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Uniform interpolation and sequent calculi in modal logic

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Abstract A method is presented that connects the existence of uniform interpolants to the existence of certain sequent calculi. This method is applied to several modal logics and is shown to cover known results from the literature, such as the existence of uniform interpolants for the modal logic K. New is the result that KD has uniform interpolation. The results imply that for modal logics K4 and S4, which are known not to have uniform interpolation, certain sequent calculi cannot exist.

Keywords uniform interpolation, sequent calculus, modal logic, propositional quantifiers

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1 Introduction

Interpolation has been studied in a variety of settings since William Craig proved in 1957 that classical predicate logic has interpolation. Interpolation is a desirable property for a logic to have because it indicates that the logic is well-behaved in a sense reminiscent to analyticity: if an implication $\varphi \to \psi$ holds in the logic, then there is a χ in the common language of φ and ψ that *interpolates*, that is, such that $\varphi \to \chi$ and $\chi \to \psi$ hold. What the common language is depends on the logic one considers. In propositional logics it typically means that all atoms in χ occur in both φ and ψ.

In 1992 it was proved by Andrew Pitts that intuitionistic propositional logic IPC, which has interpolation, also satisfies the stronger property of *uniform interpolation*: given a formula φ and an atom p, there exist *uniform interpolants*

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$\forall p\varphi$ and $\exists p\varphi$ which are formulas (in the language of IPC) that do not contain p and such that for all ψ not containing p:

$$\vdash \varphi \to \psi \;\Leftrightarrow \vdash \exists p\varphi \to \psi \qquad \vdash \psi \to \varphi \;\Leftrightarrow \vdash \psi \to \forall p\varphi.$$

This is a strengthening of interpolation in which the interpolant only depends on the premiss (in the case of $\exists$) or the conclusion (in the case of $\forall$) of the given implication: $\forall p_1 \dots \forall p_n\varphi$ and $\exists p_1 \dots \exists p_n\varphi$ interpolate any $\psi \to \varphi$ and $\varphi \to \psi$, respectively, such that $p_1, \dots, p_n$ do not occur in ψ. In case $\forall p_1 \dots \forall p_n\varphi$ or $\exists p_1 \dots \exists p_n\varphi$ contain additional variables not in the common language of φ and ψ, these have to be replaced by $\top$ or $\bot$ or a variable in the common language.

As the notation suggests, the fact that the uniform interpolants are definable in IPC also shows that the propositional quantifiers are definable in that logic. From the algebraic point of view the quantifiers are left and right adjoints of certain embeddings.

Around the time that Pitts obtained his result, Shavrukov proved, by completely different methods, that the modal logic GL has uniform interpolation [9]. Since then, uniform interpolation has been established for various other logics, including the modal logics K and KT [1,11,12]. Intriguingly, the modal logics K4 and S4 do not have uniform interpolation [1,4]. As there are only seven propositional intermediate logics with interpolation [6], the number of intermediate logics with uniform interpolation is necessarily bounded by that number. Ghilardi and Zawadowski showed that there are exactly that many [5].

Whereas in the presence of a decent analytic sequent calculus, proofs of interpolation are often relatively straightforward, proofs of uniform interpolation are in general quite complex. Moreover, it is less clear in how far, if at all, proof systems such as sequent calculi can be of help in establishing the property, as there are logics with analytic sequent calculi that have uniform interpolation (K and GL) as well as logics with analytic sequent calculi that do not (K4 and S4).

In this paper our aim is twofold: to develop a method to extract uniform interpolants from sequent calculi and to prove, using this method, that logics without uniform interpolation lack certain calculi. For both aims it holds that the more general the considered calculi are, the stronger the result. In this paper we restrict ourselves to classical propositional modal logics, but the method applies to intermediate logics as well. For such logics the method is more complicated though, since $\exists$ is not expressible in terms of $\forall$, whereas in the classical case one can just take $\neg\forall p\neg$ for $\exists p$. We plan to treat intermediate logics in a separate paper.

Our construction of uniform interpolants on the basis of calculi provides a modular approach to uniform interpolation. Namely, to establish that the extension of a given calculus with uniform interpolation by a given rule still has uniform interpolation, only certain local properties have to be verified, whereas for most other methods the proof of uniform interpolation for the extension has to be given again in full. Our method is different from but inspired by Pitts' ingenious syntactic method. Bílková used a similar method as Pitts in

combination with a clever termination strategy for the logics K, GL, KT and Grz [1]. Most other proofs of uniform interpolation are of a semantical nature.

In our method, we isolate a certain type of propositional rules called *focussed rules* and a certain type of modal rules called *focussed modal rules* and prove that any logic with a terminating balanced sequent calculus consisting of focussed and focussed modal rules has uniform interpolation. Termination means that in no rule the premisses are more complex, in a certain given ordering, than the conclusion. And a calculus is balanced if for certain combinations of left and right rules, either both rules belong to the calculus or both do not. This result then implies the well-known fact that classical propositional logic has uniform interpolation, and that so do K and KD. Whereas for K this was already known, for KD the result is new. Our results also imply that K4 and S4 cannot have sequent calculi of the above kind. Although for S4 this might be easy to infer in another way, for K4 this seems to be a novel insight.

Besides the mentioned results, uniform interpolation is obtained for various other modal logics as well. The main interest in these results lies not so much in the logics involved, but rather in the illustration they provide of the flexibility of the method developed here. The calculi covered in this paper are not the only calculi to which our method applies, or so we conjecture. It seems likely that similar reasoning applies to other calculi for modal and intermediate logics. We chose, however, to first set up the general framework in this paper because we think it is of interest in itself, and in this way we steer clear of the complexities that might arise once the existential quantifier is no longer expressible in terms of the universal quantifier, as is the case in nonclassical logics.

2 Logics and calculi

The logics we consider are modal propositional logics, formulated in a language $\mathcal{L}$ that contains constants $\top$ and $\bot$, propositional variables or atoms $p, q, r, \ldots$ and the connectives $\wedge, \vee, \neg, \rightarrow$ and the modal operator $\Box$. We assume that all logics we consider are extensions of classical propositional logic CPC and satisfy the necessitation rule, but we do not assume them to be normal. The logics are given by consequence relations denoted by $\vdash$ or $\vdash_{\mathsf{L}}$ and we assume them to be consistent, meaning to not derive $\bot$. $\mathcal{F}$ denotes the set of formulas in $\mathcal{L}$ and $\mathcal{M}$ is the set of all finite multisets of formulas in $\mathcal{F}$. Given a set of atoms $\mathcal{P}$, $\mathcal{F}(\mathcal{P})$ denotes all formulas in $\mathcal{L}$ in which all atoms belong to $\mathcal{P}$. The language $\mathcal{L}_{\mathsf{qf}}$ is defined to be the extension of $\mathcal{L}$ with propositional quantifiers $\forall p$ and $\exists p$ for every atom p, and $\mathcal{F}_{\mathsf{qf}}$ is the set of formulas in that language.

