Uniform interpolation and the existence of sequent calculi

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September 27, 2018

Abstract

This paper presents a uniform and modular method to prove uniform interpolation for several intermediate and intuitionistic modal logics. The proof-theoretic method uses sequent calculi that are extensions of the terminating sequent calculus G4ip for intuitionistic propostitional logic. It is shown that whenever the rules in a calculus satisfy certain structural properties, the corresponding logic has uniform interpolation. It follows that the intuitionistic versions of K and KD (without the diamond operator), as well as several other intuitionistic modal logics, have uniform interpolation. It also follows that no intermediate or intuitionistic modal logic without uniform interpolation has a sequent calculus satisfying those structural properties. Thereby establishing that except for the seven intermediate logics that have uniform interpolation, no intermediate logic has such a sequent calculus.

Keywords: uniform interpolation, sequent calculus, intermediate logic, intuitionistic modal logic, propositional quantifiers
MSC: 03B05, 03B45, 03F03

1 Introduction

In (Pitts, 1992), Andrew Pitts established that intuitionistic propositional logic IPC has uniform interpolation. His proof was the first syntactic or proof–theoretic proof of a result of that kind. This paper shows that several intermediate and intuitionistic modal logics have uniform interpolation by providing a direct connection, for a given logic, between the property of having uniform interpolation and the existence of sequent calculi for the logic. The method developed to prove these results is uniform, and, perhaps more importantly, provides a way to prove negative results concerning proof systems: logics without uniform interpolation cannot have sequent calculi of a certain form. The methods used in this paper are proof–theoretic, uniform and modular, and are inspired by Pitts' proof–theoretic proof from 1992.

Uniform interpolation is a strengthening of interpolation, and a logic L is said to satisfy or have the property if the propositional quantifiers $\exists p$ and $\forall p$ are definable in the logic, where

$\exists p\varphi$ and $\forall p\varphi$ are defined by requiring that they do not contain p and are such that for all ψ not containing p:

$$\vdash_{\mathsf{L}} \varphi \to \psi \;\Leftrightarrow\; \vdash \exists p\varphi \to \psi \qquad \vdash_{\mathsf{L}} \psi \to \varphi \;\Leftrightarrow\; \vdash \psi \to \forall p\varphi.$$

This implies that $\vdash \varphi \to \exists p\varphi$ and $\vdash \forall p\varphi \to \varphi$. Therefore, $\exists p_1 \dots \exists p_n\varphi$ is an interpolant for any derivable implication $\varphi \to \psi$ for which ψ does not contain any p_i and all other atoms in φ occur in ψ. This shows that uniform interpolation implies interpolation. It also shows that $\exists p_1 \dots \exists p_n\varphi$ is an interpolant that does not depend on the structure of the consequent of an implication, just on the variables it contains. Likewise, $\forall p_1 \dots \exists p_n\varphi$ is an interpolant that does not depend on the structure of the antecedent of an implication. In the literature, $\exists p$ is also called the *post* or *right* interpolant and $\forall p$ the *pre* or *left* interpolant.

Uniform interpolation as a property is stronger than interpolation, as there are modal logics, for example $\mathsf{S4}$ and $\mathsf{K4}$, that do not satisfy the stronger property, but do have interpolation (Ghilardi and Zawadowski, 1995; Bílková, 2007).

In (Iemhoff, 2018), a method has been developed to prove uniform interpolation for any modal logic with a sequent calculus consisting of so–called centered and centered modal rules[1] This method provides a single framework via which to prove in a uniform way existing and new results on uniform interpolation, such as the result from (Bílková, 2007) that K has uniform interpolation and the new result that KD has uniform interpolation. But the most important use of the method lies in its contraposition: it implies that no logic without uniform interpolation has a sequent calculus consisting of centered and centered modal rules. Since there are many modal logics without uniform interpolation, it follows that none of these logics can have a calculus of this kind.

In this paper we extend the method of (Iemhoff, 2018) to intermediate and intuitionistic modal logics, where the latter are modal logics that contain IPC. This is not a straightforward extension, since in contrast to CPC, already for IPC itself the proof of uniform interpolation is highly nontrivial. The intricate proof in (Pitts, 1992) makes use of a terminating calculus for IPC developed independently by Dyckhoff (1992) and Hudelmaier (1988, 1992, 1993) and, much earlier in a somewhat different form, by Vorob'ev (1952, 1970). In Iemhoff (2017) we have extended that calculus to terminating calculi for intuitionistic modal logics, and these are the calculi we use in this paper. Our method is uniform in the sense that it does not establish uniform interpolation based on a specific calculus, but based on certain structural properties that the calculus has to satisfy. In this way one can prove uniform interpolation for several logics at once, namely for all those that have calculi that satisfy these requirements.

We show how via our method Pitts' result can be obtained, that the method can be extended to other intermediate and intuitionistic modal logics, and show that the diamond–free fragments of what in the literature are called iK and iKD[2], have uniform interpolation.

1.1 Main aim

Rather than proving uniform interpolation for intermediate and intuitionistic modal logics, the main aim of the paper is in fact the opposite: to prove that intermediate and intuitionistic modal logics without uniform interpolation do not have certain sequent calculi. The idea is simple. We provide sufficient conditions such that whenever a calculus satisfies

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[1] In that paper we used the word *focussed* instead of *centered.*

[2] There is no complete consensus on terminology in the field of intuitionistic modal logic. References and alternative names will be provided in Section 8.

these conditions its logic has uniform interpolation. So that for a logic not having uniform interpolation it can be concluded it does not have a calculus satisfying those constraints.

Thus this enterprise can be viewed as a possible approach to establish what, if any, sequent calculi nonclassical logics can have. The calculi we are interested in here are calculi with good properties, meaning without a cut rule and satisfying some form of the subformula property. From the definition of centered and centered modal rules below it will be clear that the calculi we consider have such properties.

Although in this paper we focus on sequent calculi, we conjecture that our method can be adapted to certain other proof systems as well. The general idea being that a proof system with certain *structural properties* (such as the subformula property or closure under weakening) implies that the corresponding logic has certain *regular properties* (such as uniform interpolation). The more general the requirements on the proof system, the stronger the result. This works in both ways: If many logics have a proof system with certain structural properties, then the method establishes that many logics satisfy the corresponding regular properties, and if many logics do not satisfy certain regular properties, as in the case of uniform interpolation in intermediate logics, then the method shows that none of these many logics can have a proof system with the corresponding structural properties.

In this paper the regular property is uniform interpolation, and the proof systems are extensions of G4ip and $\mathsf{G4iK}_\Box$ by centered and centered modal rules, notions that are defined below. Since there are only seven intermediate logics with uniform interpolation, our method in particular shows (Corollary 5.0.2) that except for these seven logics, no intermediate logic has a sequent calculus of that particular form.

1.2 Related work

In the literature there is quite some work on uniform interpolation for classical modal logics and intermediate logics, but for intuitionistic modal logics far less is known. In this section we discuss some results from these areas that are relevant to our results.

For several modal logics, uniform interpolation has been established in various ways. The results on K and GL by Shavrukov (1993) and Visser (1996a,b) obtained around the same time as Pitts' result, used semantic techniques. This is in contrast to Pitts' result for IPC, which is syntactic in nature. A similar syntactic method was shown to apply to K, T, GL, and S4Grz in (Bílková, 2006, 2007) and to substructural logics in (Alizadeh et al., 2014).

An algebraic or categorical approach can be found in the work of Ghilardi and Zawadowski (1995, 2002) and van Gool et al. (2016). The former proved that S4, which has interpolation, does not have uniform interpolation, a fact used by (Bílková, 2006) to show that neither has K4. In (Maksimova, 1977) it has been shown that there are only seven propositional intermediate logics with interpolation, and Ghilardi and Zawadowski (2002) showed that there are exactly that many logics with uniform interpolation. In the algebraic setting, the quantifiers $\forall p$ and $\exists p$ can be seen to be adjoints of a certain embedding operation.

Propositional quantification in modal and intuitionistic logic has been studied in various contexts. Since there are several possible ways to define quantification, one has to be careful in comparing the different approaches. In (Połacik, 1998) it is shown that the uniform interpolants as defined above do not coincide with topological quantification. The paper (Kremer, 1997), in which it is proved that a certain version of propositional quantified intuitionistic logic is recursively isomorphic to full second order classical logic, is a good source for references to the literature on the topic.

Several intuitionistic modal logics have been introduced in the literature. Often, they consist of the modal axioms of well–known classical modal logics, but with intuitionistic logic as the underlying propositional logic (Bellin et al., 2001; Bierman and de Paiva, 2000; Božić and Došen, 1984; Došen, 1985; Simpson, 1994; Wolter and Zakharyaschev, 1999). Litak (2014) provides a nice overview of the work of the Georgian School on intuitionistic modal logic, in particular on fixed point theorems for such logics.

Contents

2 Preliminaries

2.1 Language and sequents

The logics we consider are (modal) propositional logics, formulated in a language $\mathcal{L}$ that contains constants $\top$ and $\bot$, propositional variables or atoms $p, q, r, \dots$ and the connectives $\wedge, \vee, \neg, \to$, and the modal operator $\Box$ in case of modal logics. $\mathcal{F}$ denotes the set of formulas in $\mathcal{L}$ and $\mathcal{M}$ is the set of all finite multisets of formulas in $\mathcal{F}$. Given a set of atoms $\mathcal{P}$, $\mathcal{F}(\mathcal{P})$ denotes all formulas in $\mathcal{L}$ in which all atoms belong to $\mathcal{P}$. The language $\mathcal{L}_{\mathsf{qf}}$ is defined to be the extension of $\mathcal{L}$ with propositional quantifiers $\forall p$ and $\exists p$ for every atom p, and $\mathcal{F}_{\mathsf{qf}}$ is the set of formulas in that language.

Sequents are expressions of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas in $\mathcal{F}_{\mathsf{qf}}$, which are interpreted as $I(\Gamma \Rightarrow \Delta) = (\bigwedge \Gamma \to \bigvee \Delta)$. We say that a sequent *is in* $\mathcal{L}$ when all its formulas belong to $\mathcal{L}$. In this paper we only consider single-conclusion sequents, meaning that the succedent Δ contains at most one formula. We denote finite multisets by $\Gamma, \Pi, \Delta, \Sigma$. We denote by $\Gamma \cup \Pi$ the multiset that contains only formulas φ that belong to Γ or Π and the number of occurrences of φ in $\Gamma \cup \Pi$ is the sum of the occurrences of φ in Γ and in Π. In a sequent, notation Π, Γ is short for $\Gamma \cup \Pi$. We also define (a for antecedent, s for succedent):

$$(\Gamma \Rightarrow \Delta)^a \equiv_{df} \Gamma \qquad (\Gamma \Rightarrow \Delta)^s \equiv_{df} \Delta.$$

Expression $S_0 \subseteq S_1$ denotes that $S_0^a \subseteq S_1^a$ and $S_0^s \subseteq S_1^s$, and $S_0 \subset S_1$ denotes: $S_0^a \subset S_1^a$ and $S_0^s \subseteq S_1^s$, or $S_0^s \subset S_1^s$ and $S_0^a \subseteq S_1^a$. When sequents are used in the setting of formulas, we often write S for $I(S)$, such as in $\vdash \bigvee_i S_i$, which thus means $\vdash \bigvee_i I(S_i)$. Multiplication of sequents is defined as

$$S_1 \cdot S_2 \equiv_{df} (S_1^a \cup S_2^a \Rightarrow S_1^s \cup S_2^s).$$

For a multiset Γ, $\Box\Gamma$ and $\boxdot\Gamma$ denote the multisets $\{\Box\varphi \mid \varphi \in \Gamma\}$ and $\Gamma \cup \Box\Gamma$, respectively. $\boxdot\varphi$ is short for $\varphi \wedge \Box\varphi$, but if the expression occurs as an element of a sequent it stands for $\varphi, \Box\varphi$. For example, $(\Gamma, \boxdot\varphi \Rightarrow \Delta)$ should be read as $(\Gamma, \Box\varphi, \varphi \Rightarrow \Delta)$. For a sequent S, we define

$$\Box S \equiv_{df} (\{\Box\varphi \mid \varphi \in S^a\} \Rightarrow \{\Box\psi \mid \psi \in S^s\}) \; \boxdot S \equiv_{df} (\{\boxdot\varphi \mid \varphi \in S^a\} \Rightarrow \{\boxdot\psi \mid \psi \in S^s\}).$$

This implies that $\Box(\Gamma \Rightarrow\,) = (\Box\Gamma \Rightarrow\,)$ and $\Box(\,\Rightarrow \Delta) = (\,\Rightarrow \Box\Delta)$, and similarly for $\boxdot$.

The set $\mathcal{F}_{\mathsf{ex}}$ is the smallest set of expressions that contains all formulas in the language $\mathcal{L}$, is closed under the connectives (and modal operator, if present), and if S is a sequent in $\mathcal{L}$ and p an atom, then $\forall pS$ and $\exists pS$ belong to $\mathcal{F}_{\mathsf{ex}}$. For example, when S is a sequent in $\mathcal{L}$ and φ a propositional formula, then $(\varphi \to \exists pS)$ belongs to $\mathcal{F}_{\mathsf{ex}}$, as does $\Box(\varphi \wedge \forall pS)$, but $\exists p \exists qS$

does not. The interpretation of $\mathcal{F}_{\mathsf{ex}}$ into $\mathcal{F}_{\mathsf{qf}}$ is the identity on formulas in $\mathcal{F}_{\mathsf{qf}}$, commutes with the connectives and the modal operator and interprets quantified sequents as

$$\forall p S \equiv_{df} \forall p I(S) \qquad \exists p S \equiv_{df} \exists p (\bigwedge S^a).$$

We say that a sequent *is in* $\mathcal{L}_{\mathsf{ex}}$ when all its formulas belong to $\mathcal{F}_{\mathsf{ex}}$.

2.2 Rules and instances

For a proper treatment of our proof systems we need to make a distinction between the object–language and the meta–language, where the latter is the language in which the sequent calculi will be defined. $\overline{\mathcal{L}}$ consists of infinitely many formula symbols $\overline{\varphi}, \overline{\psi}, \overline{\chi}, \overline{\varphi}_1, \overline{\varphi}_2, \dots,$ constants $\top$ and $\bot$, the connectives $\wedge, \vee, \neg, \to$, and the modal operator $\Box$ in the case of modal logics. The set $\overline{\mathcal{F}}$ of meta–formulas in this language is defined as usual: the constants and all formula symbols are meta–formulas, and if φ and ψ are meta–formulas, then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$ and $\neg\varphi$. $\overline{\mathcal{M}}$ is an infinite set of symbols for *meta–multisets*, the elements we denote by $\overline{\Gamma}, \overline{\Pi}, \overline{\Delta}, \overline{\Sigma}$. A *meta–sequent* $\overline{S}$ is an expression $\overline{S} = (X \Rightarrow Y)$, where X and Y are finite multisets consisting of elements in $\overline{\mathcal{F}} \cup \overline{\mathcal{M}}$.

A *substitution* σ is a map from $\overline{\mathcal{F}} \cup \overline{\mathcal{M}}$ to $\mathcal{F} \cup \mathcal{M}$ that maps constants to themselves, meta–formula to formulas, that commutes with the connectives and modal operator, and that maps meta–multisets to multisets of formulas. Thus $\sigma[\overline{\mathcal{F}}] \subseteq \mathcal{F}$ and $\sigma[\overline{\mathcal{M}}] \subseteq \mathcal{M}$. *Sub* is the set of all substitutions. Given finite multisets X and Y of elements in $\overline{\mathcal{F}} \cup \overline{\mathcal{M}}$, we write σX for $\{\sigma A \mid A \in X\}$, and $\sigma(X \Rightarrow Y)$ for $(\sigma X \Rightarrow \sigma Y)$. Since in this paper only single-conclusion sequents are considered, for a substitution σ that is applied to $X \Rightarrow Y$, it is tacitly assumed that in case Y consists of a meta–multiset symbol $\overline{\Delta}$, σ maps $\overline{\Delta}$ to a multiset that contains at most one formula.

2.3 Sequent calculi and rules

A *sequent calculus* is a set of *rules*, which are expressions of the form

$$\frac{\overline{S}_1 \quad \overline{S}_2 \quad \dots \quad \overline{S}_n}{\overline{S}_0}\ \mathcal{R} \tag{1}$$

for some meta-sequents $\overline{S}_0, \overline{S}_1, \dots, \overline{S}_n$. It is a *right rule* if $\overline{S}_0^s$ contains a meta–formula and a *left rule* if $\overline{S}_0^a$ does. Thus if $\overline{S}_0 = (\overline{\Gamma} \Rightarrow \overline{\Delta})$ for meta–multisets $\overline{\Gamma}$ and $\overline{\Delta}$, then the rule is neither left nor right. But if we assume, and we will do so in this paper, that no rule in a calculus has a conclusion that consists of meta–multisets only, then this possibility disappears and all rules are left or right (or both). A rule is called an *axiom* in case there are no premisses. Thus axioms are considered to be special cases of rules.

For any substitution σ, the inference

$$\frac{\sigma\overline{S}_1 \quad \sigma\overline{S}_2 \quad \dots \quad \sigma\overline{S}_n}{\sigma\overline{S}_0}\ \sigma\mathcal{R}$$

is an *instance* of $\mathcal{R}$. Throughout this paper we denote rules by $\mathcal{R}$ and instances of rules by R. Given a rule $\mathcal{R}$, $\mathcal{R}_{\mathsf{ins}}$ denotes the set of instances of $\mathcal{R}$.

An example of a rule could be

$$\frac{\overline{\Gamma} \Rightarrow \neg\neg\overline{\varphi}}{\overline{\Gamma} \Rightarrow \overline{\varphi}}$$

Two possible instances of the rule are

$$\frac{q, q, r \to p \Rightarrow \neg\neg p}{q, q, r \to p \Rightarrow p} \qquad \frac{\Rightarrow \neg\neg(r_1 \wedge r_2)}{\Rightarrow r_1 \wedge r_2}$$

with respective substitutions σ_1 and σ_2, where

$$\sigma_1(\overline{\varphi}) = p \ \ \sigma_1(\overline{\Gamma}) = \{q, q, r \to p\} \quad \sigma_2(\overline{\varphi}) = r_1 \wedge r_2 \ \ \sigma_2(\overline{\Gamma}) = \varnothing.$$

When a rule comes with a side condition, such as the axiom

$$\overline{\Gamma}, \overline{\varphi} \Rightarrow \overline{\varphi} \quad (\overline{\varphi} \text{ is an atom}),$$

the side condition has to be interpreted as a restriction on the substitutions that correspond to the instances of the rule. In the example, this would mean restricting the instances of the axiom to those substitutions that map $\overline{\varphi}$ to an atom.

A sequent S is *derivable* in a sequent calculus G, written $\vdash_{\mathsf{G}} S$, if there is a finite tree labelled with sequents such that every leaf is an instances of an axiom in G, the root is the sequent S and every node that is not a leaf is the conclusion of an instance of a rule in G and the premisses of that instance are exactly the labels of the immediate successors of the node. A sequent is *free* if it is not the conclusion of any instance of any rule.

