ and let $\bar{a} \in D_F$. The definition of I_F gives

$$f\big(\{w \in W \mid \bar{a} \in D_w,\ w \Vdash P\bar{a}\}\big) = I_F(P\bar{a}).$$

Since also $\{w \in W \mid \bar{a} \in D_w,\ w \Vdash P\bar{a}\} \subseteq \{w \in W \mid \bar{a} \in D_w\}$, this implies (5) for $A = P$.

For the connectives we only treat implication, the other cases are similar. Suppose $A\bar{x}\bar{y} = B\bar{x} \to C\bar{y}$, and consider $\bar{b}\bar{c} \in D_F$. (In order not to drown in brackets we write $A\bar{x}\bar{y}$ for $A(\bar{x}, \bar{y})$.) We have to show that

$$f(\{w \in W \mid \bar{b}\bar{c} \in D_w, w \Vdash A\bar{b}\bar{c}\}) = \inf\big(\inf_i e(b_i), \inf_j e(c_j), I_F(A\bar{b}\bar{c})\big). \tag{6}$$

Let X_b, X_c be such that $f(X_b) = I_F(B\bar{b})$, and $f(X_c) = I_F(C\bar{c})$. We distinguish two cases: $I_F(B\bar{b}) \subseteq I_F(C\bar{c})$ and $I_F(B\bar{b}) \supset I_F(C\bar{c})$, i.e. $X_b \subseteq X_c$ and $X_c \subset X_b$. Note that the induction hypothesis gives

$$\{w \in W \mid \bar{b} \in D_w, w \Vdash B\bar{b}\}) = C_{\bar{b}} \cap X_b \quad \{w \in W \mid \bar{c} \in D_w, w \Vdash C\bar{c}\}) = C_{\bar{c}} \cap X_c.$$

In the first case, $X_b \subseteq X_c$, this therefore implies that

$$\{w \in W \mid \bar{b} \in D_w, w \Vdash B\bar{b}\} \cap C_{\bar{c}} \subseteq \{w \in W \mid \bar{c} \in D_w, w \Vdash C\bar{c}\} \cap C_{\bar{b}}.$$

Hence

$$\{w \in W \mid \bar{b}\bar{c} \in D_w, w \Vdash A\bar{b}\bar{c}\} = C_{\bar{b}\bar{c}}.$$

Thus

$$f(\{w \in W \mid \bar{b}\bar{c} \in D_w, w \Vdash A\bar{b}\bar{c}\}) = \inf\big(\inf_i e(b_i), \inf_j e(c_j)\big).$$

As in this case $I_F(A\bar{b}\bar{c}) = 1$, this proves (6).

In the second case, $X_c \subset X_b$, it follows that

$$X_c \cap C_{\bar{b}\bar{c}} \subseteq X_b \cap C_{\bar{b}\bar{c}}. \tag{7}$$

In case we have $=$ in the above inclusion we use the linearity of the frame, which implies that for any two cones X and Y on F either $X \subseteq Y$ or $Y \subset X$. If

$X_b \subset C_{\bar{b}\bar{c}}$ then $X_c \cap C_{\bar{b}\bar{c}} \subseteq X_c \subset X_b \subseteq X_b \cap C_{\bar{b}\bar{c}}$, which contradicts the $=$. Thus $C_{\bar{b}\bar{c}} \subseteq X_b$. Using this, $X_c \subset C_{\bar{b}\bar{c}}$ would imply $X_c \cap C_{\bar{b}\bar{c}} \subseteq X_c \subset C_{\bar{b}\bar{c}} \subseteq X_b \cap C_{\bar{b}\bar{c}}$, which also contradicts the $=$. Therefore, we have both $C_{\bar{b}\bar{c}} \subseteq X_b$ and $C_{\bar{b}\bar{c}} \subseteq X_c$. Hence by the induction hypothesis

$$\{w \mid \bar{b}\bar{c} \in D_w, w \Vdash B\bar{b}\} = X_b \cap C_{\bar{b}\bar{c}} = C_{\bar{b}\bar{c}} = X_c \cap C_{\bar{b}\bar{c}} = \{w \mid \bar{b}\bar{c} \in D_w, w \Vdash C\bar{c}\}.$$

Therefore,

$$\{w \mid \bar{b}\bar{c} \in D_w, w \Vdash A\bar{b}\bar{c}\} = C_{\bar{b}\bar{c}} = C_{\bar{b}\bar{c}} \cap X_c.$$

This implies (6), because we were considering the case $X_c \subset X_b$, which means $I_F(C\bar{c}) \subset I_F(B\bar{b})$, and thus $I_F(A\bar{b}\bar{c}) = I_F(C\bar{c})$.