Sequents are expressions $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas, that are interpreted as $I(\Gamma \Rightarrow \Delta) = (\bigwedge \Gamma \rightarrow \bigvee \Delta)$, where $\bigwedge \varnothing$ and $\bigvee \varnothing$ are interpreted as $\top$ and $\bot$, respectively. We denote finite multisets by $\Gamma, \Pi, \Delta, \Sigma$. In a sequent, notation Π, Γ is short for $\Gamma \cup \Pi$. We also define (a for antecedent, s for succedent):

$$(\Gamma \Rightarrow \Delta)^a \equiv_{def} \Gamma \quad (\Gamma \Rightarrow \Delta)^s \equiv_{def} \Delta.$$

For sequents S and S', $S \subseteq S'$ denotes that $S^a \subseteq S'^a$ and $S^s \subseteq S'^s$. When sequents are used in the setting of formulas, we often write S for $I(S)$, such as in $\vdash \bigvee_i (S_i \Rightarrow S)$, which thus means $\vdash \bigvee_i \big(I(S_i) \to I(S)\big)$. Multiplication of sequents is defined as

$$S_1 \cdot S_2 \equiv_{def} (S_1^a \cup S_2^a \Rightarrow S_1^s \cup S_2^s).$$

The set $\mathcal{F}_{\mathsf{ex}}$ is the smallest set of expressions that contains $\mathcal{F}$, is closed under the connectives and modal operator, and if S is a sequent such that all its elements belong to $\mathcal{F}$, then $\forall pS$ and $\exists pS$ belong to $\mathcal{F}_{\mathsf{ex}}$. For example, $\Box\forall p_1(p_1 \Rightarrow p_2)$ belongs to $\mathcal{F}_{\mathsf{ex}}$, but $\forall p_1 \forall p_2(p_1 \Rightarrow p_2)$ does not. We say that a sequent is *in* $\mathcal{L}$ or *in* $\mathcal{F}$ if all formulas that occur in it are in $\mathcal{F}$, and likewise for $\mathcal{L}_{\mathsf{qf}}$, $\mathcal{F}_{\mathsf{qf}}$ and $\mathcal{F}_{\mathsf{ex}}$.

The interpretation of $\mathcal{F}_{\mathsf{ex}}$ into $\mathcal{F}_{\mathsf{qf}}$ is the identity on formulas in $\mathcal{F}$, commutes with the logical operators and interprets quantified sequents as

$$\forall pS \equiv_{def} \forall pI(S).$$

2.1 Rules and axioms

For a proper syntactic treatment of interpolation we need to make a distinction between the language and the meta-language. $\overline{\mathcal{L}}$ is a copy of $\mathcal{L}$ in which every atom p is replaced by $\overline{p}$. The set $\overline{\mathcal{F}}$ of formulas in this language is defined as usual. $\overline{\mathcal{M}}$ is an infinite set of symbols for *meta-multisets*, the elements we denote by $\overline{\Gamma}, \overline{\Pi}, \overline{\Delta}, \overline{\Sigma}$. A *meta-sequent* is an expression $X \Rightarrow Y$, where X and Y are multisets consisting of elements in $\overline{\mathcal{F}} \cup \overline{\mathcal{M}}$.

A *substitution* σ is a map from $\overline{\mathcal{F}} \cup \overline{\mathcal{M}}$ to $\mathcal{F} \cup \mathcal{M}$ that commutes with the connectives and modal operator and such that $\sigma[\overline{\mathcal{F}}] \subseteq \mathcal{F}$ and $\sigma[\overline{\mathcal{M}}] \subseteq \mathcal{M}$. *Sub* is the set of all substitutions.

A *sequent calculus* is a set of *rules*, which are expressions of the form

$$\frac{S_1 \quad S_2 \quad \dots \quad S_n}{S_0}\ \mathcal{R} \tag{1}$$

for some meta-sequents $S_0, S_1, \dots, S_n$. A rule is also called an *axiom* in case there are no premisses, so that it consists of a single meta-sequent. Thus axioms are considered to be special instances of rules.

For any substitution σ, the inference

$$\frac{\sigma S_1 \quad \sigma S_2 \quad \dots \quad \sigma S_n}{\sigma S_0}\ \sigma\mathcal{R}$$

is an *instance* of $\mathcal{R}$. Throughout this paper we denote schematic rules by $\mathcal{R}$ and instances of rules by R. Sets of rules are denoted by $\mathcal{R}$, and $\mathcal{R}_{\mathsf{ins}}$ denotes the set of instances of rules in $\mathcal{R}$. We use the same symbols for axioms but it will always be clear from the context, explicitly stated or immaterial whether the rule is an axiom or not.

As is often done implicitly in papers on sequent calculi, we will from now on confuse the meta-level with the object-level by omitting overscores and the word "meta", trusting that it will always be clear from the context (or does not matter) on which level we are. For example, an axiom such as $\overline{\Gamma}, \overline{p} \Rightarrow \overline{p}, \overline{\Delta}$ will simply be written as $\Gamma, \varphi \Rightarrow \varphi, \Delta$.

Sometimes rules come with side conditions, which formally are conditions on the substitutions that are allowed. For example, an axiom $\Gamma, \varphi \Rightarrow \varphi, \Delta$ could have the side condition that for φ only atoms may be substituted. Also in this case we follow common procedure, by writing such an axiom as $\Gamma, p \Rightarrow p, \Delta$.

For $\mathcal{R}$ as in (1) and for any meta-sequent S, $\mathcal{R}(S)$ denotes the rule

$$\frac{S \cdot S_1 \quad S \cdot S_2 \quad \dots \quad S \cdot S_n}{S \cdot S_0}\ \mathcal{R}(S)$$

For instances R and sequents S, $R(S)$ is defined similarly.

A rule is *backwards applicable* to a sequent S when there is at least one instance of the rule with S as the conclusion. An instance of a rule is *backwards applicable* to S if its conclusion is S. A sequent is *free* if it is not the conclusion of any instance of any rule and it is not an instance of any axiom.

When a logic L given by a consequence relation $\vdash_{\mathsf{L}}$ has a sequent calculus with respect to which it is sound and complete, then we assume that the consequence relation is such that for every instance $S_1 \dots S_n / S_0$ of a rule in the calculus, $I(S_1), \dots, I(S_n) \vdash_{\mathsf{L}} I(S_0)$ holds.

By $\vdash^{\mathcal{R}}_{\mathsf{CPC}}$ we denote the smallest consequence relation containing $\mathcal{R}$ and such that $\varphi_1, \dots, \varphi_n \vdash^{\mathcal{R}}_{\mathsf{CPC}} \psi$ holds whenever ψ or $(\bigwedge \varphi_i \to \psi)$ hold in CPC.