2.3.1 Principal formulas and sequents

In the definitions and proofs below we often use a case distinction based on a sequent being or not being principal for an instance of a rule, a notion that is defined as follows. Every instance of any rule in this paper comes with the notion of *principal formulas*, which are one or more formula occurrences singled out in the conclusion of the instance, and which are defined per rule. A sequent S is *principal* for an instance R of a rule if the conclusion of R is of the form $S' \cdot S$ for some sequent S' and all principal formulas of R occur in S. For example, suppose R has conclusion $(\Gamma, \varphi \Rightarrow \Delta)$ and φ is the principal formula of R, then any sequent of the form $(\Gamma', \varphi \Rightarrow \Delta')$, where $(\Gamma' \Rightarrow \Delta') \subseteq (\Gamma \Rightarrow \Delta)$, is principal for R.

2.3.2 Convention

As is often done implicitly in papers on proof systems, to keep the notation light, from now on the terminology for the object–language is also used for the meta–language: over scores and the word "meta" are omitted, trusting that it will always be clear from the context (or does not matter) which language we are concerned with. For example, an axiom such as $\overline{\Gamma}, \overline{\varphi} \Rightarrow \overline{\varphi}, \overline{\Delta}$ will simply be written as $\Gamma, \varphi \Rightarrow \varphi, \Delta$.

2.4 Logics

Logics are considered to be given as consequence relations closed under substitution, where $\vdash_{\mathsf{L}}$ denotes the consequence relation for logic L. Thus $\vdash_{\mathsf{L}}$ is a relation between sets of formulas and formulas, where $\Gamma \vdash_{\mathsf{L}} \varphi$ means that formula φ follows in L from the set of formulas Γ. If $\vdash_{\mathsf{L}} \varphi$, then φ is a *theorem* of the logic.

An *intermediate logic* is a logic in the language of propositional logic such that its set of theorems contains the theorems of IPC and is contained in the set of theorems of classical propositional logic CPC. An *intuitionistic modal logic* is a logic in the language of modal logic (the language of propositional logic plus the operator $\Box$) such that its set of theorems contains the theorems of IPC. Every logic in this paper is either an intermediate logic or an intuitionistic modal logic.

In this paper all logics are defined via sequent calculi. Given a sequent calculus G and set of sequents $\mathcal{S} \cup \{S\}$, $\mathcal{S} \vdash_{\mathsf{G}} S$ denotes that S is derivable in G from sequents $\mathcal{S}$. The *logic corresponding to* G, denoted L_{G}, is defined as the smallest consequence relation given by

$$\{I(S') \mid S' \in \mathcal{S}\} \vdash_{\mathsf{L}_{\mathsf{G}}} I(S) \ \equiv_{df} \ \mathcal{S} \vdash_{\mathsf{G}} S.$$

When a logic L has a sequent calculus with respect to which it is sound and complete, then we assume that the consequence relation is such that for every instance $S_1 \dots S_n / S_0$ of a rule in the calculus, $I(S_1), \dots, I(S_n) \vdash_{\mathsf{L}} I(S_0)$ holds. Clearly, logics of the form L_{G} have this property by definition. This requirement implies that in the case of logics with a sequent calculus that contains a rule that expresses necessitation, like $(\Rightarrow \varphi)/(\Rightarrow \Box\varphi)$, the inference $\varphi \vdash_{\mathsf{L}} \Box\varphi$ should hold for all φ, a fact that we will often use.

By $\vdash_{\mathsf{IPC}}^{\mathcal{R}}$ we denote the smallest consequence relation containing $\mathcal{R}$ and such that $\varphi_1, \dots, \varphi_n \vdash_{\mathsf{IPC}}^{\mathcal{R}} \psi$ holds whenever $(\bigwedge \varphi_i \to \psi)$ holds in IPC.

2.5 Reductive calculi

An order $\prec$ on sequents is *reductive* if

- it is well-founded;
- all proper subsequents of a sequent come before that sequent;
- whenever all formulas in S occur boxed in S', then $S \prec S'$;
- for all multisets Γ, Δ, formulas φ and atoms q: $(\Gamma, \varphi \Rightarrow \Delta) \prec (\Gamma, q \to \varphi \Rightarrow \Delta)$.

A calculus is *terminating* with respect to an order $\prec$ on sequents if

- it is finite;
- for all sequents S and all rules in the calculus there are at most finitely many instances of the rule with conclusion S;
- in every instance of a rule in the calculus the premisses come before the conclusion in the order $\prec$.

A calculus is *reductive* if it is terminating with respect to an order that is reductive.

A typical example of a rule that in general cannot belong to a reductive calculus is the cut rule, as in most common orders on sequents the premisses of that rule do not come before its conclusion. We will see that many standard cut-free calculi for modal logic are reductive.

Example 2.5.1 In all concrete examples in this paper we use the following reductive order on formulas in $\mathcal{F}$ based on a *weight* function which is a combination of the weight functions from Bílková (2007) and Dyckhoff (1992): $\varphi \prec \psi \equiv_{df} w(\varphi) < w(\psi)$, where

$$\begin{array}{ll} w(p) = w(\bot) = 1 & \\ w(\varphi \circ \psi) = w(\varphi) + w(\psi) + 1 & \circ \in \{\vee, \to\} \\ w(\varphi \wedge \psi) = w(\varphi) + w(\psi) + 2 & \\ w(\Box\varphi) = w(\varphi) + 1. & \end{array}$$

We extend the weight to multisets as in (Dershowitz and Manna, 1979): $\Delta \prec \Gamma$ iff Δ is the result of replacing one or more formulas in Γ by zero or more formulas of lower weight. Sequents inherit this ordering by defining:

$$S_0 \prec S_1 \equiv_{df} S_0^a \cup S_0^s \prec S_1^a \cup S_1^s.$$

In this paper, whenever a general result about reductive calculi is applied to a concrete calculus, the reductive order that is used is the one in this example. Although most theorems hold for any reductive order, it may be helpful to keep this concrete order in mind throughout the paper.

Returning to general reductive orders, a reductive order $\prec$ is extended to an order on formulas in $\mathcal{F}_{\mathsf{ex}}$ as follows. First, we associate the following set of formulas with a formula φ in $\mathcal{F}_{\mathsf{ex}}$: $\mathsf{qf}(\varphi)$ denotes the multiset consisting of all occurrences of subformulas of the form QpS in φ, where $Q \in \{\exists, \forall\}$. The order on multisets of the form $\mathsf{qf}(\varphi)$ again is in the style of (Dershowitz and Manna, 1979): $\mathsf{qf}(\varphi) \prec_{\mathsf{qf}} \mathsf{qf}(\psi)$ iff $\mathsf{qf}(\varphi)$ is the result of replacing one or more formulas of the form QpS in $\mathsf{qf}(\psi)$ by zero or more formulas of the form $Q'qS'$ with $S' \prec S$, where $Q, Q' \in \{\exists, \forall\}$. This order is well–defined since by definition such S and S' are sequents in $\mathcal{L}$ and therefore can be compared via $\prec$. The order on $\mathcal{F}_{\mathsf{ex}}$, that is also denoted by $\prec$, can now be defined: if $\varphi, \psi \in \mathcal{F}$, then $\varphi \prec \psi$ iff $(\Rightarrow \varphi) \prec (\Rightarrow \psi)$; if $\varphi \in \mathcal{F}$ and $\psi \notin \mathcal{F}$, then $\varphi \prec \psi$ and not $\psi \prec \varphi$; if $\varphi, \psi \notin \mathcal{F}$, then $\varphi \prec \psi$ if $\mathsf{qf}(\varphi) \prec_{\mathsf{qf}} \mathsf{qf}(\psi)$. When $\varphi \prec \psi$, we say that φ is of *lower rank* than ψ. Clearly, if the order $\prec$ on sequents is well–founded, then so is the order $\prec$ on $\mathcal{F}_{\mathsf{ex}}$.

3 Uniform interpolants

A logic has *uniform interpolation* if for any atom p and any set of atoms $\mathcal{P}$ not containing p, the embedding of $\mathcal{F}(\mathcal{P})$ into $\mathcal{F}(\mathcal{P} \cup \{p\})$ has a right and a left adjoint: For any formula φ and any atom p there exist formulas χ_r and χ_l in the language of the logic, that do not contain p and such that for all ψ not containing p:

$$\vdash \psi \to \varphi \;\Leftrightarrow\; \vdash \psi \to \chi_r \qquad \vdash \varphi \to \psi \;\Leftrightarrow\; \vdash \chi_l \to \psi.$$

These formulas are usually denoted by $\forall p\varphi$ and $\exists p\varphi$, respectively, and thus we have

$$\vdash \psi \to \varphi \;\Leftrightarrow\; \vdash \psi \to \forall p\varphi \qquad \vdash \varphi \to \psi \;\Leftrightarrow\; \vdash \exists p\varphi \to \psi.$$

Given a formula φ, its *universal uniform interpolant with respect to* $p_1 \dots p_n$ is $\forall p_1 \dots p_n\varphi$, which is short for $\forall p_1(\forall p_2(\dots(\forall p_n\varphi)\dots))$, and its *existential uniform interpolant with respect to* $p_1 \dots p_n$ is $\exists p_1 \dots p_n\varphi$, short for $\exists p_1(\exists p_2(\dots(\exists p_n\varphi)\dots))$. The requirements above could be replaced by the following four requirements.

$$\vdash \forall p\varphi \to \varphi \qquad \vdash \psi \to \varphi \;\Rightarrow \vdash \psi \to \forall p\varphi. \tag{$\forall$}$$

$$\vdash \varphi \to \exists p\varphi \qquad \vdash \varphi \to \psi \;\Rightarrow \vdash \exists p\varphi \to \psi. \tag{$\exists$}$$

In classical logic one only needs one quantifier, as $\exists p$ can be defined as $\neg\forall p\neg$ and vice versa. Although in the intuitionistic setting $\exists p$ can also be defined in terms of $\forall p$, namely as $\exists p\varphi = \forall q(\forall p(\varphi \to q) \to q)$ for a q not in φ, having it as a separate quantifier is convenient in the proof-theoretic approach presented here (we follow (Pitts, 1992), which also uses both quantifiers).

3.1 Partitions

To define uniform interpolants in the setting of sequents, we introduce the notion of a partition, which applies to sequents and to rules. The notion for sequents is treated in this section and the one for rules later on.

Intuitively, if in the statement of uniform interpolation the implication is replaced by a sequent arrow, then $(\psi \Rightarrow \forall p\varphi)$, for ψ not containing p, can be viewed as partitioning the sequent $S = (\psi \Rightarrow \varphi)$ in two sequents $S^r = (\psi \Rightarrow\ )$ and $S^i = (\ \Rightarrow \varphi)$, and applying universal quantification to the second one. Likewise for $\exists p\varphi$. The definition of a partition is a generalization of that idea to arbitrary sequents.

A *partition* of a sequent S is an ordered pair (S^r, S^i) (i for interpolant, r for rest) such that $S = S^r \cdot S^i$. It is a *p–partition* if p does not occur in S^r. For any sequent S and partition (S^i, S^r) we use the abbreviation:

$$S^r \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \varnothing) \equiv_{df} \begin{cases} S^r \cdot (\exists pS^i \Rightarrow \forall pS^i) & \text{if } S^s \neq \varnothing \text{ and } S^{rs} = \varnothing \\ S^r \cdot (\exists pS^i \Rightarrow\ ) & \text{if } S^s = \varnothing \text{ or } S^{rs} \neq \varnothing. \end{cases}$$

A *(p-)partition* of an instance $R = (S_1 \dots S_n / S_0)$ of a rule is a (p-)partition of the sequents in the rule. Given such a partition, (R^r, R^i) and $R^\star$ respectively denote the expressions

$$\frac{(S_1^r, S_1^i) \quad \dots \quad (S_n^r, S_n^i)}{(S_0^r, S_0^i)}\ (R^r, R^i) \qquad \frac{S_1^\star \quad \dots \quad S_n\star}{S_0^\star}\ R^\star$$

3.1.1 The interpolant properties

Recall from Section 2.1 that $\forall pS$ and $\exists pS$ are defined to be $\forall pI(S)$ and $\exists p(\bigwedge S^a)$, respectively. In particular, $\forall p(\Rightarrow \varphi)$ is equivalent to $\forall p\varphi$ and $\exists p(\varphi \Rightarrow)$ to $\exists p\varphi$. As will be shown in Lemma 3.1.1, ($\forall$) and ($\exists$) can be replaced by the following three requirements, the *interpolant properties*.

- ($\forall$l) For all p: $\vdash S^a, \forall pS \Rightarrow S^s$;
- ($\exists$r) For all p: $\vdash S^a \Rightarrow \exists pS$;
- ($\forall\exists$) If S is derivable, for all p and all p–partitions (S^r, S^i): $\vdash S^r \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \varnothing)$.

Properties ($\forall$l) and ($\exists$r) are the *independent* (from partitions) *interpolant properties*, and ($\forall\exists$) is the *dependent interpolant property.*

A partition (S^r, S^i) of S *satisfies* the interpolant properties if, in the case of the independent property, S satisfies them (in which case we also say that S satisfies them), and in case of the dependent property, it holds for that particular partition. A sequent *satisfies* a property if every possible partition of the sequent satisfies it.

Lemma 3.1.1 If all sequents satisfy the interpolant properties, then L has uniform interpolation.

Proof ($\exists$) Consider $S = (\varphi \Rightarrow)$. By ($\exists$r) we have $\vdash I(\varphi \Rightarrow \exists pS)$, and since $\exists p(\varphi \Rightarrow) = \exists p\varphi$, we have thereby shown $\varphi \to \exists p\varphi$ to be derivable.

Consider a ψ not containing p such that $\vdash \varphi \to \psi$. Let $S = (\varphi \Rightarrow \psi)$ and consider the p–partition (S^r, S^i), where $S^i = (\varphi \Rightarrow)$ and $S^r = (\Rightarrow \psi)$. Hence $\exists p\varphi = \exists pS^i$ by definition and $S^{rs} \neq \varnothing$. As $\vdash (\exists p\varphi \Rightarrow) \cdot (\Rightarrow \psi)$ by ($\forall\exists$), $\vdash \exists p\varphi \to \psi$ follows.

($\forall$) Consider $S = (\Rightarrow \varphi)$. By ($\forall$l) we have $\vdash I(\forall pS \Rightarrow \varphi)$, and since $\forall p(\Rightarrow \varphi) = \forall p\varphi$, we have thereby shown $\forall p\varphi \to \varphi$ to be derivable.

Consider a ψ not containing p such that $\vdash \psi \to \varphi$. Let $S = (\psi \Rightarrow \varphi)$ and consider the p–partition (S^r, S^i), where $S^i = (\Rightarrow \varphi)$ and $S^r = (\psi \Rightarrow)$. Hence $\forall p\varphi = \forall pS^i$ by definition and $S^{rs} = \varnothing$. Thus $\vdash (\psi \Rightarrow) \cdot (\exists pS^i \Rightarrow \forall pS^i)$ by ($\forall\exists$), that is, $\vdash \psi, \exists pS^i \Rightarrow \forall p\varphi$. But $\vdash (\Rightarrow \exists pS^i)$ by ($\exists$). Therefore $\vdash \psi \to \forall p\varphi$. $\square$

Fact 3.1.1 All free sequents satisfy the dependent interpolant properties.

3.2 Interpolant assignments

Let G be a sequent calculus. Recall that given a rule $\mathcal{R}$, $\mathcal{R}_{\mathsf{ins}}$ denotes the set of instances of $\mathcal{R}$ and $\mathsf{G}_{\mathsf{ins}}$ denotes the set of instances of rules in G. An *interpolant assignment* ι for G, assigns, for every atom p and sequent S, $\iota\exists pS = \top$ and $\iota\forall pS = \bot$ in case S is empty, and in case S is not empty:

- for every $R \in \mathsf{G}_{\mathsf{ins}}$ with conclusion S, to each of the expressions $\exists \overset{R}{p} S$ and $\forall \overset{R}{p} S$ a formula in $\mathcal{F}_{\mathsf{ex}}$ that is of lower rank than $\exists pS$ (or, equivalently, of lower rank than $\forall pS$), which are denoted by $\iota\exists \overset{R}{p} S$ and $\iota\forall \overset{R}{p} S$, respectively, and
- for every $\mathcal{R} \in \mathsf{G}$ such that S is not principal for at least one instance of $\mathcal{R}$, to each of the expressions $\exists \overset{\overline{\mathcal{R}}}{p} S$ and $\forall \overset{\overline{\mathcal{R}}}{p} S$ a formula in $\mathcal{F}_{\mathsf{ex}}$ that is of lower rank than $\exists pS$, which are denoted by $\iota\exists \overset{\overline{\mathcal{R}}}{p} S$ and $\iota\forall \overset{\overline{\mathcal{R}}}{p} S$, respectively.

We use the following abbreviations for certain formulas in $\mathcal{F}_{\mathsf{ex}}$. Recall that p and q range over atoms.

$$\begin{array}{lll}
\forall^+ pS & \equiv_{df} & \bigvee\{\iota\forall \overset{R}{p} S \mid R \in \mathsf{G}_{\mathsf{ins}},\ S \text{ is the conclusion of } R\} \\
\forall^- pS & \equiv_{df} & \bigvee\{\iota\forall \overset{\overline{\mathcal{R}}}{p} S \mid \mathcal{R} \in \mathsf{G},\ S \text{ is not principal for some instance of } \mathcal{R}\} \\
\exists^+ pS & \equiv_{df} & \bigwedge\{\iota\exists \overset{R}{p} S \mid R \in \mathsf{G}_{\mathsf{ins}},\ S \text{ is the conclusion of } R\} \\
\exists^- pS & \equiv_{df} & \bigwedge\{\iota\exists \overset{\overline{\mathcal{R}}}{p} S \mid \mathcal{R} \in \mathsf{G},\ S \text{ is not principal for some instance of } \mathcal{R}\} \\
\forall^{at} pS & \equiv_{df} & \bigvee\{q \in S^s \mid q \text{ an atom and } q \neq p, \text{ or } q = \top\} \vee \\
 & & \bigvee\{q \wedge \forall p(\varphi, S^a\backslash\{q \to \varphi\} \Rightarrow S^s)\} \mid (q \to \varphi) \in S^a, q \neq p\} \\
\exists^{at} pS & \equiv_{df} & \bigwedge\{q \in S^a \mid q \text{ an atom and } q \neq p, \text{ or } q = \bot\} \wedge \\
 & & \bigwedge\{q \to \exists p(\varphi, S^a\backslash\{q \to \varphi\} \Rightarrow S^s) \mid (q \to \varphi) \in S^a, q \neq p\}.
\end{array}$$

Observe that there could be more than one instance of a single rule $\mathcal{R}$ that has S as a conclusion, in which case every instance corresponds to a separate disjunct or conjunct of the interpolant assignment. The definition above is well-defined for reductive calculi, because in that case all sets over which the big conjunctions and disjunctions range are finite.