On the other hand, if the inclusion in (7) is $\subset$ we reason as follows. Pick a $v \in X_b \cap C_{\bar{b}\bar{c}}$, $v \notin X_c \cap C_{\bar{b}\bar{c}}$. Thus $\bar{b}\bar{c} \in D_v$ and $v \Vdash B\bar{b}$ and $\nVdash C\bar{c}$. Hence if for some w, $\bar{b}\bar{c} \in D_w$ and $w \Vdash A\bar{b}\bar{c}$, then $w \not\succcurlyeq v$ and $w \Vdash C\bar{c}$. Hence

$$\{w \in W \mid \bar{b}\bar{c} \in D_w, w \Vdash A\bar{b}\bar{c}\} = \{w \in W \mid \bar{b}\bar{c} \in D_w, w \Vdash C\bar{c}\}.$$

Thus by the induction hypothesis on $I_F(C\bar{c})$,

$$f(\{w \in W \mid \bar{b}\bar{c} \in D_w, w \Vdash A\bar{b}\bar{c}\}) = \inf\big(\inf_i e(b_i), \inf_j e(c_j), I_F(C\bar{c})\big).$$

As we were considering the case $X_c \subset X_b$, which means $I_F(C\bar{c}) \subset I_F(B\bar{b})$, and thus $I_F(A\bar{b}\bar{c}) = I_F(C\bar{c})$, this proves (6).

The cases for the quantifiers are a little less straightforward. First we treat the existential quantifier. We have to show that for all formulas $\exists y A(\bar{x}, y)$ in $\mathcal{L}$ and all $\bar{a} = a_1, \ldots, a_n \in D_F$:

$$f(\{w \in W \mid \bar{a} \in D_w, w \Vdash \exists y A(\bar{a}, y)\}) = \inf\big(\inf_{i \leq n} e(a_i), I_F(\exists y A(\bar{a}, y))\big).$$

This follows from the following list of equalities

$$\begin{array}{rcl}
f(\{w \in W \mid \bar{a} \in D_w, w \Vdash \exists y A(\bar{a}, y)\}) & = & \\
f(\bigcup\limits_{b \in D_F} \{w \in W \mid \bar{a}, b \in D_w, w \Vdash A(\bar{a}, b)\}) & = & \\
\sup\limits_{b \in D_F} f(\{w \in W \mid \bar{a}, b \in D_w, w \Vdash A(\bar{a}, b)\}) & = & \text{(IH)} \\
\sup\limits_{b \in D_F} \inf(\inf\limits_{i \leq n} e(a_i), e(b), I_F(A(\bar{a}, b)) & = & \\
\inf\big(\inf\limits_{i \leq n} e(a_i), \sup\limits_{b \in D_F} \inf(e(b), I_F(A(\bar{a}, b)))\big) & = & \\
\inf\big(\inf\limits_{i \leq n} e(a_i), \sup\limits_{b \in D_F} (e(b) \wedge I_F(A(\bar{a}, b)))\big) & = & \\
\inf(\inf\limits_{i \leq n} e(a_i), I_F(\exists y A(\bar{a}, y)) & &
\end{array}$$

This finishes the case of the existential quantifier.

For the universal quantifier we have to show that for all formulas $\forall y A(\bar{x}, y)$ in $\mathcal{L}$ and for all $\bar{a} = a_1, \ldots, a_n \in D_F$:

$$f(\{w \in W \mid \bar{a} \in D_w, w \Vdash \forall y A(\bar{a}, y)\}) = \inf\big(\inf_{i \leq n} e(a_i), I_F(\forall y A(\bar{a}, y))\big). \quad (8)$$

Let $f(X_b) = I_F(A(\bar{a}, b))$. The X_b exist because f is an isomorphism. First note that the induction hypothesis on $I_F(A(\bar{a}, b))$ implies

$$\{w \in W \mid \bar{a}b \in D_w, w \Vdash A(\bar{a}, b)\} = C_{\bar{a}b} \cap X_b. \quad (9)$$