2.2 Focussed rules

A rule $\mathcal{R}$ that is not an axiom is *focussed* if there are meta-sequents $S_1, \dots, S_n$, a meta-sequent S_0 consisting of exactly one meta-formula that is not an atom, and a meta-sequent $S = (\Gamma \Rightarrow \Delta)$ for two distinct meta-multisets Γ and Δ that do not occur in the S_i, such that $\mathcal{R}$ is the rule

$$\frac{S \cdot S_1 \quad S \cdot S_2 \quad \dots \quad S \cdot S_n}{S \cdot S_0}\ \mathcal{R}$$

The rule is a *right rule* in case S_0^a is empty and a *left rule* otherwise.

An axiom is *focussed* if it is either of the form $(\Gamma, r \Rightarrow r, \Delta)$ or $(\Gamma, \bot \Rightarrow \Delta)$ or $(\Gamma \Rightarrow \top, \Delta)$. Thus when we speak of a focussed rule, this either means a focussed rule that is not an axiom, or a focussed axiom. A calculus is *focussed* if every rule in it is focussed.

Typical focussed rules are the left and right rules of many Gentzen calculi. The right conjunction rule

$$\frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta} \tag{2}$$

is clearly focussed, as one can take $(\Gamma \Rightarrow \Delta)$ for S, $(\Rightarrow \varphi \wedge \psi)$ for S_0 and $(\Rightarrow \varphi)$ and $(\Rightarrow \psi)$ for S_1 and S_2, respectively.

2.3 Terminating calculi

A calculus is defined to be a collection of axioms and rules. If G is a calculus, $\vdash_{\mathsf{G}} S$ denotes that sequent S is derivable in G. G is *closed under weakening* if whenever a sequent S is derivable, the sequent $S \cdot S'$ is derivable for any sequent S'. It is *terminating* if it is finite and for all sequents S and rules in the calculus there are at most finitely many instances of the rule backwards applicable to S, and there is a well-founded order $\prec$ on sequents such that in every rule of the calculus the premisses come before the conclusion in this order, all proper subsequents of a sequent come before that sequent, $(\Gamma \Rightarrow) \prec (\Box\Gamma \Rightarrow)$ and $(\Rightarrow \Gamma) \prec (\Rightarrow \Box\Gamma)$ for any multiset Γ, and no sequent comes before the empty sequent.

A typical example of a rule that in general cannot belong to a terminating calculus is the cut rule, as there can be infinitely many instances of the rule with the same conclusion. We will see that many standard cut-free calculi for modal logic are terminating.

All calculi in this paper are assumed to derive at least all sequents for which the interpretation holds in classical propositional logic.

2.4 Partitions

The method to construct uniform interpolants that we are going to develop uses partitions of sequents and rules. A *partition* of a sequent S is an ordered pair (S^r, S^i) (i for interpolant, r for rest) such that $S = S^r \cdot S^i$, where $=$ denotes equality on multisets. It is a *p-partition* if p does not occur in S^r. It is *trivial* if S^r is the empty sequent, and whence $S^i = S$.

A *(p-)partition* $\mathsf{pt}(\mathcal{R})$ of a rule

$$\frac{S_1 \quad \dots \quad S_n}{S_0}\ \mathcal{R}$$

is an expression of the form

$$\frac{S_1^r \cdot S_1^i \quad \dots \quad S_n^r \cdot S_n^i}{S_0^r \cdot S_0^i}\ \mathsf{pt}(\mathcal{R})$$

where (S_j^r, S_j^i) is a (p-)partition of S_j. The (p-)partitions $(S_1^r, S_1^i), \dots, (S_n^r, S_n^i)$ of the premisses of $\mathcal{R}$ are said to *$\mathcal{R}$-correspond* to (S_0^r, S_0^i) under pt.

The way the partition of a rule is denoted above is slightly ambiguous. Although $S \cdot S'$ is equal to $S' \cdot S$ as a sequent, in denoting the partition of a rule as is done above, $S \cdot S'$ refers to the partition (S, S') of sequent $S \cdot S'$, while $S' \cdot S$ refers to the partition (S', S) of the same sequent $S \cdot S'$. We allow this ambiguity in order to keep the notation of partitions of rules simple.

For $\star \in \{i, r\}$ the rule $\mathcal{R}^\star$ is defined as

$$\frac{S_1^\star \quad \dots \quad S_n^\star}{S_0^\star}\ \mathcal{R}^\star$$

Note that for every p-partition of S_0 there exists at least one p-partition of $\mathcal{R}$ that partitions S_0 in exactly the same way, for example by choosing for the S_i^r $(i > 0)$ the empty sequents.

2.5 Rank

Given an order $\prec$ on sequents we define another order, the rank, on $\mathcal{F}_{\mathsf{ex}}$ as follows. We first associate the following set of formulas with a formula. Given a formula φ in $\mathcal{F}_{\mathsf{ex}}$, let $\mathsf{qf}(\varphi)$ be the multiset consisting of all occurrences of subformulas of the form QpS in φ. The order on multisets of the form $\mathsf{qf}(\varphi)$ is in the style of [2]: $\mathsf{qf}(\varphi) \prec_{\mathsf{qf}} \mathsf{qf}(\psi)$ iff $\mathsf{qf}(\varphi)$ is the result of replacing one or more formulas of the form QpS in $\mathsf{qf}(\psi)$ by zero or more formulas of the form QqS' with $S' \prec S$. This order is well–defined since by definition such S are sequents in $\mathcal{L}$, to which $\prec$ applies.

The order on $\mathcal{F}_{\mathsf{ex}}$, that we will also denote with $\prec$, is defined by cases. If $\varphi, \psi \in \mathcal{F}$, then $\varphi \prec \psi$ iff $(\Rightarrow \varphi) \prec (\Rightarrow \psi)$. If $\varphi \in \mathcal{F}$ and $\psi \notin \mathcal{F}$, then $\varphi \prec \psi$ holds and $\psi \prec \varphi$ does not. If $\varphi, \psi \notin \mathcal{F}$, then $\varphi \prec \psi$ holds if $\mathsf{qf}(\varphi) \prec_{\mathsf{qf}} \mathsf{qf}(\psi)$. When $\varphi \prec \psi$, we say that φ is of *lower rank* than ψ. Clearly, if the order $\prec$ on sequents is well–founded, then so is the order $\prec$ on $\mathcal{F}_{\mathsf{ex}}$.