We define a rewrite relation $\rightarrowtail$ on $\mathcal{F}_{\mathsf{ex}}$ that is the smallest relation on $\mathcal{F}_{\mathsf{ex}}$ that preserves the logical operators and satisfies:

$$\forall pS \ \rightarrowtail\ \forall^+ pS \vee \forall^- pS \vee \forall^{at} pS \qquad \exists pS \ \rightarrowtail\ \exists^+ pS \wedge \exists^- pS \wedge \exists^{at} pS.$$

Example 3.2.1 Suppose the calculus only contains the rule $\mathcal{R}$ for conjunction on the right:

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}$$

Consider the sequent $S = (\Rightarrow \varphi_1 \wedge \psi_1, \varphi_2 \wedge \psi_2)$. Let R_i stand for the instance of $\mathcal{R}$ with $\varphi_i \wedge \psi_i$ as the principal formula, and define sequents $S_{\varphi_1} = (\Rightarrow \varphi_1, \varphi_2 \wedge \psi_2)$ and $S_{\psi_1} = (\Rightarrow \psi_1, \varphi_2 \wedge \psi_2)$, and similarly for S_{φ_2} and S_{ψ_2}. By the above definition,

$$\forall pS \rightarrowtail \iota\forall\overset{R_1}{p}S \vee \iota\forall\overset{R_2}{p}S \vee \iota\forall^{at}pS.$$

Using the order in Example 2.5.1, the standard interpolant assignment introduced below satisfies $\iota\forall\overset{R_i}{p}S = \forall pS_{\varphi_i} \wedge \forall pS_{\psi_i}$. This implies that

$$\forall pS \rightarrowtail (\forall pS_{\varphi_1} \wedge \forall pS_{\psi_1}) \vee (\forall pS_{\varphi_2} \wedge \forall pS_{\psi_2}) \vee \bot.$$

3.3 Reduction

The following lemma shows that every $\varphi \in \mathcal{F}_{\mathsf{ex}}$ either belongs to $\mathcal{F}$ or reduces via $\rightarrowtail$ to a unique $\psi \in \mathcal{F}$. In the latter case, the ψ will be denoted by $\delta\varphi$. Slightly abusing notation, we will mostly omit the δ, especially in the setting of derivability. For example, under this convention, $\vdash \varphi \to \bot$ abbreviates $\vdash \delta\varphi \to \bot$.

Lemma 3.3.1 In any reductive calculus, the relation $\rightarrowtail$ on $\mathcal{F}_{\mathsf{ex}}$ is confluent and strongly normalizing.

Proof Let $\prec$ be the extension to $\mathcal{F}_{\mathsf{ex}}$ of the order with respect to which the calculus is reductive, as defined in Section 2.5, and recall that it is by definition well-founded. In the terminology of (Baader and Nipkow, 1998) (Definition 4.2.2), $\rightarrowtail$ determines a rewrite relation (the set V of variables, in their sense, is empty in our setting). From the definition of interpolant assignments it follows that $\varphi \rightarrowtail \psi$ implies $\psi \prec \varphi$, and thus the rewrite relation is terminating. Since no rules overlap, it has no critical pairs (Definition 6.2.1 of the same volume), and therefore (Corollary 6.2.5) the rewrite relation is confluent. Since the relation is also normalizing (as it is terminating), it follows that $\rightarrowtail$ is strongly normalizing, implying that every term has a unique normal form. $\Box$

3.4 Explanation

As is clear from the definition above, for a sequent S and atom p, the uniform interpolants $\forall pS$ and $\exists pS$ are a disjunction, respectively conjunction of formulas of lower rank than $\exists pS$, also if S is free. The role of these expressions in a proof of the interpolant properties is as follows. Clearly, if only the dependent properties have to be satisfied, then taking $\bot$ for $\exists pS$ for all sequents S suffices. If only the independent properties have to be satisfied, then assigning $\bot$ to $\forall pS$ and $\top$ to $\exists pS$ suffices. The interplay between the independent and the dependent properties is what makes the definition of the uniform interpolants difficult. It it is based on the following observation.

For the dependent interpolant property, there are, for every derivable sequent S and p–partition (S^r, S^i), two cases given some derivation of S: for R being the last inference of the derivation and an instance of a rule $\mathcal{R}$, either S^i is principal for R or it is not. Suppose that in the first case one can show that for some instance R^i of $\mathcal{R}$ with conclusion S^i,

$\vdash S^{ra}, \exists\overset{R^i}{p}S^i \Rightarrow \forall\overset{R^i}{p}S^i$, and in the second case that $\vdash S^{ra}, \exists\overset{\overline{\mathcal{R}}}{p}S^i \Rightarrow \forall\overset{\overline{\mathcal{R}}}{p}S^i$. Then the dependent interpolant property, $\vdash (S^{ra}, \exists pS^i \Rightarrow \forall pS^i)$, holds for (S^r, S^i), as $\exists\overset{R^i}{p}S^i$ is a conjunct of $\exists pS^i$ and $\forall\overset{R^i}{p}S^i$ is a disjunct of $\forall pS^i$ in the first case, and $\exists\overset{\overline{\mathcal{R}}}{p}S^i$ is a conjunct of $\exists pS^i$ and $\forall\overset{\overline{\mathcal{R}}}{p}S^i$ is a disjunct of $\forall pS^i$ in the second case. The same strategy can be used to show that $\vdash S^{ra}, \exists pS^i \Rightarrow S^{rs}$ in case $S^s = \varnothing$ or $S^{rs} \neq \varnothing$. This is how the dependent interpolant property will be proved.

The role of the disjuncts $\forall^{at}p$ and conjuncts $\exists^{at}p$ lies in certain particular cases. For example, given an instance of an axiom $(q \Rightarrow q)$ and partition $S^r = (q \Rightarrow)$, $S^i = (\ \Rightarrow q)$, the sequent $(q, \exists pS^i \Rightarrow \forall pS^i)$ has to be derivable, and $\forall^{at}pS^i$ and $\exists^{at}pS^i$ take care of that case.

3.5 The inductive properties

In order to develop a modular method for proving uniform interpolation, we introduce the following six properties of rules, where $\varnothing \vdash \varphi$ should be read as $\vdash \varphi$. Recall that $(\Gamma \Rightarrow \Delta) \subseteq (\Gamma' \Rightarrow \Delta')$ denotes that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$ (Section 2.1). Given an instance $R = (S_1 \dots S_n / S_0)$ of a rule $\mathcal{R}$, we define

$$\begin{aligned}
\mathfrak{I}_R^p \quad &\equiv_{df} \quad \{S_j \cdot (\forall pS_j \Rightarrow), (S_j^a \Rightarrow \exists pS_j) \mid 1 \leq j \leq n\} \cup \\
&\phantom{\equiv_{df}} \quad \{S^a \Rightarrow \exists p(S^a \Rightarrow) \mid S \subset S_0 \text{ or } \Box S \subseteq S_0 \text{ or } S \subseteq S_j \text{ for some } 1 \leq j \leq n\} \\
\mathfrak{D}_R^p \quad &\equiv_{df} \quad \bigcup_{j=1}^{n} \{S_j^r \cdot (\exists pS_j^i \Rightarrow \forall pS_j^i \mid \varnothing) \mid (S_j^r, S_j^i) \text{ a } p\text{–partition of } S_j\}.
\end{aligned}$$

In $\mathfrak{I}_R^p$, requirement $\Box S \subseteq S_0$ is only included in the case of modal logic. For $\mathfrak{D}_R^p$, note that it contains the sequent $S_j^r \cdot (\exists pS_j^i \Rightarrow \forall pS_j^i \mid \varnothing)$ for *any* possible p-partition (S_j^r, S_j^i) of a premiss S_j of R. And that for S with empty succedent, $S^r \cdot (\exists pS^i \Rightarrow)$ derives $S^r \cdot (\exists pS^i \Rightarrow \forall pS^i)$. The sets $\mathfrak{I}_R^p$ and $\mathfrak{D}_R^p$ contain the sequents to which, in a proof of the interpolant properties that uses induction along $\prec$, the induction hypothesis needs to be applied. In such a proof, the assumption that the interpolant properties hold for all sequents below S implies that the sequents in $\mathfrak{I}_R^p$ and $\mathfrak{D}_R^p$ are derivable. Note that in the case that R is an instance of an axiom, both sets are empty.

$(\text{IPP})_{\mathcal{R}}^{\forall}$ $\mathfrak{I}_R^p \vdash S \cdot (\forall\overset{R}{p}S \Rightarrow)$ for every instance R of $\mathcal{R}$ with conclusion S.

$(\text{IPN})_{\mathcal{R}}^{\forall}$ If S is not principal for some instance of $\mathcal{R}$, then the assumption that all sequents below S satisfy the interpolant properties implies $\vdash S \cdot (\forall\overset{\overline{\mathcal{R}}}{p}S \Rightarrow)$.

$(\text{IPP})_{\mathcal{R}}^{\exists}$ $\mathfrak{I}_R^p \vdash (S^a \Rightarrow \exists\overset{R}{p}S)$ for every instance R of $\mathcal{R}$ with conclusion S.

$(\text{IPN})_{\mathcal{R}}^{\exists}$ If S is not principal for some instance of $\mathcal{R}$, then the assumption that all sequents below S satisfy the interpolant properties implies $\vdash (S^a \Rightarrow \exists\overset{\overline{\mathcal{R}}}{p}S)$.

$(\text{DPP})_{\mathcal{R}}$ For every sequent S that has a derivation of which the last inference is an instance R of $\mathcal{R}$, and for every p–partition (S^r, S^i) such that sequent S^i is principal for R: $\mathfrak{D}_R^p \vdash S^r \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \varnothing)$.

$(\text{DPN})_{\mathcal{R}}$ For every sequent S that has a derivation of which the last inference is an instance R of $\mathcal{R}$, and for every p–partition (S^r, S^i) such that sequent S^i is not principal for R: if all sequents that are below S satisfy the interpolant properties, then $\vdash S^r \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \varnothing)$.

These six properties are called the *inductive properties* in this paper. "IP" stands for *independent property*, "DP" for *dependent property* "P" and "N" for *principal* and *not principal*, respectively.

An interpolant assignment is *sound* for a rule $\mathcal{R}$ in a calculus, if the six inductive properties hold for $\mathcal{R}$, where $\vdash$ is derivability in the calculus. It is *sound* for a calculus if it is sound for all the rules of the calculus. Sometimes the following strengthening of $(\mathrm{DPN})_{\mathcal{R}}$ holds:

$(\mathrm{DPN})^+_{\mathcal{R}}$ For every sequent S that has a derivation of which the last inference is an instance R of $\mathcal{R}$, for every p–partition (S^r, S^i) such that sequent S^i is not principal for R: $\mathfrak{D}^p_R \vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing)$.

This is a strengthening of $(\mathrm{DPN})_{\mathcal{R}}$ because under the assumption that all sequents lower than S satisfy the interpolant properties, all sequents in the set $\mathfrak{D}^p_R$ become derivable.

Remark 3.5.1 The following observation will be used to prove $(\mathrm{DPP})_{\mathcal{R}}$ and $(\mathrm{DPN})_{\mathcal{R}}$. Consider a sequent S with partition (S^r, S^i), which has a derivation of which the last inference is an instance $R = (S_1 \dots S_n / S)$ of $\mathcal{R}$. To prove $(\mathrm{DPP})_{\mathcal{R}}$, thus in case S^i is principal for R, in order to prove

$$\mathfrak{D}^p_R \vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing)$$

it suffices to show that

$$\mathfrak{D}^p_R \vdash S^r \cdot (\exists \overset{R^i}{p} S^i \Rightarrow \forall \overset{R^i}{p} S^i \mid \varnothing)$$

for some partition (R^r, R^i) of R with conclusion (S^r, S^i) such that R^i is an instance of $\mathcal{R}$. The reason being that for such an R^i, $\exists \overset{R^i}{p} S^i$ is a conjunct of $\exists p S^i$ and $\forall \overset{R^i}{p} S^i$ a disjunct of $\forall p S^i$. Likewise, to prove $(\mathrm{DPN})_{\mathcal{R}}$, thus in case S^i is not principal for R, to prove that

$$\vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing)$$

it suffices to prove that

$$\vdash S^r \cdot (\exists \overset{\overline{\mathcal{R}}}{p} S^i \Rightarrow \forall \overset{\overline{\mathcal{R}}}{p} S^i \mid \varnothing).$$

3.6 Soundness

Lemma 3.6.1 If a logic L has a reductive calculus for which there exists a sound interpolant assignment, then all sequents satisfy the interpolant properties.

Proof We use induction along the well-founded order $\prec$ on sequents with respect to which the calculus is reductive. Therefore assume that all sequents lower than S satisfy the interpolant properties. We have to show that so does S. Note that all sequents in the sets $\mathfrak{I}^p_R$ are derivable because they express interpolant properties of sequents that come before S in the order.

$(\forall \mathrm{l})$ We have to show that

1. $\vdash S^a, \forall^{at} p S \Rightarrow S^s$,
2. $\vdash S^a, \forall \overset{R}{p} S \Rightarrow S^s$ for all instances R with conclusion S,
3. $\vdash S^a, \forall \overset{\overline{\mathcal{R}}}{p} S \Rightarrow S^s$ for all rules $\mathcal{R}$ such that some instance is not backwards applicable to S.

2. follows from $(\mathrm{IPP})^{\forall}_{\mathcal{R}}$ and 3. from $(\mathrm{IPN})^{\forall}_{\mathcal{R}}$. For 1., first consider its disjuncts of the form q for some $q \neq p$ that belongs to S^s. Then $S^a, q \Rightarrow S^s$ clearly holds. Second, consider disjuncts of the form $(q \wedge \forall p(\Gamma, \varphi \Rightarrow \Delta))$, where $S = (\Gamma, q \to \varphi \Rightarrow \Delta)$ for some $q \neq p$. Let $S' = (\Gamma, \varphi \Rightarrow \Delta)$. Since $S' \prec S$, the assumption that all sequents lower than S satisfy the interpolant properties implies that $(\Gamma, \varphi, \forall p S' \Rightarrow \Delta)$ is derivable. Thus so is $(\Gamma, q \to \varphi, q \wedge \forall p S' \Rightarrow \Delta)$.

$(\exists\mathrm{r})$ We have to show that

1. $\vdash S^a \Rightarrow \exists^{at} pS$,
2. $\vdash S^a \Rightarrow \exists \overset{R}{p} S$ for all instances R with conclusion S,
3. $\vdash S^a \Rightarrow \exists \overset{\overline{\mathcal{R}}}{p} S$ for all rules $\mathcal{R}$ such that some instance is not backwards applicable to S.

2. follows from $(\mathrm{IPP})^{\exists}_{\mathcal{R}}$ and 3. from $(\mathrm{IPN})^{\exists}_{\mathcal{R}}$. For 1., first consider conjuncts of the form q, where $q \in S^a$ and $q \neq p$. Then $\vdash S^a \Rightarrow q$ clearly holds. The remaining conjuncts of $\exists^{at} pS$ are of the form $(q \to \exists p(\Gamma, \varphi \Rightarrow \Delta))$, where $S = (\Gamma, q \to \varphi \Rightarrow \Delta)$ and $q \neq p$. Let $S' = (\Gamma, \varphi \Rightarrow \Delta)$. Since $S' \prec S$, the assumption that all sequents lower than S satisfy the interpolant properties implies that $(\Gamma, \varphi \Rightarrow \exists p S')$ is derivable. Thus $(\Gamma, q \to \varphi \Rightarrow q \to \exists p S')$ is derivable.

$(\forall\exists)$ Assume that S is derivable and let $R = (S_1 \dots S_n / S)$ be the last inference of some derivation of S. Suppose R is an instance of rule $\mathcal{R}$. Consider an arbitrary p–partition (S^r, S^i) of S. Either S^i is principal for R or it is not. Since all sequents in $\mathfrak{D}^p_R$ are derivable, $\vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing)$ follows from $(\mathrm{DPP})_{\mathcal{R}}$ or $(\mathrm{DPN})_{\mathcal{R}}$. □

Theorem 3.6.1 If a logic L has a reductive calculus for which there exists a sound interpolant assignment, then L has uniform interpolation.

Proof This follows from Lemmas 3.1.1 and 3.6.1. □

3.7 Modularity

Note that when a rule $\mathcal{R}$ in a calculus G satisfies $(\mathrm{IPP})^{\exists}_{\mathcal{R}}$, $(\mathrm{IPP})^{\forall}_{\mathcal{R}}$, $(\mathrm{IPN})^{\exists}_{\mathcal{R}}$, $(\mathrm{IPN})^{\forall}_{\mathcal{R}}$ for an interpolant assignment, then it does so in all extensions of G under the same interpolant assignment. In other words, the four properties are *modular*. This does not hold for the dependent properties, because a sequent not derivable in the original calculus can become derivable in the extension, and therefore has to be treated in a proof of $(\mathrm{DPP})_{\mathcal{R}}$ or $(\mathrm{DPN})_{\mathcal{R}}$. However, for all rules $\mathcal{R}$ treated in this paper, that is the rules of $\mathsf{G4iK}_\Box$ and all centered (modal) rules, $(\mathrm{DPP})_{\mathcal{R}}$ and $(\mathrm{DPN})_{\mathcal{R}}$ are modular too: they hold not only in the main calculus, $\mathsf{G4iK}_\Box$, but in any balanced extension of it.

4 Centered rules

In this section we introduce the class of *one-sided unary thinnable* rules and their *standard* interpolant assignment, which is sound for these rules. Many well–known rules of sequent calculi are of this form.

4.1 Properties of rules

A rule $\mathcal{R}$ that is not an axiom is *thinnable* if it is of the form

$$\frac{S\cdot S_1 \quad S\cdot S_2 \quad \dots \quad S\cdot S_n}{S\cdot S_0} \tag{2}$$

where $S_0, S_1, \dots, S_n$ are meta–sequents such that

- if $S^s = \varnothing$, then $S = (\Gamma \Rightarrow \Delta)$ for two distinct meta–multisets Γ and Δ that do not occur in any of $S_0, S_1, \dots, S_n$,
- if $S^s \neq \varnothing$, then $S = (\Gamma \Rightarrow\ )$ for a meta–multiset Γ that does not occur in any of $S_0, S_1, \dots, S_n$.

A thinnable rule (2) is *unary* if moreover:

- S_0 consists of exactly one meta–formula, which is not an atom, and in the setting of modal logic it is not boxed either,
- any variable in any of $S_1, \dots, S_n$ occurs in S_0.

Note that a thinnable unary rule is either a left rule or a right rule, and not both. A unary thinnable rule (2) is *one-sided* if moreover:

- if $\mathcal{R}$ is a left rule, the succedents of all $S_0, \dots, S_n$ are empty,
- if $\mathcal{R}$ is a right rule, the antecedents of all $S_0, \dots, S_n$ are empty.