Define

$$Y \equiv_{def} \{b \in D_F \mid I_F(A(\bar{a}, b)) < e(b)\} = \{b \in D_F \mid X_b \subset C_b\}.$$

As observed in Section 1.2,

$$Y = \emptyset \ \Rightarrow \ I_F(\forall x A(\bar{a}, x)) = 1$$
$$Y \neq \emptyset \ \Rightarrow \ I_F(\forall x A(\bar{a}, x)) = \inf\{I_F(A(\bar{a}, b)) \mid b \in Y\}.$$

Thus to show (8) it suffices to show that

$$Y = \emptyset \ \Rightarrow \ \{w \in W \mid \bar{a} \in D_w, w \Vdash \forall y A(\bar{a}, y)\} = C_{\bar{a}} \quad (10)$$
$$Y \neq \emptyset \ \Rightarrow \ \{w \in W \mid \bar{a} \in D_w, w \Vdash \forall y A(\bar{a}, y)\} = C_{\bar{a}} \cap \bigcap\{X_b \mid X_b \subset C_b\}. \quad (11)$$

To prove (10) assume $Y = \emptyset$. This implies $C_b \subseteq X_b$ for all b. Hence (9) implies that for all b:

$$\{w \in W \mid \bar{a}b \in D_w, K, w \Vdash A(\bar{a}, b)\} = C_{\bar{a}b}.$$

Thus $\bar{a}b \in D_w$ implies $w \Vdash A(\bar{a}, b)$. Hence $\{w \in W \mid \bar{a} \in D_w, w \Vdash \forall y A(\bar{a}, y)\} = C_{\bar{a}}$, which proves (10).

To prove (11) assume $Y \neq \emptyset$. Thus now there is a $c \in D_F$ such that $X_c \subset C_c$. We prove

$$\{w \in W \mid \bar{a} \in D_w, w \Vdash \forall y A(\bar{a}, y)\} = C_{\bar{a}} \cap \bigcap\{X_b \mid b \in D_F, X_b \subset C_b\}.$$

as follows.

$\subseteq$: Suppose $\bar{a} \in D_w$ and $w \Vdash \forall y A(\bar{a}, y)$. Clearly $w \in C_{\bar{a}}$. We have to show that $w \in X_b$ for all X_b such that $X_b \subset C_b$. Now $X_b \subset C_b$ implies the existence of a node $v \in W$ such that $v \in C_b$ and $v \notin X_b$. Thus $b \in D_v$. Hence by the induction hypothesis (9), $\bar{a} \notin D_v$ or $v \nVdash A(\bar{a}, b)$. Note that in both cases it follows that $v \preccurlyeq w$. Hence $b \in D_w$, and thus $w \Vdash A(\bar{a}, b)$. Whence $w \in X_b$ by (9).

$\supseteq$: Assume $w \in C_{\bar{a}} \cap \bigcap\{X_b \mid b \in D_F, X_b \subset C_b\}$. To show $w \Vdash \forall y A(\bar{a}, y)$, assume $w \preccurlyeq v$ and $b \in D_v$. Hence $v \in C_{\bar{a}b}$. To show $v \Vdash A(\bar{a}, b)$ we distinguish two cases. If $C_b \subseteq X_b$, then by (9) we have $v \Vdash A(\bar{a}, b)$. If $X_b \subset C_b$, then $w \in X_b$, and whence $v \in X_b$, since $w \preccurlyeq v$. Thus by (9) $v \Vdash A(\bar{a}, b)$ also in this case.

This finishes the case of the universal quantifier. Whence (5) is proved, and thereby it is shown that (a) holds.