3 Uniform interpolants

Let $\mathcal{R}$ be a terminating calculus. Recall that $\mathcal{R}_{\mathsf{ins}}$ denotes the set of instances of rules in $\mathcal{R}$. A *universal interpolant assignment* for $\mathcal{R}$, assigns, for every atom p, sequent S in $\mathcal{L}$ and for every R being either $\varnothing$ or an instance of a rule in $\mathcal{R}$ backwards applicable to S, to the expression $\forall^R_p S$ a formula in $\mathcal{F}_{\mathsf{ex}}$ that is of lower rank than $\forall pS$. For free sequents S the interpolant assignment assigns formulas in $\mathcal{F}_{\mathsf{ex}}$ of lower rank than $\forall pS$ to expressions $\forall^{\varnothing}_p S$.

If φ is the formula chosen for $\forall^R_p S$, then we write $\forall^R_p S \sim_0 \varphi$. We define an equivalence $\sim$ on formulas in $\mathcal{L}_{\mathsf{ex}}$ that is the smallest equivalence relation that commutes with the connectives and modal operator and furthermore satisfies:

$$\forall pS \sim \begin{cases} \forall^{\varnothing}_p S & \text{if } S \text{ is free} \\ \forall^{\mathrm{at}}_p S \bigvee \{\forall^R_p S \mid R \in \mathcal{R}_{\mathsf{ins}} \text{ backwards applicable to } S\} & \text{if } S \text{ is not free.} \end{cases}$$

The first disjunct is defined as

$$\forall^{\mathrm{at}}_p S \equiv_{def} I(\{q \in S^a \mid q \neq p\} \Rightarrow \{q \in S^s \mid q \neq p\}).$$

Observe that there could be more than one instance of a single rule R which has S as a conclusion, in which case every instance corresponds to a separate disjunct of the interpolant.

Remark 1 Because of the definition of universal interpolant assignments, for every sequent S minimal in order $\prec$, the formula assigned to $\forall pS$ belongs to

$\mathcal{L}$. Therefore one can show by induction along $\prec$ that for every sequent S and atoms $p_1 \dots p_n$ there exists a formula in $\mathcal{L}$ denoted by $\ulcorner \forall p_1 \dots p_n S \urcorner$ such that $\forall p_1 \dots p_n S \sim \ulcorner \forall p_1 \dots p_n S \urcorner$.

Example 1 Suppose the calculus only contains the rule $\mathcal{R}$ for conjunction on the right as given in (2) and there exists an ordering in which sequents with more connectives come after sequents with less connectives. Let S be the sequent $(\Rightarrow \varphi_1 \wedge \psi_1, \varphi_2 \wedge \psi_2)$ and R_i stand for the instance of $\mathcal{R}$ with $\varphi_i \wedge \psi_i$ as the main formula. By the above definition,

$$\forall p S \sim \forall_p^{R_1} S \vee \forall_p^{R_2} S$$

Let $S_{\varphi_1} = (\Rightarrow \varphi_1, \varphi_2 \wedge \psi_2)$ and $S_{\psi_1} = (\Rightarrow \psi_1, \varphi_2 \wedge \psi_2)$, and similarly for S_{φ_2} and S_{ψ_2}. The standard interpolant assignment introduced below, assigns $\forall p S_{\varphi_1} \wedge \forall p S_{\psi_1}$ to $\forall_p^{R_1} S$ and $\forall p S_{\varphi_2} \wedge \forall p S_{\psi_2}$ to $\forall_p^{R_2} S$. This implies that

$$\forall p S \sim \forall_p^{\mathrm{at}} S \vee (\forall p S_{\varphi_1} \wedge \forall p S_{\psi_1}) \vee (\forall p S_{\varphi_2} \wedge \forall p S_{\psi_2}).$$

3.1 The interpolant properties

A given formula φ in $\mathcal{F}$ has *uniform interpolants* if for every atom p there exist two formulas in $\mathcal{F}$, denoted $\forall p \varphi$ and $\exists p \varphi$, such that for all ψ not containing p:

$$\vdash \psi \to \varphi \;\Leftrightarrow \vdash \psi \to \forall p \varphi \qquad \vdash \varphi \to \psi \;\Leftrightarrow \vdash \exists p \varphi \to \psi.$$

Given a formula φ, its *universal uniform interpolant with respect to* $p_1 \dots p_n$ is $\forall p_1 \dots p_n (\Rightarrow \varphi)$, which we write as $\forall p_1 \dots p_n \varphi$. As we only consider classical logics we can take $\neg \forall \neg$ for $\exists$ and the above is equivalent to

$$\vdash \forall p \varphi \to \varphi \qquad \vdash \psi \to \varphi \;\Rightarrow \vdash \psi \to \forall p \varphi. \tag{$\forall$}$$

Because in this paper we express most results in terms of universal interpolants, we usually omit the word *universal* and just speak of *uniform interpolants*.

A logic has *uniform interpolation* if all its formulas have uniform interpolants. This is equivalent to the statement that for every atom p and for any set of atoms $\mathcal{P}$ not containing p, the embedding of $\mathcal{F}(\mathcal{P})$ into $\mathcal{F}(\mathcal{P} \cup \{p\})$ has a right and a left adjoint.

As will be shown in Lemma 1, in the approach via sequents, ($\forall$) is replaced by the following requirements, the *interpolant properties*.

($\forall$l) for all p: $\vdash S^a, \forall p S \Rightarrow S^s$;
($\forall$r) for all p: $\vdash S^r \cdot (\Rightarrow \forall p S^i)$ if S is derivable and S^r does not contain p.

A calculus G has *uniform interpolation* if ($\forall$l) and ($\forall$r) hold, where $\vdash$ denotes $\vdash_{\mathsf{G}}$.

Property ($\forall$l) is the *independent* (from partitions) *interpolant property*, and ($\forall$r) is the *dependent interpolant property*. For every such property and any

$R \in \mathcal{R}_{\mathsf{ins}} \cup \{\varnothing\}$, the *R-variant* is the result of replacing quantifier $\forall p$ in the property by $\forall^R_p$. A partition (S^r, S^i) of S *satisfies* the interpolant properties if, in the case of the independent property, S satisfies it (in which case we also say that S satisfies it), and in case of the dependent property, it holds for that particular partition. A sequent *satisfies* a property if every possible partition of the sequent satisfies it.

Remark 2 Observe that the independent interpolant property holds for a sequent S if and only if the R-variant of the independent interpolant property holds for all R backwards applicable to S and the independent interpolant property in which $\forall p$ is replaced by $\forall^{\mathrm{at}}_p$ holds. If the R-variant of the dependent interpolant property holds for partition (S^r, S^i) for at least one R that is backwards applicable to S^i, then the property holds for (S^r, S^i).