A rule is *centered* if either it is a one-sided unary thinnable rule that is not an axiom, or it is an axiom of the form $(\Gamma, r \Rightarrow r)$, $(\Gamma, \bot \Rightarrow \Delta)$ or $(\Gamma \Rightarrow \top)$, with Γ a meta–multiset.[3]
In an instance $R = (S\cdot S_1 \dots S\cdot S_n / S\cdot S_0)$ of $\mathcal{R}$, the *principal* formula of R is the formula in S_0. All other occurrences in R of the formula in S_0 are not principal. Thus although we speak of the principal formula, it is in fact an occurrence of a formula that is principal. In axiom $(\Gamma, r \Rightarrow r)$ both occurrences of r are *principal* and in $(\Gamma, \bot \Rightarrow \Delta)$ and $(\Gamma \Rightarrow \top)$ the indicated occurrence of $\bot$ and $\top$, respectively, are *principal*.

Example 4.1.1 Typical centered rules are the left and right rules of Gentzen calculi. The right conjunction rule

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}\ \mathrm{R}\wedge$$

is clearly centered, as one can take $(\Gamma \Rightarrow\ )$ for S, $(\ \Rightarrow \varphi\wedge\psi)$ for S_0 and $(\ \Rightarrow \varphi)$ and $(\ \Rightarrow \psi)$ for S_1 and S_2, respectively. Note that what we defined to be the principal formula of an instance of R$\wedge$ coincides with what is usually called the principal formula of such an instance. An example of a less standard rule that is centered is the rule

$$\frac{\Gamma \Rightarrow \neg\chi \to \varphi \vee \psi}{\Gamma \Rightarrow (\neg\chi \to \varphi) \vee (\neg\chi \to \psi)}$$

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[3] In an earlier paper on uniform interpolation in classical modal logic (Iemhoff, 2018) unary thinnable rules were called *focussed*. Because that name is already in use in the field of linear logic, we have changed our terminology.

A rule that is not centered is the right implication rule

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}$$

as it is a right rule, but the antecedent of $S_1 = (\varphi \Rightarrow \psi)$ is not empty. This does not mean that this rule blocks uniform interpolation, it just means that it is not covered by the general treatment that we develop for centered rules, and it therefore has to be treated separately. A similar phenomenon occurs for two implication rules in the calculus `G4ip` (Dyckhoff, 1992), which are treated in Section 5.

4.2 Partition of centered rules

Given an instance $R = \big(S \cdot S_1 \dots S \cdot S_n / S \cdot S_0\big)$ of a centered rule $\mathcal{R}$ and a p-partition of R, where each $S \cdot S_j$ is partitioned in $\big((S \cdot S_j)^r, (S \cdot S_j)^i\big)$, then this partition is *standard* if either R^i is equal to

$$\frac{S^i \cdot S_1 \quad \dots \quad S^i \cdot S_n}{S^i \cdot S_0}\ R^i$$

and $(S \cdot S_j)^r = S^r$ for all $j = 0, \dots, n$, or R^r is equal to

$$\frac{S^r \cdot S_1 \quad \dots \quad S^r \cdot S_n}{S^r \cdot S_0}\ R^r$$

and $(S \cdot S_j)^i = S^i$ for all $j = 0, \dots, n$. The following lemma implies that centered rules are modular.

Lemma 4.2.1 For any instance $R = \big(S \cdot S_1 \dots S \cdot S_n / S \cdot S_0\big)$ of a centered rule $\mathcal{R}$ and any p-partition $((S \cdot S_0)^r, (S \cdot S_0)^i)$ of $S \cdot S_0$, there exists exactly one standard p-partition of R with conclusion $((S \cdot S_0)^r, (S \cdot S_0)^i)$ such that either the principal formula belongs to S^i and R^i is an instance of $\mathcal{R}$ or the principal formula belongs to S^r and R^r is an instance of $\mathcal{R}$.

Proof Since there is only one principal formula, the one in S_0, there exists a p-partition (S^r, S^i) of S such that either $(S \cdot S_0)^r = S^r \cdot S_0$ and $(S \cdot S_0)^i = S^i$, or $(S \cdot S_0)^i = S^i \cdot S_0$ and $(S \cdot S_0)^r = S^r$.

Given partition (S^r, S^i), a partition of the premisses of R is defined as follows:

$$\begin{cases} (S \cdot S_j)^i = S^i \cdot S_j & (S \cdot S_j)^r = S^r \qquad \text{if } (S \cdot S_0)^i = S^i \cdot S_0 \\ (S \cdot S_j)^i = S^i & (S \cdot S_j)^r = S^r \cdot S_j \quad \text{otherwise.} \end{cases}$$

Note that the partition is well-defined, standard, and $(S \cdot S_j)^r$ and $(S \cdot S_j)^i$ indeed form a partition of $S \cdot S_j$. That it is a p-partition of the premisses follows from the assumption that all atoms in the S_j must occur in S_0.

As $\mathcal{R}$ is centered, in the first case of the definition of the partition, R^i is an instance of $\mathcal{R}$ and in the second case R^r is, which completes the proof. □

Example 4.2.1 Consider the following instance R of the rule L$\vee$ for disjunction on the left:

$$\frac{S_1 \quad S_2}{S_0} = \frac{\Gamma, \varphi_1 \Rightarrow \Delta \quad \Gamma, \varphi_2 \Rightarrow \Delta}{\Gamma, \varphi_1 \vee \varphi_2 \Rightarrow \Delta}$$

Then for the partition $(S_0^r, S_0^i) = \big((\Gamma \Rightarrow \Delta), (\varphi_1 \vee \varphi_2 \Rightarrow)\big)$ of the conclusion S_0, the following is the standard partition of the rule given this partition.

$$\frac{(\varphi_1 \Rightarrow) \quad (\varphi_1 \Rightarrow)}{(\varphi_1 \vee \varphi_2 \Rightarrow)}\; R^i \qquad \frac{(\Gamma \Rightarrow \Delta) \quad (\Gamma \Rightarrow \Delta)}{(\Gamma \Rightarrow \Delta)}\; R^r$$

If the partition of the conclusion is, for example, $(S_0^r, S_0^i) = \big((\Gamma, \varphi_1 \vee \varphi_2 \Rightarrow), (\Rightarrow \Delta)\big)$, then the standard partition of R with that particular partition of the conclusion is

$$\frac{(\Gamma, \varphi_1 \Rightarrow) \quad (\Gamma, \varphi_1 \Rightarrow)}{(\Gamma, \varphi_1 \vee \varphi_2 \Rightarrow)}\; R^r \qquad \frac{(\Rightarrow \Delta) \quad (\Rightarrow \Delta)}{(\Rightarrow \Delta)}\; R^i$$

4.3 Standard interpolant assignment for centered rules

For a centered rule $\mathcal{R}$, the *standard interpolant assignment* ι is defined as follows. If $\mathcal{R}$ is not an axiom, then for an instance

$$\frac{S_1 \quad S_2 \quad \dots \quad S_n}{S}\; R$$

of $\mathcal{R}$ we define

$$\iota\exists\overset{R}{p}S \equiv_{df} \textstyle\bigvee_{i=1}^n \exists p S_i \quad \iota\forall\overset{R}{p}S \equiv_{df} \textstyle\bigwedge_{i=1}^n (\exists p S_i \to \forall p S_i).$$

If $\mathcal{R}$ is an axiom, and R is an instance of it which consists of sequent S, then

$$\iota\forall\overset{R}{p}S \equiv_{df} \top \quad \iota\exists\overset{R}{p}S \equiv_{df} \bigwedge\{\varphi \in S^a \mid \varphi \text{ does not contain } p\}.$$

For S such that some instance of $\mathcal{R}$ is not backwards applicable to S we define

$$\iota\exists\overset{\overline{\mathcal{R}}}{p}S \equiv_{df} \top \quad \iota\forall\overset{\overline{\mathcal{R}}}{p}S \equiv_{df} \bot.$$

Although in this case the assignments $\exists\overset{\overline{\mathcal{R}}}{p}S$ and $\forall\overset{\overline{\mathcal{R}}}{p}S$ do not depend on $\mathcal{R}$ and are moreover trivial, this will no longer be the case for later rules. In order to provide a uniform approach we chose to define the assignments $\exists\overset{\overline{\mathcal{R}}}{p}S$ and $\forall\overset{\overline{\mathcal{R}}}{p}S$ for every rule $\mathcal{R}$ seperately also in this case.
An interpolation assignment is *standard* if it is standard for all centered rules.

4.4 Soundness of the standard interpolant assignment

In this section we prove that the standard interpolant assignment for centered rules is sound, by proving the six inductive properties (Section 3.5).

Lemma 4.4.1 For any centered rule $\mathcal{R}$ in a reductive calculus with a standard interpolant assignment, $(\mathrm{IPP})^{\exists}_{\mathcal{R}}$ and $(\mathrm{IPN})^{\exists}_{\mathcal{R}}$ hold.

Proof That $(\mathrm{IPN})^{\exists}_{\mathcal{R}}$ holds is clear. We treat with $(\mathrm{IPP})^{\exists}_{\mathcal{R}}$. Consider sequents $S_0, S_1, \dots, S_n$ such that $R = (S_1 \dots S_n / S_0)$ is an instance of $\mathcal{R}$. We have to show that $\mathfrak{I}^p_R \vdash S_0^a \Rightarrow \exists\overset{R}{p}S_0$. The case that R is an axiom is immediate from the definition of interpolant assignments for centered axioms. Therefore assume it is not an axiom. We distinguish the cases that $\mathcal{R}$ is a left rule and a right rule.

If $\mathcal{R}$ is a left rule, there are S'_i and S' such that the succedents of the S'_i are empty and

$$\frac{S_1 \quad \dots \quad S_n}{S_0} = \frac{S' \cdot S'_1 \quad \dots \quad S' \cdot S'_n}{S' \cdot S'_0}$$

and for all S the following is an instance of $\mathcal{R}$:

$$\frac{S \cdot S'_1 \dots S \cdot S'_n}{S \cdot S'_0}$$

This holds in particular for $S = (S'^a \Rightarrow \bigvee_{i=1}^n \exists p S_i)$. Since $\mathfrak{I}^p_R$ derives $(S^a_i \Rightarrow \exists p S_i)$ and $\iota \exists \overset{R}{p} S_0 = \bigvee_{i=1}^n \exists p S_i$, $\mathfrak{I}^p_R$ derives $(S^a_i \Rightarrow \exists \overset{R}{p} S_0) = ((S' \cdot S'_i)^a \Rightarrow \exists \overset{R}{p} S_0) = S \cdot S'_i$, for all $i = 1, \dots, n$. An application of $\mathcal{R}$ shows that $\mathfrak{I}^p_R \vdash S \cdot S'_0$, which implies the desired.

If $\mathcal{R}$ is a right rule, there are S'_i and S' such that the antecedents of the S'_i are empty and

$$\frac{S_1 \quad \dots \quad S_n}{S_0} = \frac{S' \cdot S'_1 \quad \dots \quad S' \cdot S'_n}{S' \cdot S'_0}$$

This implies that all S^a_i are equal. And since $(S^a_i \Rightarrow \exists p S_i)$ belongs to $\mathfrak{I}^p_R$ for all $i = 1, \dots, n$, $\mathfrak{I}^p_R$ derives $(S^a_0 \Rightarrow \exists \overset{R}{p} S_0)$. □

Lemma 4.4.2 For any instance $S_1 \dots S_n / S_0$ of a centered rule and any formulas $\varphi_1, \dots, \varphi_n$:

$$\{S_j \cdot (\varphi_j \Rightarrow) \mid j = 1, \dots, n\} \vdash^{\mathcal{R}}_{\mathsf{IPC}} S_0 \cdot (\bigwedge_{j=1}^{n} \varphi_j \Rightarrow).$$

Proof Clearly, $\{S_1, \dots, S_n\} \vdash^{\mathcal{R}}_{\mathsf{IPC}} S_0$. Let $S = (\bigwedge_{j=1}^n \varphi_j \Rightarrow)$. Since $\mathcal{R}$ is centered, we have $\{S \cdot S_1, \dots, S \cdot S_n\} \vdash^{\mathcal{R}}_{\mathsf{IPC}} S \cdot S_0$. Since $S_j \cdot (\varphi_j \Rightarrow) \vdash^{\mathcal{R}}_{\mathsf{IPC}} S_j \cdot S$, the desired follows. □

Lemma 4.4.3 For any centered rule $\mathcal{R}$ in a reductive calculus with a standard interpolant assignment, $(\mathrm{IPP})^{\forall}_{\mathcal{R}}$ and $(\mathrm{IPN})^{\forall}_{\mathcal{R}}$ hold.

Proof That $(\mathrm{IPN})^{\forall}_{\mathcal{R}}$ holds is clear. For $(\mathrm{IPP})^{\forall}_{\mathcal{R}}$ we reason as follows. Let $R = (S_1 \dots S_n / S_0)$ be an instance of a centered rule $\mathcal{R}$. If $\mathcal{R}$ is an axiom, S_0 is derivable and $(\mathrm{IPP})^{\forall}_{\mathcal{R}}$ clearly holds, as centered axioms are closed under left weakening. If not, $\iota \forall \overset{R}{p} S_0 = \bigwedge_{i=1}^n (\exists p S_j \to \forall p S_j)$. Since for each j,

$$\{S_j \cdot (\forall p S_j \Rightarrow), (S^a_j \Rightarrow \exists p S_j)\} \vdash S_j \cdot (\exists p S_j \to \forall p S_j \Rightarrow),$$

we can use Lemma 4.4.2 to obtain the desired result. □

Lemma 4.4.4 For all formulas $\varphi_1, \dots, \varphi_n$ and any partition (S^r, S^i) of the conclusion of an instance $R = (S_1 \dots S_n / S)$ of a centered rule that is not an axiom and such that S^i is principal for R, for the standard partition of R:

$$\{S^r_j \cdot (\Rightarrow \varphi_j) \mid j = 1, \dots, n\} \vdash^{\mathcal{R}}_{\mathsf{IPC}} S^r \cdot (\Rightarrow \bigwedge_{j=1}^{n} \varphi_j).$$

Proof As S^i contains the principal formula of R, $S^r = S^r_j$, which immediately implies the desired. $\Box$

Lemma 4.4.5 For any centered rule $\mathcal{R}$ in a reductive calculus with a standard interpolant assignment, $(\mathrm{DPP})_{\mathcal{R}}$ holds.

Proof Consider a sequent S for which there exists a derivation of which the last inference is an instance $R = (S_1 \dots S_n/S)$ of $\mathcal{R}$, and let (S^r, S^i) be a partition of S such that S^i is principal for R. We have to show that

$$\mathfrak{D}^p_R \vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing). \tag{3}$$

If $\mathcal{R}$ is an axiom, then since S^i contains the principal formulas of R, S^i is an instance of it, which we denote by R^i. As the axiom is centered, $\iota\forall\overset{R^i}{p}S^i = \top$ and thus $\vdash \forall p S^i$. Therefore (3) clearly holds, at least in case that S^{rs} is empty and S^s is not. If this does not hold, which means if S^{rs} is not empty or S^s is empty, then we have to prove that $\mathfrak{D}^p_R \vdash S^r \cdot (\exists p S^i \Rightarrow)$. Note that in both cases, S^{is} is empty. Hence $\mathcal{R}$ has to be the axiom $\mathrm{L}\bot$. For if it would be one of the other centered axioms, then the fact that S^i is an instance of it implies that S^{is} is not empty. Thus S and S^i are instances of $\mathrm{L}\bot$. Therefore S^{ia} contains $\bot$. Hence $\exists^{at} p S^i$, which is a conjunct of $\exists p S^i$, contains $\bot$ as a conjunct. Therefore (3) holds. This completes the case that $\mathcal{R}$ is an axiom.

If $\mathcal{R}$ is not an axiom, consider the standard p-partition (R^r, R^i) of R with conclusion (S^r, S^i). Since S^i contains the principal formula of R, Lemma 4.2.1 implies that R^i is an instance of $\mathcal{R}$. Let the partition of the premisses S_j be denoted by (S^r_j, S^i_j). The definition of standard partition implies that $S^r_j = S^r$ for all $j = 1, \dots, n$. Therefore $\iota\exists\overset{R^i}{p}S^i = \bigvee_{j=1}^n \exists p S^i_j$ and $\iota\forall\overset{R^i}{p}S = \bigwedge_{j=1}^n (\exists p S^i_j \to \forall p S^i_j)$ by the definition of the standard interpolant assignment for centered rules.

We distinguish the case that $S^{rs} = \varnothing$ and $S^s \neq \varnothing$ from the case that this does not hold. In the first case, $S^{rs}_j = \varnothing$ holds for all premisses S_j because $S^r_j = S^r$, as observed in the previous paragraph. Hence $S^r \cdot (\exists p S^i_j \Rightarrow \forall p S^i_j \mid \varnothing)$ is in $\mathfrak{D}^p_R$. We show $\mathfrak{D}^p_R$ derives $S^r \cdot (\exists p S^i_j \Rightarrow \forall p S^i_j)$. For those j such that $S^s_j \neq \varnothing$, this holds by the definition of $S^r \cdot (\exists p S^i_j \Rightarrow \forall p S^i_j \mid \varnothing)$. And if $S^s_j = \varnothing$, then by definition $S^r \cdot (\exists p S^i_j \Rightarrow \forall p S^i_j \mid \varnothing) = (S^{ra}, \exists p S^i_j \Rightarrow)$, and thus $\mathfrak{D}^p_R$ derives $(S^{ra}, \exists p S^i_j \Rightarrow \chi)$ for any formula χ, in particular for $\chi = \forall p S^i_j$. This proves that also in the case $S^s_j = \varnothing$, $\mathfrak{D}^p_R \vdash S^r \cdot (\exists p S^i_j \Rightarrow \forall p S^i_j)$.

The above shows that $\mathfrak{D}^p_R \vdash (S^{ra} \Rightarrow \exists p S^i_j \to \forall p S^i_j)$ for all j. An application of Lemma 4.4.4 shows that $\mathfrak{D}^p_R \vdash S^r \cdot (\Rightarrow \forall\overset{R^i}{p}S^i)$, which implies $\mathfrak{D}^p_R \vdash S^r \cdot (\exists\overset{R^i}{p}S^i \Rightarrow \forall\overset{R^i}{p}S^i)$. From Remark 3.5.1 we conclude that this implies (3).