To prove (b) we proceed in a similar way. Let (D, e) be a domain assignment to V and let I be a Scott logic interpretation on (V, D, e). We define a Kripke model (F, D_V, I_V) on F as follows. $D_V = \{D_w \mid w \in W\}$ where

$$D_w \equiv_{\scriptscriptstyle def} \{a \in D \mid w \in g(e(a))\}.$$

For n-ary predicates P and sequences $\bar{a}$ in D_w,

$$(I_V)_w(P) \;\equiv_{\scriptscriptstyle def}\; \{\bar{a} \in D_w \mid w \in g(I(P)(\bar{a}))\}.$$

For function symbols h and sequences $\bar{a} \in D_w$, put $(I_V)_w(h)(\bar{a}) = I(h)(\bar{a})$. A point of notation: for $\mathcal{L}$-formulas $A\bar{x}$ and $\bar{a} \in D$, let

$$I_V(A\bar{a}) \;\equiv_{\scriptscriptstyle def}\; \{w \in W \mid \bar{a} \in D_w,\; (F, D_V, I_V), w \Vdash A\bar{a}\}.$$

To show (b), it suffices to show that D_V is a domain assignment to F, that (F, D_V, I_V) is a Kripke model, and that (3) holds. To show that D_V is a domain assignment to F it suffices to show that

$$\bigcap_{w \in W} D_w = \bigcap_{w \in W} \{a \in D \mid w \in g(e(a))\} \neq \emptyset$$

and that $w \preccurlyeq v$ implies $D_w \subseteq D_v$. That the intersection of the D_w is nonempty follows from the fact that $g(e(a)) = W$ for some a, which again follows from the fact that $e(a) = 1$ for some a. The persistency follows from the fact that the $g(e(a))$ are cones.

Next we show that (F, D_V, I_V) is a Kripke model. That for n-ary predicates P, $(I_V)_w(P)$ is a n-ary predicate on D_w^n, and that $(I_V)_w(a) = a$ for all $a \in D$ follows from the definition of the $(I_V)_w$. It is similarly easy to see that the $(I_V)_w$ satisfy the persistency requirements. Thus it remains to show that for nodes w and n-ary functions h, $(I_V)_w(h)$ is a function from D_w^n to D_w. But this is so, since if $a_1, \ldots, a_n = \bar{a} \in D_w^n$, then $w \in g(e(a_i))$ for all i. Since by the definition of Scott interpretations,

$$\inf_{i \leq n} e(a_i) \leq e(I(h)(\bar{a})),$$

this implies $w \in \bigcap_i g(e(a_i)) = g(\inf_{i \leq n} e(a_i)) \subseteq g(e(I(h)(\bar{a})))$. And whence $(I_F)_w(h)(\bar{a}) = I(h)(\bar{a}) \in D_w$.

It remains to show (3). For this it suffices to show that for all $\mathcal{L}$-formulas A, for all $\bar{a} = a_1, \ldots, a_n \in D$,

$$\bigcap_{i=1}^{n} g(e(a_i)) \cap g\big(I(A\bar{a})\big) = I_V(A\bar{a}). \tag{12}$$

We again use formula induction. The atomic case and the connective cases are similar as in case (a), and left to the reader. Observe that

$$a \in D_w \;\Rightarrow\; w \in g(e(a)). \tag{13}$$

For the existential quantifier, by the induction hypothesis we have that

$$I_V(\exists x A(\bar{a},x)) = \bigcup_{b\in D} I_V(A(\bar{a},b)) = \bigcup_{b\in D}\bigcap_{i=1}^{n} g(e(a_i)) \cap g(e(b)) \cap gI(A(\bar{a},b)).$$

That (12) holds for the existential quantifier follows from the following equalities:

$$\begin{aligned}
&I_V(\exists x A(\bar{a},x)) = \\
&\bigcup_{b\in D}\bigcap_{i=1}^{n} g(e(a_i)) \cap g(e(b)) \cap gI(A(\bar{a},b)) = \\
&\bigcap_{i=1}^{n} g(e(a_i)) \cap (\bigcup_{b\in D} g(e(b)) \cap gI(A(\bar{a},b))) = \\
&\bigcap_{i=1}^{n} g(e(a_i)) \cap g\big(\sup_{b\in D}\big(\inf\big(e(b), I(A(\bar{a},b))\big)\big)\big) = \\
&\bigcap_{i=1}^{n} g(e(a_i)) \cap g\big(\sup_{b\in D} e(b) \wedge I(A(\bar{a},b))\big) = \\
&\bigcap_{i=1}^{n} g(e(a_i)) \cap g\big(I(\exists x A(\bar{a},x))\big).
\end{aligned}$$