Lemma 1 *If all sequents satisfy the interpolant properties, then* L *has uniform interpolation.*

Proof We have to show that $(\forall)$ holds. Consider $S = (\Rightarrow \varphi)$. Hence $\forall p\varphi = \forall pS$ by definition. By $(\forall\mathrm{l})$, $\vdash I(\forall p\varphi \Rightarrow \varphi)$, that is, $\forall p\varphi \to \varphi$ is derivable.

Consider a ψ not containing p such that $\vdash \psi \to \varphi$. Let $S = (\psi \Rightarrow \varphi)$ and consider the partition (S^r, S^i), where $S^r = (\psi \Rightarrow)$ and $S^i = (\Rightarrow \varphi)$. Hence $\forall p\varphi = \forall pS^i$ by definition. And $\vdash (\psi \Rightarrow) \cdot (\Rightarrow \forall p\varphi)$ by $(\forall\mathrm{r})$. Therefore $(\psi \to \forall p\varphi)$ is derivable.

A fact that we will often be used is that all free sequents satisfy the dependent interpolant properties.

3.2 Soundness

In order to divide the proof of the main theorems in separate parts we introduce the following three properties of rules $\mathcal{R}$.

- $(\mathrm{IP})_{\mathcal{R}}$ $\{S_j \cdot (\forall pS_j \Rightarrow) \mid 1 \leq j \leq n\} \vdash S_0 \cdot (\forall^R_p S_0 \Rightarrow)$ for every instance $R = S_1 \dots S_n/S_0$ of $\mathcal{R}$.
- $(\mathrm{DPB})_{\mathcal{R}}$ For every instance $S_1 \dots S_n/S_0$ of $\mathcal{R}$ and every p–partition (S^r_0, S^i_0) of S_0 such that $\mathcal{R}$ is backwards applicable to S^i_0, there exists a partition of $\mathcal{R}$ such that $\{S^r_j \cdot (\Rightarrow \forall pS^i_j) \mid 1 \leq j \leq n\} \vdash S^r_0 \cdot (\Rightarrow \forall^R_p S^i_0)$, where R is an instance of $\mathcal{R}$ with conclusion S^i_0.
- $(\mathrm{DPN})_{\mathcal{R}}$ For every instance $S_1 \dots S_n/S_0$ of $\mathcal{R}$ and every p–partition (S^r_0, S^i_0) of S_0 such that S^i_0 is not empty and $\mathcal{R}$ is not backwards applicable to S^i_0, if all sequents lower than S_0 satisfy the interpolant properties, then the sequent $S^r_0 \cdot (\Rightarrow \forall pS^i_0)$ is derivable.

((IP) stands for *independent property*, (DP) for *dependent property*, B for *backwards*.)

- (FP) If S is free and all sequents lower than S satisfy the interpolant properties, then S satisfies the independent interpolant property.

An interpolant assignment is *sound* for a given calculus if (FP) holds and $(\mathrm{IP})_{\mathcal{R}}$, $(\mathrm{DPB})_{\mathcal{R}}$ and $(\mathrm{DPN})_{\mathcal{R}}$ hold for every rule $\mathcal{R}$ in the calculus.

Since free sequents trivially satisfy the dependent interpolant property, this seems to suggest that one could always take $\bot$ for the interpolant assignment of free sequents. Free sequents do, however, also play a role in the dependent interpolant property, because the partition of a sequent might be such that S^i is a free sequent. Therefore this simple interpolant assignment for free sequents might not always be possible.

Lemma 2 *If a logic* L *has a terminating calculus that is closed under weakening and for which there exists a sound interpolant assignment, then all sequents satisfy the interpolant properties.*

Proof We have to show that under the assumptions in the lemma, the interpolant properties hold for any p–partition (S^r, S^i) of any sequent S. Free sequents satisfy the dependent property in a trivial way, as they are not derivable. Therefore, using Remark 2, it suffices to show that for any sequent S and any p-partition (S^r, S^i) of S:

1. $\vdash S^a, \forall_p^{\mathrm{at}} S \Rightarrow S^s$ if S is not free,
2. $\vdash S^a, \forall_p^{R} S \Rightarrow S^s$ for all R backwards applicable to S, if S is not free,
3. $\vdash S^r \cdot (\Rightarrow \forall p S^i)$ in case S is derivable,
4. S satisfies the independent interpolant property if S is free.

For 1, note that in case $\forall_p^{\mathrm{at}} S$ is empty, 1 clearly holds. Therefore assume $\forall_p^{\mathrm{at}} S$ is not empty, say it is of the form $\bigwedge q_i \to \bigvee r_j$, where the disjunct or conjunct may be empty, but not both. Then $S^a, \forall_p^{\mathrm{at}} S \Rightarrow S^s$ is clearly derivable.

Statements 2 and 3 are proved by induction on the well-founded order $\prec$ on $\mathcal{F}_{\mathsf{ex}}$ defined in Section 2.5. Recall that in every rule of the calculus the premisses come before the conclusion in this order, and all proper subsequents of a sequent come before that sequent.

Sequents S that are not free and minimal in the order have to be axioms. Thus 2 follows from $(\mathrm{IP})_{\mathcal{R}}$ as the set corresponding to the premisses is empty since an axiom has no premisses. For 3, since S is minimal in the order, $S^i = S$ or S^i is the empty sequent. In the first case, for the same reason as in the case of $(\mathrm{IP})_{\mathcal{R}}$, property $(\mathrm{DPB})_{\mathcal{R}}$ implies that $\vdash S^r \cdot (\Rightarrow \forall_p^R S^i)$. In the last case, because $S^r = S$ and S is derivable, also the weakening $S^r \cdot (\Rightarrow \forall p S^i)$ is derivable. Hence 3 holds.

For the induction step, assume that all sequents lower than S satisfy the interpolant properties. For 2, consider an instance $R = (S_1 \dots S_n / S)$ of a rule. The induction hypothesis implies that the interpolant properties hold for $S_1, \dots, S_n$. Therefore 2 follows from $(\mathrm{IP})_{\mathcal{R}}$. For 3, assume that S is derivable and let $\mathcal{R}$ be a rule or axiom that is backwards applicable to S, and consider a partition (S^r, S^i) of S and a corresponding partition of $\mathcal{R}$. In case S^i is the empty sequent, $S^r = S$ and 3 clearly holds. Thus assume S^i is not empty. In case $\mathcal{R}$ is backwards applicable to S^i, then 3 follows from $(\mathrm{DPB})_{\mathcal{R}}$. In case $\mathcal{R}$ is not backwards applicable to S^i, $(\mathrm{DPN})_{\mathcal{R}}$ applies.

Finally, 4 is proved by induction to $\prec$. If S is minimal in the ordering, then it satisfies the independent interpolant property by (FP). In the induction step we assume that all sequents lower than S satisfy the interpolant properties. Hence we can apply (FP) again to show that the independent interpolant property holds for S.