We turn to the case that $S^{rs} \neq \varnothing$ or $S^s = \varnothing$, where the former implies $S^{rs}_j \neq \varnothing$ for all $j = 1, \dots, n$, as $S^r = S^r_j$, and the latter implies $S^s_j = \varnothing$, for all $j = 1, \dots, n$, by the definition of centered rules. Using that $S^r \cdot (\exists p S^i_j \Rightarrow)$ belongs to $\mathfrak{D}^p_R$, we conclude that $\mathfrak{D}^p_R$ derives $S^r \cdot (\exists\overset{R^i}{p}S^i \Rightarrow)$. Again, Remark 3.5.1 implies that (3) holds. $\Box$

Lemma 4.4.6 For the standard partition of any instance $R = (S_1 \dots S_n/S_0)$ of any centered rule $\mathcal{R}$ that is not an axiom and such that S^i_0 is not principal for R: $S^i_j = S^i_0$ for all $j = 1, \dots, n$ and for all sequents S,

$$\{S^r_j \cdot S \mid j = 1, \dots, n\} \vdash^{\mathcal{R}}_{\mathsf{IPC}} S^r_0 \cdot S.$$

Proof As S_0^i does not contain the principal formula of R, R^r is an instance of $\mathcal{R}$ and $S_j^i = S_0^i$ for all $j = 1, \dots, n$ by Lemma 4.2.1. As $\mathcal{R}$ is centered, $(S \cdot S_1^r, \dots, S \cdot S_n^r / S \cdot S_0^r)$ is an instance of $\mathcal{R}$, which implies that what we had to show. ☐

Lemma 4.4.7 For any centered rule $\mathcal{R}$ in a reductive calculus with a standard interpolant assignment, $(\mathrm{DPN})_{\mathcal{R}}^+$ holds.[4]

Proof Consider a sequent S_0 for which there exists a derivation of which the last inference is an instance $R = (S_1 \dots S_n / S_0)$ of $\mathcal{R}$ and let (S_0^r, S_0^i) be a partition of S_0 such that S_0^i is not principal for R. We have to show that

$$\mathfrak{D}_R^p \vdash S_0^r \cdot (\exists p S_0^i \Rightarrow \forall p S_0^i \mid \varnothing). \tag{4}$$

First consider the case that $\mathcal{R}$ is an axiom. If the axiom is of the form $(\Gamma, \bot \Rightarrow \Delta)$ or $(\Gamma \Rightarrow \top)$, then the fact that S_0^i is not instance of it implies that S^{ra} contains $\bot$ or S^{rs} consists of $\top$. In both cases (4) holds. Therefore consider the remaining case that the axiom is of the form $(\Gamma, q \Rightarrow q)$. Since S_0^i is not instance of R, $(q \Rightarrow q)$ cannot be a subsequent of S_0^i. If $(q \Rightarrow q)$ is a subsequent of S_0^r, then $S_0^r \cdot (\exists p S_0^i \Rightarrow)$ is derivable, and we are done. If $(q \Rightarrow q)$ is neither a subsequent of S_0^i nor of S_0^r, either $q \in S_0^{ra}$, $S_0^{is} = \{q\}$, and $S_0^{rs} = \varnothing$, or $q \in S_0^{ia}$ and $S_0^{rs} = \{q\}$. Since S_0^r does not contain p we have that $q \neq p$. Hence $\vdash_{\mathsf{IPC}}^{\mathcal{R}} S_0^r \cdot (\exists^{at} p S_0^i \Rightarrow \forall^{at} p S_0^i)$ in the first case and $\vdash_{\mathsf{IPC}}^{\mathcal{R}} S_0^r \cdot (\exists^{at} p S_0^i \Rightarrow)$ in the second. As $\exists^{at} p S_0^i$ is a conjunct of $\exists p S_0^i$ and $\forall^{at} p S_0^i$ is a disjunct of $\forall p S_0^i$, this implies (4).

The case that $\mathcal{R}$ is not an axiom remains. We have to show that $S_0^r \cdot (\exists p S_0^i \Rightarrow \forall p S_0^i \mid \varnothing)$ is derivable from $\mathfrak{D}_R^p$. By Lemma 4.4.6, the fact that S_0^i does not contain the principal formula of R implies that for the standard partition of R: $\{S \cdot S_j^r \mid j = 1, \dots, n\} \vdash_{\mathsf{IPC}}^{\mathcal{R}} S \cdot S_0^r$ for any S and $S_0^i = S_j^i$. Thus $\exists p S_j^i = \exists p S_0^i$ and $\forall p S_j^i = \forall p S_0^i$. Therefore, $S_j^r \cdot (\exists p S_0^i \Rightarrow \forall p S_0^i \mid \varnothing)$ belongs to $\mathfrak{D}_R^p$.

If $S_0^{rs} = \varnothing$ and $S_0^s \neq \varnothing$, we have to show that $\mathfrak{D}_R^p \vdash S_0^r \cdot (\exists p S_0^i \Rightarrow \forall p S_0^i)$. By the observation above for $S = (\exists p S_0^i \Rightarrow \forall p S_0^i)$, it suffices to show that $\mathfrak{D}_R^p \vdash S_j^r \cdot S$ for all j. The assumption on S_0 implies that $S_0^{is} = S_j^{is} \neq \varnothing$ for all j. Thus $S_j^{rs} = \varnothing$. Hence $S_j^r \cdot (\exists p S_0^i \Rightarrow \forall p S_0^i \mid \varnothing) = S_j^r \cdot (\exists p S_0^i \Rightarrow \forall p S_0^i)$, which proves that $\mathfrak{D}_R^p \vdash S_j^r \cdot S$.

If $S_0^{rs} \neq \varnothing$ or $S_0^s = \varnothing$, we have to show that $\mathfrak{D}_R^p \vdash S_0^r \cdot (\exists p S_0^i \Rightarrow)$. By the observation above, for $S = (\exists p S_0^i \Rightarrow)$, it suffices to show that $\mathfrak{D}_R^p \vdash S_j^r \cdot S$ for all j. Since $\mathcal{R}$ is centered, R is of the form

$$\frac{S_1 \quad \dots \quad S_n}{S_0} = \frac{S \cdot S_1' \quad \dots \quad S \cdot S_n'}{S \cdot S_0'}$$

where S_0' consists of one formula and either $S_j'^a = \varnothing$ for all $j = 0, \dots, n$ or $S_j'^s = \varnothing$ for all $j = 0, \dots, n$. Therefore, if $S_0^s = \varnothing$, then $S_j^s = \varnothing$ for all j. Hence $S_j^r \cdot (\exists p S_0^i \Rightarrow \forall p S_0^i \mid \varnothing) = S_j^r \cdot (\exists p S_0^i \Rightarrow)$ belongs to $\mathfrak{D}_R^p$, and we are done. If $S_0^{rs} \neq \varnothing$, then $S_0^{is} = S_j^{is} = \varnothing$. We distinguish the cases that $\mathcal{R}$ is a right and a left rule. If $\mathcal{R}$ is a right rule, none of the $S_j'^s$ is empty. Thus the S_j^{rs} are all not empty, and $S_j^r \cdot (\exists p S_0^i \Rightarrow \forall p S_0^i \mid \varnothing) = S_j^r \cdot (\exists p S_0^i \Rightarrow)$ for all j, which is what we had to show. If, on the other hand, $\mathcal{R}$ is a left rule, then $S_0'^s = \varnothing$, which implies that $S_0^{rs} \subseteq S^s$. Hence $S_j^{rs} \neq \varnothing$ by the definition of the standard partition, and again $S_j^r \cdot (\exists p S_0^i \Rightarrow \forall p S_0^i \mid \varnothing) = S_j^r \cdot (\exists p S_0^i \Rightarrow)$ for all j follows. ☐

Theorem 4.4.1 A logic L with a reductive calculus with a standard interpolant assignment that is sound with respect to all rules that are not centered, has uniform interpolation.

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[4] $(\mathrm{DPN})_{\mathcal{R}}^+$ is the strengthening of $(\mathrm{DPN})_{\mathcal{R}}$ given in Section 3.5.

Proof By Theorem 3.6.1 it suffices to prove that the interpolant assignment is sound. This follows from 4.4.1, 4.4.3, 4.4.5, 4.4.7. $\square$

Corollary 4.4.1 A logic L with a reductive calculus that consists of centered rules only, has uniform interpolation.

Corollary 4.4.2 Any intermediate logic that does not have uniform interpolation cannot have a reductive calculus consisting of centered rules.

We do not know whether there are examples of interesting logics that have calculi that consist solely of centered rules. But in the next section we show that IPC has a calculus consisting of centered and noncentered rules, for which there exists a sound interpolant assignment, and this brings us to the class of logics that we set out to study: the intermediate and intuitionistic modal logics.

5 Intuitionistic logic

As a first application of the method developed thus far we establish that intuitionistic propositional logic has uniform interpolation, a fact first proved by Pitts (1992). We use the same calculus as Pitts does in his article, the calculus $\mathsf{G4ip}$ developed independently by Dyckhoff (1992) and Hudelmaier (1992, 1993) and given in Figure 1. The calculus has no structural rules, but they are admissible in it, as is the cut rule. Recall that sequents are assumed to have at most one formula in the succedent. Thus $|\Delta| \leq 1$ for the Δ in Figure 1. The interpolant assignments for the noncentered rules $\mathrm{R}{\to}, \mathrm{L}_1{\to}, \mathrm{L}_4{\to}$ of $\mathsf{G4ip}$ as defined in the proof of Theorems 5.0.1 are called the *standard* interpolant assignments for these rules.

Theorem 5.0.1 For any extension of the calculus $\mathsf{G4ip}$ there exists for any of the rules of $\mathsf{G4ip}$ a sound interpolant assignment that is standard for centered rules.

Proof As explained in Section 3.7, we have to extend the standard interpolant assignment to the rules in the calculus that are not centered and show that the assignment is sound with respect to the new rules, that is, that any new rule $\mathcal{R}$ satisfies the six inductive properties (Section 3.5).

The three rules in question are $\mathrm{R}{\to}$, $\mathrm{L}_1{\to}$ and $\mathrm{L}_4{\to}$: for $\mathrm{R}{\to}$ ($\mathrm{L}_4{\to}$) the requirement for centered rules that in right (left) rules the antecedents (succedents) of the sequents should be empty is violated, and in $\mathrm{L}_1{\to}$ the requirement that S_0 consists of one formula is violated. For all three rules the assignment $\iota\forall\overset{\overline{\mathcal{R}}}{p}S \equiv_{df} \bot$ is as for centered rules, Section 4.3, and $(\mathrm{IPN})^{\forall}_{\mathcal{R}}$ is easily seen to hold. Assignments of the form $\iota\exists\overset{\overline{\mathcal{R}}}{p}S$, $\iota\forall\overset{R}{p}S$, $\iota\exists\overset{R}{p}S$ are defined as follows, where we first treat $\mathrm{R}{\to}$, then $\mathrm{L}_1{\to}$, and then $\mathrm{L}_4{\to}$.

Suppose $\mathcal{R} = \mathrm{R}{\to}$. For an instance $R = (S_1/S) = (\Gamma, \varphi \Rightarrow \psi)/(\Gamma \Rightarrow \varphi \to \psi)$ of $\mathcal{R}$ define $\iota\exists\overset{\overline{\mathcal{R}}}{p}S \equiv_{df} \top$ as for centered rules and furthermore

$$\begin{array}{lll}
\iota\exists\overset{R}{p}S \equiv_{df} \varphi \to \exists p S_1 & \iota\forall\overset{R}{p}S \equiv_{df} \varphi \to \forall p S_1 & \text{if } p \text{ does not occur in } \varphi \\
\iota\exists\overset{R}{p}S \equiv_{df} \top & \iota\forall\overset{R}{p}S \equiv_{df} \exists p S_1 \to \forall p S_1 & \text{if } p \text{ occurs in } \varphi.
\end{array}$$

Clearly, $(\mathrm{IPN})^{\exists}_{\mathcal{R}}$ holds. We have to prove the remaining four properties.

$$\Gamma, q \Rightarrow q \quad \text{At} \ \ (q \text{ an atom}) \qquad\qquad \Gamma, \bot \Rightarrow \Delta \quad \text{L}\bot$$

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}\ \text{R}\wedge \qquad\qquad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ \text{L}\wedge$$

$$\frac{\Gamma \Rightarrow \varphi_i}{\Gamma \Rightarrow \varphi_0 \vee \varphi_1}\ \text{R}\vee \ (i = 0, 1) \qquad\qquad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ \text{L}\vee$$

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\ \text{R}{\to}$$

$$\frac{\Gamma, q, \psi \Rightarrow \Delta}{\Gamma, q, q \to \psi \Rightarrow \Delta}\ \text{L}_1{\to} \ (q \text{ an atom}) \qquad \frac{\Gamma, \varphi \to (\psi \to \gamma) \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \to \gamma \Rightarrow \Delta}\ \text{L}_2{\to}$$

$$\frac{\Gamma, \varphi \to \gamma, \psi \to \gamma \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \to \gamma \Rightarrow \Delta}\ \text{L}_3{\to} \qquad\qquad \frac{\Gamma, (\psi \to \gamma) \Rightarrow \varphi \to \psi \quad \gamma, \Gamma \Rightarrow \Delta}{\Gamma, (\varphi \to \psi) \to \gamma \Rightarrow \Delta}\ \text{L}_4{\to}$$

Figure 1: The Gentzen calculus G4ip. In all L-rules $|\Delta| \leq 1$. In all rules except $\text{L}_1{\to}$ and the axioms, the formula displayed in the conclusion is the *principal* formula. In $\text{L}_1{\to}$ both formulas q and $q \to \psi$ in the conclusion are *principal*. In axiom At both q's are *principal*, and $\bot$ is *principal* in $\text{L}\bot$.

(IPP)$^{\exists}_{\mathcal{R}}$ We have to show that $\mathfrak{I}^p_R$ derives $\Gamma \Rightarrow \exists\overset{R}{p}S$. The case that p occurs in φ is trivial. If p does not occur in φ, then we use that $\mathfrak{I}^p_R$ contains $(\Gamma, \varphi \Rightarrow \exists p S_1)$, and thus derives $(\Gamma \Rightarrow \exists\overset{R}{p}S)$.

(IPP)$^{\forall}_{\mathcal{R}}$ We have to show that $\mathfrak{I}^p_R$ derives $(\Gamma, \forall p S \Rightarrow \varphi \to \psi)$, for which we use that $(\Gamma, \varphi, \forall p S_1 \Rightarrow \psi)$ belongs to $\mathfrak{I}^p_R$. If p occurs in φ, we use that $\mathfrak{I}^p_R$ contains sequent $(\Gamma, \varphi \Rightarrow \exists p S_1)$, and thus derives $(\Gamma, \varphi, \exists p S_1 \to \forall p S_1 \Rightarrow \psi)$, and $(\Gamma, \exists p S_1 \to \forall p S_1 \Rightarrow \varphi \to \psi)$ as well. If p does not occur in φ, then we use that $\mathfrak{I}^p_R$ derives $(\Gamma, \varphi, \varphi \to \forall p S_1 \Rightarrow \psi)$, and thus also $(\Gamma, \varphi \to \forall p S_1 \Rightarrow \varphi \to \psi)$.

For (DPP)$_{\mathcal{R}}$ and (DPN)$_{\mathcal{R}}$, consider a derivation of S of which the last inference is an instance $R = (S_1/S) = (\Gamma, \varphi \Rightarrow \psi)/(\Gamma \Rightarrow \varphi \to \psi)$ of $\mathcal{R}$, and let (S^r, S^i) be a partition of S.

(DPP)$_{\mathcal{R}}$ Suppose that S^i is principal for R. Choose the p–partition (S^r_1, S^i_1) of S_1 for which $S^r_1 = S^r$. It suffices to show that

$$S^r_1 \cdot (\exists p S^i_1 \Rightarrow \forall p S^i_1 \mid \varnothing) \vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing).$$

Since S^i contains the principal formula of R, S^{is} consists of $\varphi \to \psi$, and thus S^{rs} is empty. Let R^i be instance (S^i_1/S^i) of $\mathcal{R}$. Hence $\exists\overset{R^i}{p}S^i$ and $\forall\overset{R^i}{p}S^i$ are a conjunct and a disjunct of $\exists p S^i$ and $\forall p S^i$, respectively. Thus if suffices to show that

$$S^{ra}_1, \exists p S^i_1 \Rightarrow \forall p S^i_1 \vdash S^{ra}, \exists\overset{R^i}{p}S^i \Rightarrow \forall\overset{R^i}{p}S^i.$$

In case p does not occur in φ, the above clearly holds. In the other case, note that the left side derives $(S^{ra}_1 \Rightarrow \exists p S^i_1 \to \forall p S^i_1)$. Since $S^r_1 = S^r$, it also derives $(S^{ra}, \exists\overset{R^i}{p}S^i \Rightarrow \exists p S^i_1 \to \forall p S^i_1)$, which implies $(S^{ra}, \exists\overset{R^i}{p}S^i \Rightarrow \forall\overset{R^i}{p}S^i)$, which is what had to be shown.

(DPN)$_{\mathcal{R}}$ Suppose that S^i is not principal for R. Assume that all sequents lower than S satisfy the interpolant properties. We have to show that

$$\vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing).$$

Since S^i does not contain the principal formula of R, S^{is} is empty and S^{rs} consists of $\varphi \to \psi$. Hence p cannot occur in φ. Let (S^r_1, S^i_1) be the corresponding partition of S_1 such that $S^i_1 = S^i$. It suffices to show $\vdash S^{ra}, \exists p S^i \Rightarrow \varphi \to \psi$, which follows from $\vdash S^{ra}_1, \exists p S^i_1, \varphi \Rightarrow \psi$, which again follows from the assumption about sequents lower than S and $S^i_1 = S^i$.

We turn to the case $\mathcal{R} = \mathrm{L}_1\!\to$. For $R = (S_1/S) = (\Gamma, q, \psi \Rightarrow \Delta)/(\Gamma, q, q \to \psi \Rightarrow \Delta)$ an instance of $\mathcal{R}$, define

$$\begin{array}{lll}
\iota\forall\overset{R}{p}S \equiv_{df} q \to \forall p S_1 & \iota\exists\overset{R}{p}S \equiv_{df} q \wedge \exists p S_1 & \text{if } q \neq p \\
\iota\forall\overset{R}{p}S \equiv_{df} \forall p S_1 & \iota\exists\overset{R}{p}S \equiv_{df} \exists p S_1 & \text{if } q = p \\
\iota\exists\overset{\overline{\mathcal{R}}}{p}S \equiv_{df} \bigwedge\{q \in S^a \mid q \neq p\}. & &
\end{array}$$

It is clear that (IPN)$^{\exists}_{\mathcal{R}}$ holds. We prove the remaining four inductive properties.

We treat (IPP)$^{\exists}_{\mathcal{R}}$ and leave (IPP)$^{\forall}_{\mathcal{R}}$ to the reader. To show $\mathfrak{I}^p_R \vdash \Gamma, q, q \to \psi \Rightarrow \exists\overset{R}{p}S$, note that $(\Gamma, q, \psi \Rightarrow \exists p S_1)$ belongs to $\mathfrak{I}^p_R$. Hence $\mathfrak{I}^p_R \vdash \Gamma, q, q \to \psi \Rightarrow q \wedge \exists p S_1$, which is what we had to show.

For (DPP)$_{\mathcal{R}}$ and (DPN)$_{\mathcal{R}}$, consider a derivation of S of which the last inference is an instance $R = (S_1/S) = (\Gamma, q, \psi \Rightarrow \Delta)/(\Gamma, q, q \to \psi \Rightarrow \Delta)$ of $\mathcal{R}$, and let (S^r, S^i) be a partition of S.