Finally, we treat the universal quantifier. Define

$$X \equiv_{def} \{b \in D \mid e(b) > I(A(\bar{a},b))\}.$$

As observed in Section 1.2,

$$\begin{aligned}
X = \emptyset \;\Rightarrow\; & I(\forall x A(\bar{a},x)) = 1 \\
X \neq \emptyset \;\Rightarrow\; & I(\forall x A(\bar{a},x)) = \inf\{I(A(\bar{a},b)) \mid b \in X\}.
\end{aligned}$$

Thus to show (12) for the universal quantifier it suffices to show that

$$X = \emptyset \;\Rightarrow\; I_V(\forall x A(\bar{a},x)) = \bigcap_{i=1}^{n} g(e(a_i)) \tag{14}$$

$$X \neq \emptyset \;\Rightarrow\; I_V(\forall x A(\bar{a},x)) = \bigcap_{i=1}^{n} g(e(a_i)) \cap \bigcap_{b\in X} g(I(A(\bar{a},b))). \tag{15}$$

First observe that by the induction hypothesis

$$I_V(A(\bar{a},b)) = \bigcap_{i=1}^{n} g(e(a_i)) \cap g(e(b)) \cap g(I(A(\bar{a},b))). \tag{16}$$

This implies (14). For if $X = \emptyset$ then $g(e(b)) \subseteq g(I(A(\bar{a},b)))$ for all b. Therefore, if $w \in \bigcap_{i=1}^{n} g(e(a_i)) \cap g(e(b))$ then $w \Vdash A(\bar{a},b)$, for all w. Hence (14).

To prove (15) assume $X \neq \emptyset$. We prove

$$I_V(\forall x A(\bar{a}, x)) = \bigcap_{i=1}^{n} g(e(a_i)) \cap \bigcap_{b \in X} g(I(A(\bar{a}, b)))$$

as follows.

$\subseteq$: Consider w such that $\bar{a} \in D_w$ and $w \Vdash \forall x A(\bar{a}, x)$. We have to show that $w \in g(I(A\bar{a}, b))$ for all $b \in X$. Pick $b \in X$. Then $e(b) > I(A(\bar{a}, b))$, thus $g(e(b)) \supset g(I(A(\bar{a}, b)))$. Whence there is a $v \in g(e(b))$ such that $v \notin g(I(A(\bar{a}, b)))$. Thus $b \in D_v$, and by the induction hypothesis (16), $\bar{a} \notin D_v$ or $v \nVdash A(\bar{a}, b)$. In both cases, $v \preccurlyeq w$ follows. Thus $b \in D_w$. Hence $w \Vdash A(\bar{a}, b)$. Therefore, $b \in g(I(A(\bar{a}, b)))$ by (16), which is what we had to prove.

$\supseteq$: Consider $w \in \bigcap_{i=1}^{n} g(e(a_i)) \cap \bigcap\{g(I(A(\bar{a}, b))) \mid b \in X\}$, and $v \succcurlyeq w$ and $b \in D_v$. We have to show that $v \Vdash A(\bar{a}, b)$. By the induction hypothesis (16) it suffices to show that $v \in g(I(A(\bar{a}, b)))$, which we prove as follows. If $b \in X$, then $v \in g(I(A(\bar{a}, b)))$, as $w \in g(I(A(\bar{a}, b)))$ and $g(I(A(\bar{a}, b)))$ is a cone. On the other hand, if $b \notin X$, then $e(b) \leq I(A(\bar{a}, b))$. Since $b \in D_v$, $v \in g(I(A(\bar{a}, b)))$ follows. This proves (15), and thereby (12), and thereby (3). This proves (b) and thereby the theorem. $\square$

4 An interpretation

There is a faithful interpretation of Scott logics into Gödel existence logics that will enable us to again prove Theorem 2, and thereby the main correspondence result in Section 5.

Definition 3 We define a translation $(\cdot)^e$ from formulas in $\mathcal{L}$ to formulas in $\mathcal{L}^e$ as follows.

$\big(P(\bar{t})\big)^e = P(\bar{t})$ for atomic P and terms $\bar{t}$,

$(\cdot)^e$ commutes with the connectives,

$(\exists x A(x))^e = \exists x\big(Ex \wedge (A(x))^e\big)$,

$(\forall x A(x))^e = \forall x\big(Ex \to (A(x))^e\big)$.