Theorem 1 *If a logic* L *has a terminating calculus that is closed under weakening and for which there exists a sound interpolant assignment, then* L *has uniform interpolation.*

Proof This follows from Lemma 1 and Lemma 2.

4 Standard assignments for propositional logic

In this section we define a specific universal interpolant assignment for propositional logic inspired by Pitts' method. It is proved to be sound for focussed axioms and rules.

For an instance

$$\frac{S_1 \quad S_2 \quad \dots \quad S_n}{S_0}\ R$$

of a n-premiss rule the *standard universal interpolant assignment* is

$$\forall^R_p S_0 \sim_0 \forall p S_1 \wedge \forall p S_2 \wedge \ldots \wedge \forall p S_n.$$

If all instances of a schematic rule $\mathcal{R}$ have a standard interpolant assignment, then the interpolant assignment for $\mathcal{R}$ is *standard*.

For instances R of focussed axioms consisting of sequent S, the *standard interpolant assignment* is defined as

$$\forall^R_p S \sim_0 \top.$$

For free sequents S the *standard interpolant assignment* is

$$\forall^{\varnothing}_p S \equiv_{def} (\{\varphi \in S^a \mid \varphi \text{ does not contain } p\} \Rightarrow \{\psi \in S^s \mid \psi \text{ does not contain } p\}).$$

We fix a partitioning of instances R of focussed rules $\mathcal{R}$ with the property that if $\mathcal{R}$ is backwards applicable to $(S \cdot S_0)^i$, then R^i is an instance of $\mathcal{R}$ and if $\mathcal{R}$ is not backwards applicable to $(S \cdot S_0)^i$, then R^r is an instance of $\mathcal{R}$. Therefore we define a partition $\mathsf{pt}(R)$

$$\frac{((S \cdot S_1)^r, (S \cdot S_1)^i) \quad \dots \quad ((S \cdot S_n)^r, (S \cdot S_n)^i)}{((S \cdot S_0)^r, (S \cdot S_0)^i)}\ \mathsf{pt}(R)$$

of an instance $(S \cdot S_1), \ldots, (S \cdot S_n)/(S \cdot S_0)$ of a focussed rule to be *standard* if for all $1 \leq j \leq n$:

$$(S \cdot S_j)^i = \begin{cases} (S^i \cdot S_j) & \text{if } (S \cdot S_0)^i = S^i \cdot S_0 \text{ for some } S^i \subseteq S \\ (S \cdot S_0)^i & \text{otherwise.} \end{cases}$$

$(S \cdot S_j)^r$ is defined to be the unique sequent such that $((S \cdot S_j)^r, (S \cdot S_j)^i)$ is a partition of $(S \cdot S_j)$. If the first case in the definition of $(S \cdot S_j)^i$ does not apply, then as S_0 contains exactly one formula, $S_0 \subseteq (S \cdot S_0)^r$ and $(S \cdot S_0)^i \subseteq S$. Therefore

$$(S \cdot S_j)^r = \begin{cases} (S \cdot S_0)^r & \text{if } (S \cdot S_0)^i = S^i \cdot S_0 \text{ for some } S^i \subseteq S \\ (S^r \cdot S_j) & \text{if } (S \cdot S_0)^r = S^r \cdot S_0 \text{ for some } S^r \subseteq S. \end{cases}$$

Example 2 Consider the following instance R of the rule $\mathcal{R}$ for conjunction on the right:

$$\frac{\Gamma \Rightarrow \varphi_1, \Delta \quad \Gamma \Rightarrow \varphi_2, \Delta}{\Gamma \Rightarrow \varphi_1 \wedge \varphi_2, \Delta}$$

Thus $S_0 = (\Rightarrow \varphi_1 \wedge \varphi_2)$, $S_j = (\Rightarrow \varphi_j)$ for $j = 1, 2$ and $S = (\Gamma \Rightarrow \Delta)$. We will describe two standard partitions of the rule, one corresponding to the first case in the definition above and the other to the second. First, consider partition (S_0^r, S_0^i) of the conclusion S_0, where $(S \cdot S_j)^i = (\Rightarrow \varphi_1 \wedge \varphi_2)$. From $S_0 \subseteq (S \cdot S_j)^i$ it follows that $(S \cdot S_j)^i = (\Rightarrow \varphi_j)$ for $j = 1, 2$ and $S_j^r = S$ for $j = 0, 1, 2$.

Second, assume $\Delta \neq \varnothing$ and the formula in S_0 does not occur in Δ, and consider partition (S_0^r, S_0^i) of S_0, where $(S \cdot S_0)^i = (\Rightarrow \Delta)$. Thus $(S \cdot S_0)^r = ((\Gamma \Rightarrow) \cdot S_0)$. As $S_0 \not\subseteq (S \cdot S_0)^i$, $(S \cdot S_j)^i = S^i$ for $j = 0, 1, 2$, $(S \cdot S_0)^r = (\Gamma \Rightarrow \varphi_1 \wedge \varphi_2)$, and $(S \cdot S_j)^r = (\Gamma \Rightarrow \varphi_j)$ for $j = 1, 2$.

Observe that in the first case R^i is an instance of $\mathcal{R}$, and in the second case R^r is. Generalising this gives the following lemma.

Lemma 3 *For any instance $R = \big((S \cdot S_1) \dots (S \cdot S_n)/(S \cdot S_0)\big)$ of a focussed rule $\mathcal{R}$ with a standard partition, either $S_0 \subseteq (S \cdot S_0)^i$ and R^i is an instance of $\mathcal{R}$ or $S_0 \subseteq (S \cdot S_0)^r$ and R^r is an instance of $\mathcal{R}$.*

4.1 Properties of focussed rules

Lemma 4 *For any instance $S_1 \dots S_n/S_0$ of a focussed rule and any formulas $\varphi_1, \dots, \varphi_n$:*

$$\{S_j \cdot (\varphi_j \Rightarrow) \mid j = 1, \dots, n\} \vdash^{\mathcal{R}}_{\mathsf{CPC}} S_0 \cdot (\bigwedge_{j=1}^{n} \varphi_j \Rightarrow).$$

Proof Clearly, $\{S_1, \dots, S_n\} \vdash^{\mathcal{R}}_{\mathsf{CPC}} S_0$. Let $S = (\bigwedge_{j=1}^{n} \varphi_j \Rightarrow)$. Since $\mathcal{R}$ is focussed, we have $\{S \cdot S_1, \dots, S \cdot S_n\} \vdash^{\mathcal{R}}_{\mathsf{CPC}} S \cdot S_0$. Since $S_j \cdot (\varphi_j \Rightarrow) \vdash^{\mathcal{R}}_{\mathsf{CPC}} S_j \cdot S$, the desired follows.