$(\mathrm{DPP})_{\mathcal{R}}$ Suppose that S^i is principal for R. Thus $S^i = (\Pi, q, q \to \psi \Rightarrow \Sigma)$ for some multisets Π, Σ. Choose the p–partition (S_1^r, S_1^i) of S_1 for which $S_1^r = S^r$. Thus $S_1^i = (\Pi, q, \psi \Rightarrow \Sigma)$. It suffices to show that

$$S_1^r \cdot (\exists p S_1^i \Rightarrow \forall p S_1^i \mid \varnothing) \vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing).$$

Let R^i denote the instance S_1^i / S^i of $\mathcal{R}$ and note that $\exists \overset{R^i}{p} S^i$ and $\forall \overset{R^i}{p} S^i$ are a conjunct and a disjunct of $\exists p S^i$ and $\forall p S^i$, respectively. Thus it suffices to show that

$$S^r \cdot (\exists p S_1^i \Rightarrow \forall p S_1^i \mid \varnothing) \vdash S^r \cdot (\exists \overset{R^i}{p} S^i \Rightarrow \forall \overset{R^i}{p} S^i \mid \varnothing).$$

As R^i is an instance of $\mathrm{L}_1\!\to$, the definition of $\exists \overset{R^i}{p} S^i$ and $\forall \overset{R^i}{p} S^i$ implies the above, both in the case that $q = p$ and that $q \neq p$.

$(\mathrm{DPN})_{\mathcal{R}}$ Suppose that S^i is not principal for R. We have to show that under the assumption that all sequents lower than S satisfy the interpolant properties we have:

$$\vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing).$$

As S^i does not contain all principal formulas of R, at least one of q and $q \to \psi$ belongs to S^{ra}. As we consider a p–partition, $q \neq p$. We distinguish three cases.

If both q and $q \to \psi$ belong to S^{ra}, then $S^i = S_1^i$ and $S^r = (\Gamma_1, q, q \to \psi \Rightarrow \Delta_1)$ and $S_1^r = (\Gamma_1, q, \psi \Rightarrow \Delta_1)$ for certain multisets Γ_1, Δ_1. Clearly,

$$\frac{S_1^r \cdot (\exists p S_1^i \Rightarrow \forall p S_1^i \mid \varnothing)}{S^r \cdot (\exists p S_1^i \Rightarrow \forall p S_1^i \mid \varnothing)}$$

is an application of $\mathcal{R}$. The premiss is derivable because the interpolant properties hold for S_1 by assumption. Hence the conclusion is derivable too. Since $S^i = S_1^i$, this proves the desired.

If $q \to \psi$ belongs to S^{ra} but q does not, then $S^r = (\Gamma_1, q \to \psi \Rightarrow \Delta_1)$ and $S^i = (\Gamma_2, q \Rightarrow \Delta_2)$ for certain multisets $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$. Consider the partition of S_1 given by $S_1^r = (\Gamma_1, \psi \Rightarrow \Delta_1)$ and $S_1^i = S^i$. Let Σ be such that $S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing) = S^r \cdot (\exists p S^i \Rightarrow \Sigma)$. We have to show that $\vdash S^r \cdot (\exists p S^i \Rightarrow \Sigma)$. We have $S_1^r \cdot (\exists p S_1^i \Rightarrow \forall p S_1^i \mid \varnothing) = S_1^r \cdot (\exists p S_1^i \Rightarrow \Sigma)$ because $S_1^{rs} = S^{rs}$ and $S_1^s = S^s$. By assumption, $S_1^r \cdot (\exists p S_1^i \Rightarrow \Sigma)$ is derivable. Therefore sequent $(\Gamma_1, q, \psi \Rightarrow \Delta_1) \cdot (\exists p S_1^i \Rightarrow \Sigma)$ is derivable too. An application of $\mathcal{R}$ proves that $S^r \cdot (\exists p S_1^i, q \Rightarrow \Sigma)$ is derivable. Since $q \in S_1^{ia}$ and $q \neq p$, q is a conjunct of $\exists \overset{\overline{\mathcal{R}}}{p} S_1^i$, which is a conjunct of $\exists p S_1^i$. Thus $S^r \cdot (\exists p S_1^i \Rightarrow \Sigma)$ is derivable. Together with $S_1^i = S^i$, this establishes what we had to show.

If q belongs to S^{ra} but $q \to \psi$ does not, then for certain multisets $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$ we have $S^r = (\Gamma_1, q \Rightarrow \Delta_1)$ and $S^i = (\Gamma_2, q \to \psi \Rightarrow \Delta_2)$. Consider the partition of S_1 given by $S_1^r = S^r$ and $S_1^i = (\Gamma_2, \psi \Rightarrow \Delta_2)$. Let Σ and Σ_1 be such that $S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing) = S^r \cdot (\exists p S^i \Rightarrow \Sigma)$ and $S_1^r \cdot (\exists p S_1^i \Rightarrow \forall p S_1^i \mid \varnothing) = S_1^r \cdot (\exists p S_1^i \Rightarrow \Sigma_1)$, respectively. As $S_1^{rs} = S^{rs}$ and $S_1^s = S^s$, we have $\Sigma = \{\forall p S^i\}$ and $\Sigma_1 = \{\forall p S_1^i\}$, or $\Sigma = \Sigma_1 = \Delta_1$. We have to show that $\vdash S^r \cdot (\exists p S^i \Rightarrow \Sigma)$. By assumption, $S_1^r \cdot (\exists p S_1^i \Rightarrow \Sigma_1)$ is derivable. The definition of $\exists^{at} p S^i$ shows that $\exists p S^i$ implies the formula $q \to \exists p S_1^i$. Together with the fact that $q \in S^{ra}$ and $S^r = S_1^r$, we obtain the derivability of $S^r \cdot (\exists p S^i \Rightarrow \Sigma_1)$. In the case that $\Sigma_1 = \Sigma$, we are done. In the case that $\Sigma = \{\forall p S^i\}$ and $\Sigma_1 = \{\forall p S_1^i\}$, the definition of $\forall^{at} p S^i$ shows that $q \wedge \forall p S_1^i$ implies $\forall p S^i$. As $q \in S^{ra}$, it follows that $S^r \cdot (\exists p S^i \Rightarrow \Sigma)$ is derivable in this case as well.

We turn to the case that $\mathcal{R} = \mathrm{L_4}{\to}$. For an instance

$$\frac{S_1 \quad S_2}{S} = \frac{\Gamma, \psi \to \gamma \Rightarrow \varphi \to \psi \quad \Gamma, \gamma \Rightarrow \Delta}{\Gamma, (\varphi \to \psi) \to \gamma \Rightarrow \Delta}\; R$$

of $\mathcal{R}$ define

$$\begin{aligned} \iota\forall \overset{R}{p} S &\equiv_{df} \bigwedge_{i=1}^{2} (\exists p S_i \to \forall p S_i) \\ \iota\exists \overset{R}{p} S &\equiv_{df} \exists p S_1 \wedge (\forall p S_1 \to \exists p S_2) \end{aligned}$$

$$\iota\exists \overset{\overline{\mathcal{R}}}{p} S \equiv_{df} \begin{cases} \top & \text{if } S^s = \varnothing \\ \bigwedge \{\exists p(\Pi \Rightarrow\,) \mid \Pi \subseteq S^a\} & \text{if } S^s \neq \varnothing. \end{cases}$$

Note that $\iota\exists \overset{\overline{\mathcal{R}}}{p} S$ and $\iota\exists \overset{R}{p} S$ are well–defined since $(\Pi \Rightarrow\,) \prec S$ for all $\Pi \subseteq S^a$ in case $S^s \neq \varnothing$. Since $(\mathrm{IPN})^{\forall}_{\mathcal{R}}$ has been treated at the beginning of the proof, we have to prove the remaining five inductive properties.

$(\mathrm{IPP})^{\exists}_{\mathcal{R}}$ We have to prove that $\mathfrak{I}^p_R$ derives $(S^a, \forall p S_1 \Rightarrow \exists p S_2)$, $(S^a \Rightarrow \exists p S_1)$, and in case $S^s \neq \varnothing$ also $(S^a \Rightarrow \exists p(\Pi \Rightarrow))$ for all $\Pi \subseteq S^a$. The latter follows from the fact that for such Π, $(\Pi \Rightarrow\,) \prec S$ in case $S^s \neq \varnothing$, which implies that $(\Pi \Rightarrow \exists p(\Pi \Rightarrow))$ belongs to $\mathfrak{I}^p_R$, which thus derives $(S^a \Rightarrow \exists p(\Pi \Rightarrow))$. For the first two cases we use the obvious fact that $\vdash \bigwedge S^a \to \bigwedge S^a_1$. That $\mathfrak{I}^p_R$ derives $(S^a \Rightarrow \exists p S_1)$ thus follows from the fact that $(S^a_1 \Rightarrow \exists p S_1)$ belongs to $\mathfrak{I}^p_R$. For the other case, the fact that $\mathfrak{I}^p_R$ contains $(S^a_1, \forall p S_1 \Rightarrow S^s_1)$ implies that it derives $S^a, \forall p S_1 \Rightarrow S^s_1$. Since $S^a, S^s_1 \vdash \bigwedge S^a_2$ and $\mathfrak{I}^p_R$ contains $(S^a_2 \Rightarrow \exists p S_2)$, it follows that $\mathfrak{I}^p_R \vdash (S^a, \forall p S_1 \Rightarrow \exists p S_2)$.

$(\mathrm{IPP})^{\forall}_{\mathcal{R}}$ We have to prove that $\mathfrak{I}^p_R$ derives $(S^a, \{\exists p S_i \to \forall p S_i \mid i = 1, 2\} \Rightarrow S^s)$. As the previous case showed that $\mathfrak{I}^p_R$ derives $(S^a \Rightarrow \exists p S_1)$ and $(S^a, \forall p S_1 \Rightarrow \exists p S_2)$, it suffices to prove that $\mathfrak{I}^p_R \vdash S^a, \forall p S_1, \forall p S_2 \Rightarrow S^s$. Since $\mathfrak{I}^p_R$ contains $(S^a_j, \forall p S_j \Rightarrow S^s_j)$ for $j = 1, 2$ and $\vdash \bigwedge S^a \to \bigwedge S^a_1$, $\mathfrak{I}^p_R$ derives $S^a, \forall p S_1 \Rightarrow S^s_1$. As $S^a, S^s_1 \vdash \bigwedge S^a_2$, $\mathfrak{I}^p_R$ also derives $S^a, \forall p S_1, \forall p S_2 \Rightarrow S^s_2$, that is, $S^a, \forall p S_1, \forall p S_2 \Rightarrow S^s$.

$(\mathrm{IPN})^{\exists}_{\mathcal{R}}$ We have to show that under the assumption that all sequents lower than S satisfy the interpolant properties we have $\vdash S^a \Rightarrow \exists \overset{\overline{\mathcal{R}}}{p} S$. If $S^s \neq \varnothing$, then $(\Pi \Rightarrow\,) \prec S$ for all $\Pi \subseteq S^a$. Therefore $\vdash \Pi \Rightarrow \exists p(\Pi \Rightarrow\,)$ and thus $\vdash S^a \Rightarrow \exists p(\Pi \Rightarrow\,)$.

For $(\mathrm{DPP})_{\mathcal{R}}$ and $(\mathrm{DPN})_{\mathcal{R}}$, consider a derivation of S of which the last inference is an instance $R = (S_1\; S_2/S)$ of $\mathcal{R}$, and let (S^r, S^i) be a partition of S.

$(\mathrm{DPP})_{\mathcal{R}}$ Suppose S^i is principal for R. We have to show that

$$\mathfrak{D}^p_R \vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing). \tag{5}$$

Consider the following partition of R:

$$S^i = (\Gamma^i, (\varphi \to \psi) \to \gamma \Rightarrow \Delta^i) \quad S^i_1 = (\Gamma^i, \psi \to \gamma \Rightarrow \varphi \to \psi) \quad S^i_2 = (\Gamma^i, \gamma \Rightarrow \Delta^i)$$
$$S^r = S^r_2 = (\Gamma^r \Rightarrow \Delta^r) \quad S^r_1 = (\Gamma^r \Rightarrow).$$

As $S^s_1 \neq \varnothing = S^{rs}_1$, $\mathfrak{D}^p_R$ contains $(\Gamma^r, \exists p S^i_1 \Rightarrow \forall p S^i_1)$. Sequent $S^r_2 \cdot (\exists p S^i_2 \Rightarrow \forall p S^i_2 \mid \varnothing)$ belongs to $\mathfrak{D}^p_R$ too. Let R^i be the instance $(S^i_1\; S^i_2/S^i)$ of $\mathcal{R}$.

First we treat the case that $S^s \neq \varnothing$ and $S^{rs} = \varnothing$. Thus we have to show that $\mathfrak{D}^p_R$ derives $S^r \cdot (\exists p S^i \Rightarrow \forall p S^i) = (\Gamma^r, \exists p S^i \Rightarrow \forall p S^i)$. As $S^r = S^r_2$ and $S^{rs} = S^{rs}_2$, sequent $S^r \cdot$

$(\exists p S^i_2 \Rightarrow \forall p S^i_2 \mid \varnothing)$ is equal to $(\Gamma^r, \exists p S^i_2 \Rightarrow \forall p S^i_2)$. Thus both $(\Gamma^r, \exists p S^i_1 \Rightarrow \forall p S^i_1)$ and $(\Gamma^r, \exists p S^i_2 \Rightarrow \forall p S^i_2)$ belong to $\mathfrak{D}^p_R$. Since $\forall \overset{R^i}{p} S^i = (\exists p S^i_1 \to \forall p S^i_1) \wedge (\exists p S^i_2 \to \forall p S^i_2)$ is a disjunct of $\forall p S^i$, we have that $\mathfrak{D}^p_R$ derives $(\Gamma^r \Rightarrow \forall p S^i)$, and therefore $(\Gamma^r, \exists p S^i \Rightarrow \forall p S^i)$.

Second, we treat the case that $S^s = \varnothing$ or $S^{rs} \neq \varnothing$. Thus we have to show that $\mathfrak{D}^p_R$ derives $S^r \cdot (\exists p S^i \Rightarrow \Delta^r) = (\Gamma^r, \exists p S^i \Rightarrow \Delta^r)$. Because $S^r = S^r_2$ and $S^{rs} = S^{rs}_2$, we have $S^r \cdot (\exists p S^i_2 \Rightarrow \forall p S^i_2 \mid \varnothing) = (\Gamma^r, \exists p S^i_2 \Rightarrow \Delta^r)$. Thus both $(\Gamma^r, \exists p S^i_1 \Rightarrow \forall p S^i_1)$ and $(\Gamma^r, \exists p S^i_2 \Rightarrow \Delta^r)$ belong to $\mathfrak{D}^p_R$. Because $\exists p S^i$ has conjunct $\exists \overset{R^i}{p} S^i$, which has conjuncts $\exists p S^i_1$ and $\forall p S^i_1 \to \exists p S^i_2$, we have that $\mathfrak{D}^p_R$ derives $(\Gamma^r, \exists p S^i \Rightarrow \Delta^r)$, which is what we had to show.

$(\mathrm{DPN})_{\mathcal{R}}$ Suppose that S^i is not principal for R. We show that under the assumption that all sequents lower than S satisfy the interpolant properties, $S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing)$ is derivable. Consider the following partition of R, where $S^r = (\Gamma^r, (\varphi \to \psi) \to \gamma \Rightarrow \Delta^r)$ and $S^i = (\Gamma^i \Rightarrow \Delta^i)$:

$$S^r_1 = (\Gamma^r, \psi \to \gamma \Rightarrow \varphi \to \psi) \quad S^r_2 = (\Gamma^r, \gamma \Rightarrow \Delta^r) \quad S^i_1 = (\Gamma^i \Rightarrow\,) \quad S^i_2 = S^i = (\Gamma^i \Rightarrow \Delta^i).$$

First, we treat the case that $S^s \neq \varnothing$ and $S^{rs} = \varnothing$. Thus we have to show that sequent $S^r \cdot (\exists p S^i \Rightarrow \forall p S^i)$, which is equal to $(\Gamma^r, \exists p S^i \Rightarrow \forall p S^i)$, is derivable. As $S^{rs} = S^{rs}_2$ and $S^s = S^s_2$, sequent $S^r \cdot (\exists p S^i_2 \Rightarrow \forall p S^i_2 \mid \varnothing)$ is equal to $(\Gamma^r, \exists p S^i_2 \Rightarrow \forall p S^i_2)$. Since the sequents $(\Gamma^r, \psi \to \gamma, \exists p S^i_1 \Rightarrow \varphi \to \psi)$ and $(\Gamma^r, \gamma, \exists p S^i_2 \Rightarrow \forall p S^i_2)$ are derivable by assumption, $\mathcal{R}$ can be applied to them, showing the derivability of

$$\Gamma^r, (\varphi \to \psi) \to \gamma, \exists p S^i_1, \exists p S^i_2 \Rightarrow \forall p S^i_2.$$

As $S^i_2 = S^i$, it follows that $\Gamma^r, (\varphi \to \psi) \to \gamma, \exists p S^i_1, \exists p S^i \Rightarrow \forall p S^i$ is derivable. Thus it suffices to show that $\exists p S^i$ derives $\exists p S^i_1$. In case Δ^i is empty, $S^i_1 = S^i$, and we are done. In case Δ^i is not empty, S^s is not empty, and thus $\exists p S^i_1 = \exists p (\Gamma^i \Rightarrow\,)$ is a conjunct of $\exists \overset{\mathcal{R}}{p} S^i$, which is a conjunct of $\exists p S^i$. Thus also in this case $\exists p S^i$ derives $\exists p S^i_1$. $\square$

Note that for the above result one cannot use the propositional part of Gentzen's LK or other calculi that contain the Cut Rule, as it is not clear that such calculi are reductive.

Theorems 4.4.1 and 5.0.1 imply the following.

Corollary 5.0.1 (Pitts, 1992) Intuitionistic propositional logic has uniform interpolation.

Since IPC, Sm, LC, GSc, KC, Bd_2, CPC are the only intermediate logics with uniform interpolation, the contraposition of Theorem 5.0.1 and Theorem 4.4.1 imply the following.

Corollary 5.0.2 If an intermediate logic is not equal to one of the seven logics IPC, Sm, LC, GSc, KC, Bd_2, CPC, then it does not have a reductive calculus that is an extension of G4ip by centered rules.