Lemma 4 For any Gödel set V, $(\cdot)^e$ is a faithful translation of S_V into G_V, i.e. for all $\mathcal{L}$-sentences A

$$S_V \models A \;\Leftrightarrow\; G_V^e \models A^e.$$

Proof $\Rightarrow$: given a Gödel domain assignment D to V and a Gödel existence logic interpretation I on (V, D) we construct a Scott domain assignment (D, e) and a Scott logic interpretation I' such that for all $\mathcal{L}_D$-sentences A

$$I'(A) = I(A^e). \tag{17}$$

Define $e : D \to V$

$$e = I(E).$$

For predicates $P \neq E$ and for all functions h in $\mathcal{L}$ define

$$I'(P) = I(P) \qquad I'(h) = I(h).$$

By definition there is an $a \in D$ such that $I(Ea) = 1$. Hence $e(a) = 1$ for some $a \in D$. Also, for all n-ary function symbols h in $\mathcal{L}$, for all sequences $\bar{a} = a_1, \ldots, a_n \in D$

$$I(\bigwedge_{i \leq n} Ea_i) \leq I(Eh(\bar{a})).$$

Hence

$$\inf_i e(a_i) \leq e(I'(h)(\bar{a})).$$

The last two observations show that I' indeed is a Scott logic interpretation on (V, D, e). The proof that (17) holds we leave to the reader.

$\Leftarrow$: given a domain assignment (D, e) to V and a Scott interpretation I on (V, D, e) we construct a Gödel existence logic interpretation I' on (V, D) such that for all $\mathcal{L}_D$-sentences A

$$I(A) = I'(A^e). \tag{18}$$

For predicates $P \neq E$ and for all functions h in $\mathcal{L}$ define

$$I'(P) = I(P) \qquad I'(h) = I(h).$$

For E define

$$I'(E) = e.$$

The proof that I' indeed is a Gödel existence logic interpretation on (V, D) and that (18) holds are left to the reader. $\Box$

Lemma 5 For any frame F, $(\cdot)^e$ is faithful translation of L_F into L_F^{cd}, i.e. for all $\mathcal{L}$-sentences A

$$L_F \models A \;\Leftrightarrow\; L_F^{cde} \models A^e.$$

Proof $\Rightarrow$: Let $K = (F, D, I)$ be an arbitrary Kripke existence model on F with constant domains. It suffices to show that we can construct a Kripke model $K' = (F, D', I')$ such that for all nodes w for all $\mathcal{L}_{D'_w}$-sentences A

$$K', w \Vdash A \;\Leftrightarrow\; K, w \Vdash A^e. \tag{19}$$

Define

$$D'_w = \{a \in D \mid K, w \Vdash Ea\} \qquad D' = \{D'_w \mid w \in w\}.$$

Define for predicates P and functions h in $\mathcal{L}$

$$(I')_w = I_w \uparrow D'_w,$$

where $I_w \upharpoonright D'_w$ denotes the restriction of I_w to D'_w. Note that $K' = (F, D', I')$ indeed is a Kripke model: $(I')_w$ interprets functions h indeed as functions from $(D')^n_w$ to D_w because of the extra conditions of Kripke existence models, the D_w are nonempty, etc. We leave the proof of (19) to the reader.

$\Leftarrow$: Let $K = (F, D, I)$ be an arbitrary Kripke model on F. We show how to construct a Kripke existence model $K' = (F, D', I')$ with constant domains. Then we complete the proof by showing that for all nodes w for all $\mathcal{L}_{D_w}$-sentences A

$$K, w \Vdash A \;\Leftrightarrow\; K', w \Vdash A^e. \tag{20}$$

Observe that the linearity of F implies that for every $\bar{a}$ there exists a $v_{\bar{a}} \in W$ such that $\bar{a} \in D_{v_{\bar{a}}}$. Define

$$D'_w = \bigcup_{v \in W} D_v \qquad I'_w(E) \equiv_{def} D_w.$$

Furthermore, let $I'_w(P) = I_w(P)$ on $\mathcal{L}$ for predicates P, and put $I'_w(h)(\bar{a}) = I_{v_{\bar{a}}}(h)(\bar{a})$. Observe that $I'_w(P) = I_w(P)$ implies that $K', w \nVdash P\bar{a}$ if $\bar{a} \notin D_w$. However, this will not contradict (20), as there only $\mathcal{L}_{D_w}$-sentences are considered. Note that the persistency requirements on K imply that I' is well-defined. We leave the proof that $K' = (F, D', I')$ indeed is a Kripke existence model with constant domains and that (20) holds to the reader. $\Box$