Lemma 5 *For all formulas $\varphi_1, \dots, \varphi_n$ and any partition (S^r, S^i) of the conclusion of an instance $S_1 \dots S_n/S$ of a focussed rule $\mathcal{R}$ which is backwards applicable to S^i:*

$$\{S_j^r \cdot (\Rightarrow \varphi_j) \mid j = 1, \dots, n\} \vdash^{\mathcal{R}}_{\mathsf{CPC}} S^r \cdot (\Rightarrow \bigwedge_{j=1}^{n} \varphi_j).$$

Proof Consider the standard partition of $\mathcal{R}$. As $\mathcal{R}$ is backwards applicable to S^i, $S^r = S^r_j$, which immediately implies what we had to show.

Lemma 6 *For all sequents S and any standard partition of an instance $R = S_1 \dots S_n/S_0$ of a focussed rule $\mathcal{R}$ which is not backwards applicable to S^i_0:*

$$\{S^r_j \cdot S \mid j = 1, \dots, n\} \vdash^{\mathcal{R}}_{\mathsf{CPC}} S^r_0 \cdot S.$$

Proof As $\mathcal{R}$ is not backwards applicable to S^i_0, R^r is an instance of $\mathcal{R}$ by Lemma 3, using the standard partition of $\mathcal{R}$. Because the rule $\mathcal{R}$ is focussed, $\{S \cdot S^r_1, \dots, S \cdot S^r_n\}/S \cdot S^r_0$ is an instance of $\mathcal{R}$, which is what we had to show.

Lemma 7 *For any focussed rule $\mathcal{R}$ in any calculus for a logic, (IP)$_{\mathcal{R}}$ holds.*

Proof Let $R = S_1 \dots S_n/S_0$ be an instance of $\mathcal{R}$. Then $\forall^R_p S_0 \sim \bigwedge^n_{j=1} \forall p S_j$. Thus by letting $\varphi_j = \forall p S_j$, Lemma 4 and the fact that $\vdash^{\mathcal{R}}_{\mathsf{CPC}} \subseteq \vdash$ can be applied to obtain (IP)$_{\mathcal{R}}$.

Lemma 8 *For any focussed rule $\mathcal{R}$ in any terminating calculus with a standard interpolant assignment, (DPB)$_{\mathcal{R}}$ holds.*

Proof Consider an instance $R = S_1 \dots S_n/S$ of $\mathcal{R}$ and let (S^r, S^i) be a partition of S such that $\mathcal{R}$ is backwards applicable to S^i. If $\mathcal{R}$ is an axiom, then $\forall^R_p S \sim \top$ and thus $\forall p S \sim \top$. Similarly for $\forall p S^i$. Therefore (DPB)$_{\mathcal{R}}$ clearly holds for sequents that are instances of axioms. Suppose $\mathcal{R}$ is not an axiom. By Lemma 3 for the standard partition of $\mathcal{R}$ there is an instance R of $\mathcal{R}$ with conclusion S^i and premisses $S^i_1, \dots, S^i_n$. Therefore $\forall^R_p S^i \sim \bigwedge^n_{j=1} \forall p S^i_j$. Thus by letting $\varphi_j = \forall p S^i_j$, Lemma 5 can be applied to obtain (DPB)$_{\mathcal{R}}$.

Lemma 9 *For every instance $S_1 \dots S_n/S_0$ of a focussed rule $\mathcal{R}$ in any terminating calculus with a standard interpolant assignment and every p–partition (S^r_0, S^i_0) of S_0 such that S^i_0 is not empty and $\mathcal{R}$ is not backwards applicable to S^i_0:*

$$\{S^r_j \cdot (\Rightarrow \forall p S^i_j) \mid 1 \leq j \leq n\} \vdash^{\mathcal{R}}_{\mathsf{CPC}} S^r_0 \cdot (\Rightarrow \forall p S^i_0).$$

Proof First consider the case that $\mathcal{R}$ is an axiom. If it is of the form $(\Gamma, \bot \Rightarrow \Delta)$ or $(\Gamma \Rightarrow \top, \Delta)$, then the fact that S^i_0 is not instance of it implies that $S^r_0 \cdot (\Rightarrow \forall p S^i_0)$ is an instance of the axiom, and we are done. If it is of the form $(\Gamma, q \Rightarrow q, \Delta)$, then $q \neq p$. For if $q = p$, as S^r_0 does not contain p, $(q \Rightarrow q)$ would be a subsequent of S^i_0, quod non. If $q \Rightarrow q$ is a subsequent of S^r_0, then $S^r_0 \cdot (\Rightarrow \forall p S^i_0)$ is an instance of the axiom, and we are done again. Therefore consider the case that $q \Rightarrow q$ is neither a subsequent of S^i_0 nor of S^r_0. If S^i_0 is free, the standard interpolant assignment implies that $\vdash^{\mathcal{R}}_{\mathsf{CPC}} S^r_0 \cdot (\Rightarrow \forall p S^i_0)$. If S^i_0 is not free, it suffices to prove that $\vdash^{\mathcal{R}}_{\mathsf{CPC}} S^r_0 \cdot (\Rightarrow \forall^{\mathrm{at}}_p S^i_0)$, which follows from the fact that $S^r_0 \cdot (\Rightarrow \forall^{\mathrm{at}}_p S^i_0)$ contains the sequent $q \Rightarrow q$.

The case that $\mathcal{R}$ is not an axiom remains. By Lemma 6, the fact that $\mathcal{R}$ is not backwards applicable to S^i_0 implies that for any standard partition and any sequent S: $\{S \cdot S^r_j \mid j = 1, \dots, n\} \vdash^{\mathcal{R}}_{\mathsf{CPC}} S \cdot S^r_0$. Lemma 3 implies that $S^i_0 = S^i_j$, and thus $(\Rightarrow \forall p S^i_j) = (\Rightarrow \forall p S^i_0)$ for all $j = 1, \dots, n$, from which the lemma follows.

Lemma 10 *For any focussed rule $\mathcal{R}$ in any terminating calculus with a standard interpolant assignment, $(DPN)_{\mathcal{R}}$ holds.*

Proof By Lemma 9.

Lemma 11 *In any terminating calculus with a standard interpolant assignment (FP) holds.*

Proof Let S be a free sequent. We have to show that S satisfies the independent interpolant property, that is, that $S^a, \forall pS \Rightarrow S^s$ is derivable. By definition $\forall pS = \forall^{\varnothing} pS$, where

$$\forall^{\varnothing} pS \sim I(\{\varphi \in S^a \mid \varphi \text{ does not contain } p\} \Rightarrow \{\psi \in S^s \mid \psi \text{ does not contain } p\}).$$

Therefore $S^a, \forall pS \Rightarrow S^s$ is equivalent to a sequent $S^a, (\bigwedge \Gamma \to \bigvee \Delta) \Rightarrow S^s$ for some $\Gamma \subseteq S^a$ and $\Delta \subseteq S^s$, which is clearly derivable, even if Γ or Δ is empty.