6 Intuitionistic modal logic

In this and the next section we extend the method developed thus far to intuitionistic modal logics by extending the class of rules to which Theorem 4.4.1 applies. To this end we first develop a reductive calculus based on G4ip for the intuitionistic normal modal logic iK without the diamond operator. Let $\mathrm{G4iK}_\Box$ be the calculus G4ip but then for the language of propositional modal logic, extended by the following two rules, where Π *ranges over multisets that do not contain boxed formulas*:

$$\frac{\Gamma \Rightarrow \varphi}{\Pi, \Box\Gamma \Rightarrow \Box\varphi}\ \mathcal{R}_K \qquad \frac{\Gamma \Rightarrow \varphi \quad \Pi, \Box\Gamma, \psi \Rightarrow \Delta}{\Pi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \Delta}\ \mathrm{L}_{\Box}{\to}$$

The *principal formula* in $\mathcal{R}_K$ is $\Box\varphi$ and in $\mathrm{L}\Box\to$ it is $\Box\varphi \to \psi$. Note that $\mathsf{G4iK}_{\Box}$ again is a reductive calculus in the order on sequents defined in Section 2.5. The following are two well–known modal rules.

$$\frac{\Gamma, \varphi \Rightarrow}{\Pi, \Box\Gamma, \Box\varphi \Rightarrow \Delta}\ \mathcal{R}_D \qquad \frac{\boxdot\Gamma \Rightarrow \varphi}{\Pi, \Box\Gamma \Rightarrow \Box\varphi}\ \mathcal{R}_{K4}$$

The calculus $\mathsf{G3iK}_{\Box}$ consists of $\mathsf{G3i}$ (for the language of modal logic) plus the rule $\mathcal{R}_K$, and $\mathsf{G3iKD}_{\Box}$ is $\mathsf{G3iK}_{\Box}$ plus the rules $\mathcal{R}_D$. The calculus $\mathsf{G4iKD}_{\Box}$ is the extension of $\mathsf{G4iK}_{\Box}$ by $\mathcal{R}_D$. Recall that we use the convention that

$$\Box S = (\{\Box\varphi \mid \varphi \in S^a\} \Rightarrow \{\Box\psi \mid \psi \in S^s\}) \quad \boxdot S = (\{\boxdot\varphi \mid \varphi \in S^a\} \Rightarrow \{\boxdot\psi \mid \psi \in S^s\}).$$

This implies that $\Box(\Gamma \Rightarrow\,) = (\Box\Gamma \Rightarrow\,)$ and $\Box(\Rightarrow \Delta) = (\Rightarrow \Box\Delta)$, and similarly for $\boxdot$.

In (Iemhoff, 2017) it is shown that for $\mathsf{X} \in \{\mathsf{K}, \mathsf{KD}\}$, the calculi $\mathsf{G3iX}_{\Box}$ and $\mathsf{G4iX}_{\Box}$ are equivalent. Section 8 discusses the other names under which these logics occur in the literature.

Theorem 6.0.1 (Iemhoff, 2017) For $\mathsf{X} \in \{\mathsf{K}, \mathsf{KD}\}$: $\mathsf{G4iX}_{\Box}$ is a reductive sequent calculus (with respect to the order in Example 2.5.1) in which the cut rule and the structural rules are admissible.

6.1 Interpolant assignment for $\mathrm{L}_{\Box}{\to}$

Before considering other modal rules, we extend the interpolant assignment to the new implication rule $\mathrm{L}_{\Box}{\to}$. For this, we first define the *standard partition* of the rule. Given an instance R of $\mathrm{L}_{\Box}{\to}$

$$\frac{S_1 \qquad S_2}{S} = \frac{\Gamma \Rightarrow \varphi \qquad \Pi, \Box\Gamma, \psi \Rightarrow \Delta}{\Pi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \Delta} \tag{6}$$

and a partition $S = (S^r, S^i)$ of the conclusion, the standard partition is defined as follows.

$$\begin{array}{lll} S_1^i = (\Gamma^i \Rightarrow \varphi) & S_2^i = (\Pi^i, \Box\Gamma^i, \psi \Rightarrow \Delta^i) & \text{if } S^{ia} = (\Pi^i, \Box\Gamma^i, \Box\varphi \to \psi \Rightarrow \Delta^i) \\ S_1^i = (\Gamma^i \Rightarrow\,) & S_2^i = S^i & \text{if } \Box\varphi \to \psi \notin S^{ia} = (\Pi^i, \Box\Gamma^i \Rightarrow \Delta^i). \end{array}$$

Given such a partition, R^i and R^r denote $S_1^i \ S_2^i / S^i$ and $S_1^r \ S_2^r / S^r$, respectively. The following lemma is easy to prove.

Lemma 6.1.1 For any instance (6) of $\mathcal{R} = \mathrm{L}_{\Box}{\to}$ and any partition (S^r, S^i) of S, for the standard partition of R, R^i is an instance of $\mathcal{R}$ if the principal formula of R belongs to S^i, and R^r is an instance of $\mathcal{R}$ otherwise.

For an instance R of (6) and for $\mathcal{R}$ denoting $\mathrm{L}_{\Box}{\to}$, the *standard* interpolant assignment is defined as follows.

$$\begin{array}{lll} \iota\exists\overset{R}{p}S & \equiv_{df} & \Box\exists p S_1 \wedge (\Box\forall p S_1 \to \exists p S_2) \wedge \bigwedge\{\Box\exists p(\Sigma \Rightarrow\,) \mid \Sigma \subseteq S_1^a\} \\ \iota\forall\overset{R}{p}S & \equiv_{df} & \Box\forall p S_1 \wedge \forall p S_2 \\ \iota\forall\overset{\overline{\mathcal{R}}}{p}S & \equiv_{df} & \bot \\ \iota\exists\overset{\overline{\mathcal{R}}}{p}S & \equiv_{df} & \Box\exists p S_1. \end{array}$$

6.2 Soundness of the interpolant assignment for $\mathrm{L}_\Box\!\to$

Lemma 6.2.1 $(\mathrm{IPP})^\exists_\mathcal{R}$ and $(\mathrm{IPN})^\exists_\mathcal{R}$ hold for $\mathcal{R} = \mathrm{L}_\Box\!\to$ in any extension of $\mathsf{G4iK}_\Box$.

Proof For $(\mathrm{IPP})^\exists_\mathcal{R}$, consider an instance R as in (6). For conjuncts of $\exists\overset{R}{p}S$ of the form $\Box\exists p(\Sigma \Rightarrow)$ for some $\Sigma \subseteq S_1^a$, note that $(\Sigma \Rightarrow \exists p(\Sigma \Rightarrow))$ belongs to $\mathfrak{I}^p_R$ by definition, as $(\Box\Sigma \Rightarrow) \subseteq S$. An application of $\mathcal{R}_K$ shows that $\mathfrak{I}^p_R$ derives $(\Box\Sigma \Rightarrow \Box\exists p(\Sigma \Rightarrow))$, and thus also $(S^a \Rightarrow \Box\exists p(\Sigma \Rightarrow))$.

For the conjunct $\Box\exists pS_1$ of $\exists\overset{R}{p}S$, note that $\mathfrak{I}^p_R$ contains $(\Gamma \Rightarrow \exists pS_1)$. An application of $\mathcal{R}_K$ gives $\mathfrak{I}^p_R \vdash (\Pi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \Box\exists pS_1)$. For the conjunct $(\Box\forall pS_1 \to \exists pS_2)$ of $\exists\overset{R}{p}S$, note that $\mathfrak{I}^p_R$ contains $(\Gamma, \forall pS_1 \Rightarrow \varphi)$ and $(\Pi, \Box\Gamma, \psi \Rightarrow \exists pS_2)$. An application of $\mathrm{L}_\Box\!\to$ shows that $\mathfrak{I}^p_R$ derives $(\Pi, \Box\Gamma, \Box\varphi \to \psi, \Box\forall pS_1 \Rightarrow \exists pS_2)$. This implies that $\mathfrak{I}^p_R$ derives $(\Pi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \Box\forall pS_1 \to \exists pS_2)$.

For $(\mathrm{IPN})^\exists_\mathcal{R}$, assume that all sequents lower than S satisfy the interpolant properties. Thus $\vdash \Gamma \Rightarrow \exists pS_1$. The presence of $\mathcal{R}_K$ implies that $\vdash \Pi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \Box\exists pS_1$, and since $\Box\exists pS_1 = \exists\overset{\overline{\mathcal{R}}}{p}S$, we are done. □

Lemma 6.2.2 In any extension of $\mathsf{G4iK}_\Box$, $(\mathrm{IPP})^\forall_\mathcal{R}$ and $(\mathrm{IPN})^\forall_\mathcal{R}$ hold for $\mathcal{R} = \mathrm{L}_\Box\!\to$.

Proof It is easy to see that $(\mathrm{IPN})^\forall_\mathcal{R}$ holds. For $(\mathrm{IPP})^\forall_\mathcal{R}$, consider an instance R as in (6) and note that $(\Gamma, \forall pS_1 \Rightarrow \varphi)$ and $(\Pi, \Box\Gamma, \psi, \forall pS_2 \Rightarrow \Delta)$ belong to $\mathfrak{I}^p_R$. This implies that $(\Pi, \Box\Gamma, \Box\forall pS_1, \psi, \forall pS_2 \Rightarrow \Delta)$ is derivable from $\mathfrak{I}^p_R$. The presence of $\mathrm{L}_\Box\!\to$ and the fact that $\forall pS_2$ is not a boxed formula, shows that $(\Pi, \Box\Gamma, \Box\varphi \to \psi, \Box\forall pS_1, \forall pS_2 \Rightarrow \Delta)$ is derivable from $\mathfrak{I}^p_R$. This implies that $(\mathrm{IPP})^\forall_\mathcal{R}$ holds. □

Lemma 6.2.3 In any extension of $\mathsf{G4iK}_\Box$, $(\mathrm{DPP})_\mathcal{R}$ holds for $\mathcal{R} = \mathrm{L}_\Box\!\to$.

Proof Consider a sequent S with a derivation of which the last inference is an instance R of $\mathcal{R}$ as in (6). Let (S^r, S^i) be a p–partition of S such that S^i is principal for R. Consider the standard partition of R such that $R^i = S_1^i \; S_2^i / S^i$ is an instance of $\mathcal{R}$, which exists by Lemma 6.1.1. We have to prove that $\mathfrak{D}^p_R \vdash S^r \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \varnothing)$. We distinguish the following two cases.

First, assume S^{rs} is empty and S^s is not. We have to show that $\mathfrak{D}^p_R \vdash S^r \cdot (\exists pS^i \Rightarrow \forall pS^i)$. Note that S_2^{rs} is empty and S_2^s is not. Hence $\mathfrak{D}^p_R$ contains sequent $S^r \cdot (\exists pS_2^i \Rightarrow \forall pS_2^i)$. Since S_1^{rs} is empty while S_1^s is not, $\mathfrak{D}^p_R$ contains $S_1^r \cdot (\exists pS_1^i \Rightarrow \forall pS_1^i)$ as well. This implies that $\mathfrak{D}^p_R$ derives sequent $S^r \cdot (\Box\exists pS_1^i, \exists pS_2^i \Rightarrow \Box\forall pS_1^i \wedge \forall pS_2^i)$. As sequent $\exists\overset{R^i}{p}S^i$ derives $\Box\exists pS_1^i \wedge (\Box\forall pS_1^i \to \exists pS_2^i)$ and $\forall\overset{R^i}{p}S^i = \Box\forall pS_1^i \wedge \forall pS_2^i$, it follows that $\mathfrak{D}^p_R$ derives sequent $S^r \cdot (\exists\overset{R^i}{p}S^i \Rightarrow \forall\overset{R^i}{p}S^i)$. Remark 3.5.1 then gives the desired conclusion.

Second, assume that S^{rs} is not empty or S^s is empty. Therefore we have to show that $\mathfrak{D}^p_R \vdash S^r \cdot (\exists pS^i \Rightarrow)$. As in the previous case, $\mathfrak{D}^p_R$ derives $S^r \cdot (\Box\exists pS_1^i \Rightarrow \Box\forall pS_1^i)$. As it contains $S^r \cdot (\exists pS_2^i \Rightarrow)$, it derives $S^r \cdot (\Box\exists pS_1^i, \exists pS_2^i \Rightarrow)$. As $\exists\overset{R^i}{p}S^i$ derives $\Box\exists pS_1^i \wedge (\Box\forall pS_1^i \to \exists pS_2^i)$, sequent $S^r \cdot (\exists\overset{R^i}{p}S^i \Rightarrow)$ is derivable from $\mathfrak{D}^p_R$.

□

Lemma 6.2.4 In any extension of $\mathsf{G4iK}_\Box$, $(\mathrm{DPN})_\mathcal{R}$ holds for $\mathcal{R} = \mathrm{L}_\Box\!\to$.

Proof Consider a sequent S with a derivation of which the last inference is an instance R of $\mathcal{R}$ as in (6). Let (S^r, S^i) be a p–partition of S such that S^i is not principal for R. Thus

$$S^r = (\Pi^r, \Box\Gamma^r, \Box\varphi \to \psi \Rightarrow \Delta^r) \quad S^i = (\Pi^i, \Box\Gamma^i \Rightarrow \Delta^i).$$

Assuming that all sequents below S in the ordering $\prec$ satisfy the interpolant properties, we have to show that $\vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing)$. Consider the standard partition of the rule, which in this case means that

$$S_1^i = (\Gamma^i \Rightarrow\ ) \quad S_2^i = S^i \quad S_1^r = (\Gamma^r \Rightarrow \varphi) \quad S_2^r = (\Pi^r, \Box\Gamma^r, \psi \Rightarrow \Delta^r).$$

We distinguish the case that $S^{rs} = \varnothing$ and $S^s \neq \varnothing$ both hold and that they do not both hold.
In the first case, $\Delta^r = \varnothing$ and $\Delta \neq \varnothing$, we have by assumption that

$$\vdash \Gamma^r, \exists p S_1^i \Rightarrow \varphi \qquad \vdash \Pi^r, \Box\Gamma^r, \psi, \exists p S_2^i \Rightarrow \forall p S_2^i.$$

$\mathcal{R}$ can be applied, and the fact that $S^i = S_2^i$ shows that

$$\vdash \Pi^r, \Box\Gamma^r, \Box\varphi \to \psi, \Box\exists p S_1^i, \exists p S^i \Rightarrow \forall p S^i.$$

Since $\Box\exists p S_1^i = \exists \overset{\overline{\mathcal{R}}}{p} S^i$ is a conjunct of $\exists p S^i$, we have reached the desired conclusion. Case $\Delta^r \neq \varnothing$ and case $\Delta = \varnothing$ are analogous. $\Box$

7 Centered modal rules

In this section centered modal rules and their standard interpolant assignment are introduced, and it is shown that this interpolant assignment is sound for these rules.

7.1 Definition of centered modal rules

A rule $\mathcal{R}$ is a *centered modal rule* if it is of the following form:

$$\frac{S_u}{S_l} = \frac{\bigcirc S_1 \cdot S_0}{S_2 \cdot \Box S_1 \cdot \Box S_0} \tag{7}$$

where S_0, S_1, S_2 are meta-sequents and

- S_0 consists of exactly one meta–formula,
- $\bigcirc S_1$ denotes either S_1 or $\boxdot S_1$,
- S_1 is of the form $(\Gamma \Rightarrow\ )$ or empty, for a distinct meta–multiset Γ that does not occur in S_0 and S_2,
- S_2 is of the form $(\Pi \Rightarrow \Delta)$ in case S_0^s is empty and of the form $(\Pi \Rightarrow\ )$ otherwise, where Π and Δ range over distinct meta–multisets that *do not contain boxed formulas* and that do not occur in S_0 or S_1.

A centered modal rule is a *centered* $\Box$*–rule* in case $\bigcirc S_1 = S_1$, and a *centered* $\boxdot$*–rule* in case $\bigcirc S_1 = \boxdot S_1$. Thus $\mathcal{R}_K$ and $\mathcal{R}_D$ below are centered $\Box$–rules, and $\mathcal{R}_{K4}$ is a centered $\boxdot$–rule. Note that if S_1 is the empty sequent, the rule is both a $\Box$–rule and a $\boxdot$–rule. The requirement that sequents have at most one formula on the right implies that the multiset $\Box S_0^s \cup \Box S_1^s \cup S_2^s$ consists of at most one formula.

Given an instance of a centered modal rule as in (7), the *lower sequent* is $S_2 \cdot \Box S_1 \cdot \Box S_0$ and denoted by S_l, the *upper sequent* S_u is the premiss $\bigcirc S_1 \cdot S_0$ of the rule. The formula in $\Box S_0$ is the *principal formula* of the instance.

A centered modal rule (7) is an r–*rule* if $S_0^s \neq \varnothing$, and an l–*rule* otherwise. It is an R-*rule* if S_1^s is not empty, and an L-*rule* if S_1^a is not empty. An Lr–*rule* is a rule that is both an L–rule and an r–rule, and an $\mathsf{l}\Box$–rule is a $\Box$–rule that is an l–rule, and likewise for all other combinations. As we consider only sequent calculi with at most one formula in the succedent, Rr–rules do not occur.

The rules $\mathcal{R}_K, \mathcal{R}_D, \mathcal{R}_{K4}$ that were defined at the beginning of Section 6 are a centered modal $\mathsf{Lr}\Box$-rule, $\mathsf{Ll}\Box$-rule, and $\mathsf{Lr}\boxdot$-rule, respectively. The following is an example of an $\mathsf{l}\Box$-rule.

$$\frac{\neg\neg\varphi \Rightarrow}{\Pi, \Box\neg\neg\varphi \Rightarrow \Delta}$$

In the following we mainly consider extensions of $\mathsf{G4iK}_\Box$ that are balanced, where a calculus is *balanced* if

- it is reductive;
- it does not contain l–rules that are not L–rules;
- contains $\mathcal{R}_{K4}$ whenever it contains some $\boxdot$–rule.

7.2 Standard interpolant assignment for centered modal rules

Lemma 7.2.1 For any instance $R = (S_u/S_l) = \big(\bigcirc S_1 \cdot S_0 / S_2 \cdot \Box S_1 \cdot \Box S_0\big)$ of a centered modal rule $\mathcal{R}$ and any p-partition (S_l^r, S_l^i) of S_l, there is a *standard partition* of R and p-partition (S_u^i, S_u^r) of S_u such that either S^i contains the principal formula, R^i is equal to

$$\frac{\bigcirc S_1^i \cdot S_0}{S_2^i \cdot \Box S_1^i \cdot \Box S_0}\ R^i$$

and $S_l^r = S_2^r \cdot \Box S_1^r$ and $S_u^r = \bigcirc S_1^r$, or S^r contains the principal formula, R^r is equal to

$$\frac{\bigcirc S_1^r \cdot S_0}{S_2^r \cdot \Box S_1^r \cdot \Box S_0}\ R^r$$

and $S_l^i = S_2^i \cdot \Box S_1^i$ and $S_u^i = \bigcirc S_1^i$. In the first case R^i is an instance of $\mathcal{R}$ and in the second case R^r is.

Proof Since S_0 contains exactly one formula, for some partition (S_u^r, S_u^i) of S_u, either $S_l^r = S_2^r \cdot \Box S_1^r \cdot \Box S_0$ and $S_l^i = S_2^i \cdot \Box S_1^i$, or vice versa ($i$ and r interchanged). We leave it to the reader to check that these partitions indeed satisfy the lemma. $\Box$

The *standard interpolant assignment* for an instance R of a centered modal rule $\mathcal{R}$ as in (7) is defined as follows.

$$\begin{array}{lll}
\iota\forall\overset{R}{p}S_l & \equiv_{df} & \Box\forall p S_u \\
\iota\exists\overset{R}{p}S_l & \equiv_{df} & \begin{cases} \Box\exists p S_u & \text{if } S_u^s \neq \varnothing \\ \Box\exists p S_u \wedge \Box\neg\forall p S_u & \text{if } S_u^s = \varnothing. \end{cases} \\
\iota\forall\overset{\overline{\mathcal{R}}}{p}S & \equiv_{df} & \bot \\
\iota\exists\overset{\overline{\mathcal{R}}}{p}S & \equiv_{df} & \Box\exists p S_u.
\end{array}$$

7.3 Soundness of the standard interpolant assignment

Lemma 7.3.1 For any centered modal rule $\mathcal{R}$ in any balanced extension of $\mathsf{G4iK}_\Box$, $(\mathrm{IPP})^{\exists}_{\mathcal{R}}$ and $(\mathrm{IPN})^{\exists}_{\mathcal{R}}$ hold.