As mentioned above, the given interpretation of Scott logics into Gödel logics enables us to again prove Theorem 2. We will only state here the main ingredients of the alternative proof and leave the details to the reader. Like in [3], the proof uses a correspondence between cone structures, frames, and truth value logics, where cone structures are defined as follows.

Definition 6 Given a frame $F = (W, \preccurlyeq)$ let $\mathcal{F}$ denote the *cone structure* $\big(Cones(F), \subseteq, \bigcap, \bigcup\big)$ of F. Given such a cone structure $\mathcal{F}$, an *existence interpretation* on $\mathcal{F}$ is a pair (D, I), where D is a nonempty set and I is a map from $\mathcal{L}^e_D$-sentences to $Cones(F)$ that satisfies the following requirements:

$I(\bot) = \emptyset$, $I(\top) = W$, $I(Ea) = W$ for some $a \in D$,

$\forall \bar{a} \in D:\ \bigcap_i I(Ea_i) \subseteq I(Eh(\bar{a}))$,

$I(A \wedge B) = I(A) \cap I(B)$,

$I(A \vee B) = I(A) \cup I(B)$,

$$I(A \to B) = \begin{cases} 1 & \text{if } I(A) \subseteq I(B) \\ I(B) & \text{otherwise,} \end{cases}$$

$I(\exists x A x) = \bigcup_{a \in D} I(At)$,

$I(\forall x A x) = \bigcap_{a \in D} I(At)$.

The logic $L^e_{\mathcal{F}}$ of $\mathcal{F}$ consists of those $\mathcal{L}^e$-sentences A such that $I(A) = W$ for all existence interpretations (D, I) on $\mathcal{F}$.

Given this definition we have the following lemma's, the proofs of which we leave to the reader.

Lemma 7 For every frame F, $L_F^{cde} = L_{\mathcal{F}}^e$.

Lemma 8 If $\big(Cones(F), \subseteq, \bigcap, \bigcup\big) \cong (V, \leq, \inf, \sup)$, then $L_{\mathcal{F}}^e = G_V^e$.

4.1 Second proof of Theorem 2

All these lemma's provide a short proof of Theorem 2: By Lemma's 7 and 8, $\big(Cones(F), \subseteq, \bigcap, \bigcup\big) \cong (V, \leq, \inf, \sup)$ implies that $L_F^{cde} = G_V^e$. Whence by Lemma 4 and 5, $L_F = S_V$ follows.

5 Scott logics and linear frames

In this section we prove the main theorem of the paper on the correspondence between Scott logics and linear frame logics. We start with the following observation.

Let $\mathcal{F}$ be a class of frames and let $\mathcal{V}$ be a set of Gödel sets. Suppose that for every countable Gödel set $V \in \mathcal{V}$ there is a countable linear frame $F \in \mathcal{F}$ such that

$$\Big(Cones(F), \subseteq, \bigcap, \bigcup\Big) \cong (V, \leq, \inf, \sup). \tag{21}$$

and vice versa for every countable linear frame $F \in \mathcal{F}$ there is a Gödel set $V \in \mathcal{V}$ such that (21). Then from Theorem 2 the following correspondence between truth value logics and frame logics follows:

> For every countable Gödel set $V \in \mathcal{V}$ there exists a countable linear frame $F \in \mathcal{F}$ such that $L_F = S_V$.
>
> For every countable linear frame $F \in \mathcal{F}$ there exists a Gödel set $V \in \mathcal{V}$ such that $L_F = S_V$.

This observation and the following theorem by A. Beckmann and N. Preining provide the correspondence between Scott logics based and logics of frames.

Theorem 9 (A. Beckmann and N. Preining [3]) For every countable Gödel set V there is a countable linear frame F such that

$$\Big(Cones(F), \subseteq, \bigcap, \bigcup\Big) \cong (V, \leq, \inf, \sup). \tag{22}$$

For every countable linear frame F there is a (not necessarily countable) Gödel set V such that (22).