Proof For $(\mathrm{IPP})^{\exists}_{\mathcal{R}}$ we have to show that $\mathfrak{I}^p_R \vdash S^a \Rightarrow \Box\exists p S_u$, and $\mathfrak{I}^p_R(S) \vdash S^a \Rightarrow \Box\neg\forall p S_u$ in case $S^s_u = \varnothing$. For the first part, it suffices to show that for any instance S_u/S of $\mathcal{R}$ and any formula φ:

$$S^a_u \Rightarrow \varphi \vdash_{\mathsf{L}} S^a \Rightarrow \Box\varphi,$$

where we will be interested in the case that $\varphi = \exists p S_u$. In case $\mathcal{R}$ is a $\Box$–rule, we apply $\mathcal{R}_K$ to the sequent $(S^a_u \Rightarrow \varphi)$ and obtain $(S^a \Rightarrow \Box\varphi)$. In case $\mathcal{R}$ is a $\boxdot$-rule we use $\mathcal{R}_{\mathsf{K4}}$ instead. This proves $\mathfrak{I}^p_R \vdash S^a \Rightarrow \Box\exists p S_u$. To prove that also $\mathfrak{I}^p_R \vdash S^a \Rightarrow \Box\neg\forall p S_u$ in case $S^s_u = \varnothing$, note that $S^a_u, \forall p S_u \Rightarrow S^s_u$ belongs to $\mathfrak{I}^p_R$, and since $S^s_u = \varnothing$, $\mathfrak{I}^p_R$ derives $S^a_u \Rightarrow \neg\forall p S_u$. An application of $\mathcal{R}_K$ (or $\mathcal{R}_{\mathsf{K4}}$ if $\mathcal{R}$ is an $\boxdot$-rule) proves that $\mathfrak{I}^p_R(S) \vdash S^a \Rightarrow \Box\neg\forall p S_u$.

For $(\mathrm{IPN})^{\exists}_{\mathcal{R}}$, assume that all sequents lower than S satisfy the interpolant properties. We have to show that $\vdash S^a \Rightarrow \Box\exists p S_u$. Since $(S^a_u \Rightarrow \exists p S_u)$ is derivable by the assumption on all sequents lower than S, an application of $\mathcal{R}_K$ (or $\mathcal{R}_{K4}$ if $\mathcal{R}$ is an $\boxdot$-rule) proves that $\vdash S^a \Rightarrow \Box\exists p S_u$. □

Lemma 7.3.2 For any centered modal rule $\mathcal{R}$ in any balanced extension of $\mathsf{G4iK}_\Box$, $(\mathrm{IPP})^{\forall}_{\mathcal{R}}$ and $(\mathrm{IPN})^{\forall}_{\mathcal{R}}$ hold.

Proof Property $(\mathrm{IPN})^{\forall}_{\mathcal{R}}$ follows immediately from the fact that $\iota\overset{\overline{\mathcal{R}}}{\forall p}S = \bot$. For $(\mathrm{IPP})^{\forall}_{\mathcal{R}}$, consider a sequent S that is the conclusion of an instance $R = (S_u/S)$ of $\mathcal{R}$. This implies that $S_u \cdot (\forall p S_u \Rightarrow)$ belongs to $\mathfrak{I}^p_R$. It suffices to show that $S_u \cdot (\forall p S_u \Rightarrow) \vdash S \cdot (\Box\forall p S_u \Rightarrow)$ as $\overset{R}{\forall p}S = \Box\forall p S_u$. In case $\mathcal{R}$ is a right $\Box$-rule, $\mathcal{R}_K$ can be applied to $S_u \cdot (\forall p S_u \Rightarrow)$ to obtain $S \cdot (\Box\forall p S_u \Rightarrow)$. In case $\mathcal{R}$ is a right $\boxdot$-rule, $\mathcal{R}_{K4}$ can be used. If $\mathcal{R}$ is a left rule, it has to be an Ll-rule because the calculus is balanced. Thus it can be applied to $S_u \cdot (\forall p S_u \Rightarrow)$ to obtain $S \cdot (\Box\forall p S_u \Rightarrow)$. □

Lemma 7.3.3 For any centered modal rule $\mathcal{R}$ in any balanced extension of $\mathsf{G4iK}_\Box$, $(\mathrm{DPP})_{\mathcal{R}}$ holds.

Proof Consider a sequent S with a derivation of which the last inference is an instance $R = (S_u/S)$ of $\mathcal{R}$ as in (6). Let (S^r, S^i) be a p–partition of S such that S^i is principal for R.

Consider the standard partition of S_u/S given the partition of S, and let (S^r_u, S^i_u) denote the partition of S_u. We have to show that $\mathfrak{D}^p_R$ derives $S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing)$. We treat the case that $\mathcal{R}$ is a $\Box$-rule, the proof for a $\boxdot$-rule is analogous.

If $\mathcal{R}$ is a left rule, it is a Ll-rule and there are $\Pi^r, \Pi^i, \Gamma^r, \Gamma^i$ such that

$$S^r = (\Pi^r, \Box\Gamma^r \Rightarrow \Delta^r) \quad S^i = (\Pi^i, \Box\Gamma^i, \Box\varphi \Rightarrow \Delta^i) \quad S^r_u = (\Gamma^r \Rightarrow) \quad S^i_u = (\Gamma^i, \varphi \Rightarrow).$$

We distinguish the case that $\Delta^r = \varnothing$ and $\Delta \neq \varnothing$ from the case that this does not hold. In the first case, we have to show that $\mathfrak{D}^p_R$ derives $(\Pi^r, \Box\Gamma^r, \exists p S^i \Rightarrow \forall p S^i)$. Since $S^s_u = \varnothing$, $\mathfrak{D}^p_R$ contains $(\Gamma^r, \exists p S^i_u \Rightarrow)$, and thus derives $(\Gamma^r, \exists p S^i_u \Rightarrow \forall p S^i_u)$. An application of $\mathcal{R}_K$ proves that $\mathfrak{D}^p_R$ derives $(\Pi^r, \Box\Gamma^r, \Box\exists p S^i_u \Rightarrow \Box\forall p S^i_u)$. By Lemma 7.2.1, R^i is an instance of $\mathcal{R}$, and thus $\Box\exists pS^i_u$ is a conjunct of $\exists\overset{R^i}{p}S^i$, which is a conjunct of $\exists pS^i$, and $\Box\forall pS^i_u = \forall\overset{R^i}{p}S^i$ is a disjunct of $\forall pS^i$. This proves the desired.

In case $\Delta^r \neq \varnothing$ or $\Delta = \varnothing$, we have to show that $\mathfrak{D}^p_R$ derives $(\Pi^r, \Box\Gamma^r, \exists pS^i \Rightarrow \Delta^r)$. Again we use that $\mathfrak{D}^p_R$ contains $(\Gamma^r, \exists pS^i_u \Rightarrow)$, and conclude from this that $\mathfrak{D}^p_R$ derives $(\Pi^r, \Box\Gamma^r, \Box\exists pS^i_u \Rightarrow \Delta^r)$ by an application of $\mathcal{R}_K$. Then the same reasoning as in the case that Δ^r is empty can be applied to obtain $\mathfrak{D}^p_R \vdash (\Pi^r, \Box\Gamma^r, \exists pS^i \Rightarrow \Delta^r)$.

If $\mathcal{R}$ is a right rule, there are $\Pi^r, \Pi^i, \Gamma^r, \Gamma^i$ such that

$$S^r = (\Pi^r, \Box\Gamma^r \Rightarrow) \quad S^i = (\Pi^i, \Box\Gamma^i \Rightarrow \Box\varphi) \quad S^r_u = (\Gamma^r \Rightarrow) \quad S^i_u = (\Gamma^i \Rightarrow \varphi),$$

and we have to show that $\vdash \Pi^r, \Box\Gamma^r, \exists pS^i \Rightarrow \forall pS^i$. Since $(\Gamma^r, \exists pS^i_u \Rightarrow \forall pS^i_u)$ belongs to $\mathfrak{D}^p_R$, an application of $\mathcal{R}_K$ shows that $\mathfrak{D}^p_R$ derives $(\Pi^r, \Box\Gamma^r, \Box\exists pS^i_u \Rightarrow \Box\forall pS^i_u)$. By Lemma 7.2.1, R^i is an instance of $\mathcal{R}$, and thus $\Box\exists pS^i_u$ is a conjunct of $\exists\overset{R^i}{p}S^i$, which is a conjunct of $\exists pS^i$, and $\Box\forall pS^i_u = \forall\overset{R^i}{p}S^i$ is a disjunct of $\forall pS^i$, which implies what we had to show. □

Lemma 7.3.4 For any centered modal rule $\mathcal{R}$ in any balanced extension of G4iK$_\Box$, $(\mathrm{DPN})_{\mathcal{R}}$ holds.

Proof Assume that S has a derivation of which the last inference is an instance

$$\frac{S_u}{S} = \frac{\bigcirc S_1 \cdot S_0}{S_2 \cdot \Box S_1 \cdot \Box S_0}\ R$$

of a centered modal rule $\mathcal{R}$. Assume that all sequents lower than S satisfy the interpolant properties. Let (S^r, S^i) be a p–partition of S such that S^i is not principal for R and consider the standard partition of S_u/S given the partition of S, where (S^r_u, S^i_u) denotes the partition of the upper sequent S_u. We have to prove the derivability of $S^r \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \varnothing)$. If S^i is empty, then $S^r = S$, and thus S^r is derivable, and so is any weakening $S^r \cdot S'$. Therefore assume S^i is not empty.

If S^i_u is empty, then since $S^r_u \cdot (\exists pS^i_u \Rightarrow \forall pS^i_u \mid \varnothing)$ is derivable by assumption and $\exists pS^i_u$ is equivalent to $\top$ and $\forall pS^i_u$ to $\bot$, S^r_u is derivable. An application of $\mathcal{R}$ shows that so is S^r, and thus also $S^r \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \varnothing)$.

If S^i_u is not empty, we distinguish the cases that $\mathcal{R}$ is and is not an r-rule. First suppose it is an r-rule. We treat the case that it is a $\Box$-rule, the case that it is a $\boxdot$-rule is similar. Since S^i does not contain the principal formula of R, there are $\Pi^r, \Pi^i, \Gamma^r, \Gamma^i$ such that

$$S^r = (\Pi^r, \Box\Gamma^r \Rightarrow \Box\varphi) \quad S^i = (\Pi^i, \Box\Gamma^i \Rightarrow) \quad S^r_u = (\Gamma^r \Rightarrow \varphi) \quad S^i_u = (\Gamma^i \Rightarrow).$$

Thus we have to show that $\vdash \Pi^r, \Box\Gamma^r, \exists pS^i \Rightarrow \Box\varphi$. By assumption we have $\vdash \Gamma^r, \exists pS^i_u \Rightarrow \varphi$, and an application of $\mathcal{R}_K$ gives $\vdash \Pi^r, \Box\Gamma^r, \Box\exists pS^i_u \Rightarrow \Box\varphi$. Since $\Box\exists pS^i_u$ is a conjunct of $\exists\overset{\mathcal{R}}{p}S^i$, which is a conjunct of $\exists pS^i$, this implies what we had to show.

Suppose that $\mathcal{R}$ is an l-rule, and thus an Ll-rule. Since S^i does not contain the principal formula of R, there are $\Pi^r, \Pi^i, \Gamma^r, \Gamma^i$ such that

$$S^r = (\Pi^r, \Box\Gamma^r, \Box\varphi \Rightarrow \Delta^r) \quad S^i = (\Pi^i, \Box\Gamma^i \Rightarrow \Delta^i) \quad S^r_u = (\Gamma^r, \varphi \Rightarrow) \quad S^i_u = (\Gamma^i \Rightarrow).$$

Since $S^s_u = \varnothing$, $\vdash (\Gamma^r, \varphi, \exists pS^i_u \Rightarrow)$ holds by assumption on all sequents lower than S. Thus an application of $\mathcal{R}$ proves that $(\Pi^r, \Box\Gamma^r, \Box\varphi, \Box\exists pS^i_u \Rightarrow \Sigma)$ is derivable for any multiset Σ with $|\Sigma| \leq 1$. In particular, $S^r \cdot (\Box\exists pS^i_u \Rightarrow)$ is derivable in case $\Delta^r \neq \varnothing$ or $\Delta = \varnothing$, and $S^r \cdot (\Box\exists pS^i_u \Rightarrow \forall pS^i)$ is derivable otherwise. As $\Box\exists pS^i_u$ is a conjunct of $\exists\overset{\mathcal{R}}{p}S^i$, which is a conjunct of $\exists pS^i$, this proves $\vdash S^r \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \varnothing)$. □

8 Main theorems

Theorem 8.0.1 Any intuitionistic modal logic L with a balanced calculus that contains G4iK$_\Box$ and has an interpolant assignment that is sound with respect to all rules that neither are centered, nor modal centered, nor belong to G4iK$_\Box$, has uniform interpolation.

Proof Consider the interpolant assignment that is standard for all centered (modal) rules as well as the rules R$\to$, L$_1$$\to$, L$_4$$\to$, L$_\Box$$\to$, and for all other rules is as given in the theorem. By Theorem 3.6.1 it suffices to prove that this interpolant assignment is sound. This follows from Theorem 5.0.1, Lemmas 6.2.1–6.2.4 and Lemmas 7.3.1–7.3.4. $\Box$

Corollary 8.0.1 Any intuitionistic modal logic L with a balanced calculus that consists of G4iK$_\Box$, centered rules, and modal centered rules, has uniform interpolation.

Recall that for intermediate logics, besides reproving Pitts' theorem, the following has been obtained.

Corollary 5.0.2 If an intermediate logic is not equal to one of the seven logics IPC, Sm, LC, GSc, KC, Bd$_2$, CPC, then it does not have a reductive calculus that is an extension of G4ip by centered rules.

For intuitionistic modal logics, Theorem 8.0.1 has the following similar corollary.

Corollary 8.0.2 No intuitionistic modal logic L that does not have uniform interpolation has a balanced calculus that is an extension of G4iK$_\Box$ by centered and centered modal rules.

It is not hard to see that calculi G4iK$_\Box$ and G4iKD$_\Box$ are balanced with respect to the order defined in Section 2.5. As proved in (Iemhoff, 2017), they are equivalent to G3iK$_\Box$ and G3iKD$_\Box$, respectively. Therefore Theorem 8.0.1 implies the following.

Theorem 8.0.2 The logics L$_{\mathsf{G3iK}_\Box}$ and L$_{\mathsf{G3iKD}_\Box}$ have uniform interpolation.

Logic L$_{\mathsf{G3iK}_\Box}$ is equivalent to logic iK from (Simpson, 1994), where it is defined via the Hilbert system that has as rules Necessitation and Modus Ponens and as axioms all substitution instances of formulas valid in IPC plus the axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$. As mentioned in the introduction, L$_{\mathsf{G3iK}_\Box}$ and L$_{\mathsf{G3iKD}_\Box}$ are called **HK**$_\Box$ and **HD**$_\Box$ in (Božić and Došen, 1984; Došen, 1985), and K$^{\mathsf{i}}$ and NV$^{\mathsf{i}}$ in (Litak, 2014), where L$_{\mathsf{G3iK4}_\Box}$ is called K4$^{\mathsf{i}}$. In (Wolter and Zakharyaschev, 1999), L$_{\mathsf{G3iK}_\Box}$ and L$_{\mathsf{G3iK4}_\Box}$ are denoted by **IntK**$_\Box$ and **IntK4**$_\Box$.

For a proper treatment of transitive logics, logics that contain $\mathcal{R}_{K4}$, the rule $L\Box\to$ has to be replaced by a corresponding implication rule:

$$\frac{\Gamma \Rightarrow \varphi \quad \Pi, \boxdot\Gamma, \psi \Rightarrow \Delta}{\Pi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \Delta}$$

Let G4iK4$_\Box$ be G4iK$_\Box$ in which $\mathcal{R}_K$ is replaced by $\mathcal{R}_{K4}$ and $L\Box\to$ by the rule above. In (Iemhoff, 2017) it is shown that the calculi G3iK4$_\Box$ (G3iK$_\Box$ plus $\mathcal{R}_{K4}$) and G4iK4$_\Box$ are equivalent.

Since $\mathcal{R}_{K4}$ is a centered modal rule and the new implication rule can be given a sound interpolant assignment, all the observations of Section 6 apply. However, it is yet unclear whether there is an ordering on sequents under which the calculus G4iK4$_\Box$ is reductive, it certainly is not so under the ordering defined in Section 2.5. Therefore, Corollary 8.0.1 does not apply and we do not know whether $\mathsf{L}_{\mathsf{G3iK4}_\Box}$ has uniform interpolation or not. Since Bílková (2007) and Ghilardi and Zawadowski (1995) have shown that $\mathsf{K4}$ and $\mathsf{S4}$ do not have uniform interpolation, one wonders whether the same holds for $\mathsf{L}_{\mathsf{G3iK4}_\Box}$, which would imply that the calculus $\mathsf{G4iK4}_\Box$ cannot be reductive.

9 Conclusion

We have presented a uniform modular method to prove that certain intermediate and intuitionistic modal logics (without the diamond operator $\Diamond$) have uniform interpolation. Using this method, we have proved that the intuitionistic versions of K and KD have uniform interpolation. The modularity of the method guarantees that when $\mathsf{G4iK}_\Box$ is extended by new rules, then in order to establish that uniform interpolation is preserved in the extension (in the case that the extension indeed has that property), only the new rules have to be proven sound for some interpolant assignment that is standard for the rules of $\mathsf{G4iK}_\Box$.

The contraposition of these results lead to the main theorem of the paper, namely that for any intermediate or intuitionistic modal logic that does not have uniform interpolation, it follows that it cannot have a reductive calculus that is an extension of $\mathsf{G4ip}$ by centered rules or a balanced extension of $\mathsf{G4iK}_\Box$ by centered (modal) rules, respectively. Applying this to the infinitely many intermediate logics without uniform interpolation, shows that extensions of $\mathsf{G4ip}$ by centered rules are excluded as proof systems for these logics.

9.1 Future work

There are many directions for future research. Something that should be explored is the extension of the framework to other intuitionistic modal logics, such as iGL, and to modal logics that contain the diamond operator. Another natural continuation of the work presented here would be the extension of the method to other classes of logics, such as the substructural logics, where one could try to develop a method similar to the one in this paper to prove and generalize the results in Alizadeh et al. (2014). It would also be useful to extend our method to hypersequent calculi, as there are logics with nice hypersequent calculi that have uniform interpolation, for example KC. Moreover, not having uniform interpolation would, for a given logic, exclude not only certain sequent calculi but even certain hypersequent calculi as sound and complete proof systems for the logic.

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