Proof From the proofs of respectively Lemma 23 and Theorem 20 in [3]. □

Using this theorem, Theorem 2 and the observation above, we have the following corollary.

Corollary 10 For every countable Gödel set V there exists a countable linear frame F such that $L_F = S_V$.

For every countable linear frame F there exists a Gödel set V such that $L_F = S_V$.

Let F_n be the linear frame with n elements. Then we also have the following corollary.

Corollary 11 When V is a Gödel set with $n+1$ elements, then $S_V = L_{F_n}$. In particular, Scott logics based on finite Gödel sets with the same number of elements are equal.

6 Questions on axiomatization, equality and countability

Up till now we considered the correspondence between logics defined via Gödel sets and logics defined via frames. One can view this as the relation between logics defined via Gödel sets and logics defined via first-order semantics. And indeed, one can view logics based on Gödel sets as intermediate logics. The other natural question is what the relation between logics based on Gödel sets and logics based axiom systems is. The question here is whether for a given Gödel set V there is an axiom system with respect to which S_V is sound and complete, and similar questions for G_V and G^e_V.

6.1 Gödel logics

In the case of Gödel logics this question has been solved completely by N. Preining [8]. For $G_{[0,1]}$ completeness was obtained before by M. Takano [10], M. Titani and G. Takeuti [11].

In [8] it has been shown that for countable infinite V, the logic G_V is not recursively enumerable, whence has no r.e. axiomatization. On the other hand, for some other Gödel logics there do exist nice axiomatizations. For example, $G_{[0,1]}$ is axiomatized by the system H [10]:

$$\begin{array}{ll} & \text{axioms of } \mathsf{IQC} \\ \mathit{LIN} & (A \to B) \vee (B \to A) \\ \mathit{QS} & \forall x(A \vee B(x)) \to (A \vee \forall x B(x)) \ (x \text{ not free in } A). \end{array}$$

The Gödel logics of finite Gödel sets $\{0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-2}{n-1}, 1\}$ have the following nice axiomatization $\mathsf{H_n}$ [8]: H plus the axiom *FIN(n)*:

$$A_1 \vee (A_1 \to A_2) \vee \ldots \vee (A_{n-2} \to A_{n-1}) \vee \neg A_{n-1}.$$

For uncountable Gödel sets the axiomatizability depends on certain properties of the sets. We will not treat the details here, they can be found in [8].

6.2 Scott logics

It is not clear whether the Scott logics are complete with respect to some r.e. axiom system in the language of first-order logic. Clearly, the results in Section 4 imply that when a Gödel existence logic G_V^e is decidable or r.e., then so, respectively, is S_V. However, the problem one encounters when trying to prove completeness for Scott logics via the standard method via prime sets, is to define domains on the basis of prime sets of formulas.

It would be nice to have answers to questions on Scott logics that are analogues of the following solved problems about Gödel logics:

- Is $S_{[0,1]}$ complete with respect to the system $\mathsf{IQCE} + \mathsf{LIN}$, i.e. H without *QS*?
- Is the logic $S_{\{0,\frac{1}{2},\frac{2}{3},\ldots,\frac{n-2}{n-1},1\}}$ complete with respect to $\mathsf{IQCE}+\mathsf{LIN}+\mathsf{FIN(n)}$, i.e. $\mathsf{H_n}$ without *QS*?

We conjecture the answers to be yes. Here results by G. Corsi [4, 5, 6] on the axiomatization of linear frames might provide answers or be helpful.

There are a lot more questions, e.g. the ones related to equality. Intuitionistic logic sometimes behaves strangely when adding equality. Skolemization is an example [7]. Also, when equality is interpreted as real equality in the domains it becomes decidable, which might not always be natural in an intuitionistic setting. A natural question here is whether the correspondence results in this paper hold when we add equality to the language.

Finally, note that in Theorem 1 by A. Beckmann and N. Preining there is a correspondence between Gödel logics and countable linear frames. In this paper a correspondence between Scott logics based on *countable* Gödel sets and linear frames is established. We do not know whether this result can be extended to arbitrary Gödel sets, as in the case of Gödel logics.

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