Theorem 2 *A logic* L *with a terminating calculus in which all rules and axioms are focussed has uniform interpolation.*

Proof First observe that such a calculus is closed under weakening. By Theorem 1 it suffices to prove that the interpolant assignment is sound. This follows from Lemmas 7, 8, 10, and 11.

4.2 Classical logic

To establish that classical propositional logic has uniform interpolation is not hard. More generally, it is known that if a logic has interpolation and is locally tabular[1], properties that indeed hold for CPC, then it has uniform interpolation. However, with the methods developed in the previous sections one can also easily infer that CPC has uniform interpolation from the existence of a terminating calculus for the logic in which all axioms and rules are focussed. The propositional part, G3p, of the calculus G3 from [10] which is given in Figure 4.2 has these properties. It is terminating (using the natural ordering given by the number of symbols in a sequent) and the axioms and rules are clearly focussed. The calculus has no structural rules, but they are admissible in it, as is the cut rule.

[1] For any finite set of variables there are only finitely many nonequivalent formulas in those variables.

$$\Gamma, p \Rightarrow p, \Delta \quad \text{At} \quad (\Gamma \cup \Delta \text{ atoms}) \qquad \Gamma, \bot \Rightarrow \Delta \quad L\bot$$

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}\ \mathrm{R}\wedge \qquad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ \mathrm{L}\wedge$$

$$\frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\ \mathrm{R}\vee \qquad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ \mathrm{L}\vee$$

$$\frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ \mathrm{R}\to \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ \mathrm{L}\to$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta}\ \mathrm{R}\neg \qquad \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma, \neg\varphi \Rightarrow \Delta}\ \mathrm{L}\neg$$

Fig. 1 The Gentzen calculus G3p

Theorem 3 *Classical propositional logic has uniform interpolation.*

Proof By Theorem 2 and the fact that the terminating calculus G3p is sound and complete for CPC.

Note that for the above result one cannot use the propositional part of Gentzen's LK or other calculi that contain the Cut Rule, as such calculi may not be terminating.

5 Standard assignment for modal logics

In this section we extend the method developed above to modal logic by extending the class of rules to which Theorem 2 applies. We use the convention that $\Box S = (\{\Box\varphi \mid \varphi \in S^a\} \Rightarrow \{\Box\psi \mid \psi \in S^s\})$, implying that $\Box(\Gamma \Rightarrow) = (\Box\Gamma \Rightarrow)$ and $\Box(\Rightarrow \Delta) = (\Rightarrow \Box\Delta)$.

A *focussed* rule is defined as before, in Section 2.2, except that it is required that the formula in S_0 is not boxed. A rule $\mathcal{R}$ is a *focussed modal rule* if it is of the form

$$\frac{\Box S_1 \cdot S_0}{S_2 \cdot \Box S_1 \cdot \Box S_0}\ \mathcal{R} \tag{3}$$

for certain sequents S_0, S_1, S_2 such that

- S_2^a, S_2^s both consist of a meta-multiset;
- S_1^a, S_1^s both consist of at most one element, which is a meta-multiset;
- S_0^a, S_0^s both consist of meta-multisets and meta–atoms;
- S_0 contains exactly one meta–atom;
- no meta–multiset occurs in $S_2 \cdot S_1 \cdot S_0$ more than once;
- if S_i^a contains a meta–multiset for $i = 0$ or $i = 1$, then the meta–multiset in S_2^a ranges over multisets of atoms;
- if S_i^s contains a meta–multiset for $i = 0$ or $i = 1$, then the meta–multiset in S_2^s ranges over multisets of atoms.

Thus S_1 may be empty, but S_2 and S_0 may not. Note that S_2 is of the form $\Lambda \Rightarrow \Theta$ for two meta-multiset symbols Λ and Θ that do not occur in S_1 or S_0. Given a focussed modal rule $\mathcal{R}$, let $\mathcal{R}^+$ denote the rule obtained from $\mathcal{R}$ by omitting the last two restrictions about the multisets in S_2. In Lemma 21 it is shown that $\mathcal{R}^+$ is not stronger than $\mathcal{R}$. The reason to work with the restricted rules is to simplify the proofs in this section.

Up to the renaming of meta-symbols there are eight possibilities for S_0 and four for S_1. To each possibility corresponds a name, which is a set consisting of the elements L,l,R,r,W,E, the lower case letters referring to the position of the atom in S_0 and the upper case letters to the presence and position of the multisets in S_0 (left and right) and in S_1 (west and east). For brevity we denote the sets as strings, so that, for example, Lr should be read as $\{\mathsf{L}, \mathsf{r}\}$.

name	S_0	name	S_0	name	S_1
LlR	$\Gamma, p \Rightarrow \Delta$	LrR	$\Gamma \Rightarrow p, \Delta$	$\varnothing$	$(\Rightarrow)$
Ll	$\Gamma, p \Rightarrow$	Lr	$\Gamma \Rightarrow p$	W	$(\Pi \Rightarrow)$
lR	$p \Rightarrow \Delta$	rR	$\Rightarrow p, \Delta$	E	$(\Rightarrow \Sigma)$
l	$p \Rightarrow$	r	$\Rightarrow p$	WE	$(\Pi \Rightarrow \Sigma)$

The terminology for the 32 focussed modal rules is as follows. Consider a focussed modal rule as in (3) and let x and y be the names in the list above for S_0 and S_1, respectively. Then the name of the rule with this S_0 and S_1 is $\mathcal{R}_{\mathsf{x}\cup\mathsf{y}}$, where $\mathsf{x}\cup\mathsf{y}$ is again abbreviated as a string. For example, $\mathcal{R}_{\mathsf{WlR}}$ and $\mathcal{R}_{\mathsf{Lr}}$ denote the following rules, where the latter is the well-known rule for K.

$$\frac{\Box\Pi, p \Rightarrow \Delta}{\Lambda, \Box\Pi, \Box p \Rightarrow \Box\Delta, \Theta}\ \mathcal{R}_{\mathsf{WlR}} \qquad \frac{\Gamma \Rightarrow p}{\Lambda, \Box\Gamma \Rightarrow \Box p, \Theta}\ \mathcal{R}_{\mathsf{Lr}}$$

$\mathcal{R}_{\mathsf{Lr}}$ is also denoted by $\mathcal{R}_{\mathsf{K}}$, and a calculus containing that rule is called *normal*. For subsets $\mathsf{x}, \mathsf{y} \subseteq \{\mathsf{W}, \mathsf{L}, \mathsf{l}, \mathsf{r}, \mathsf{R}, \mathsf{E}\}$, $\mathcal{R}_{\mathsf{x}}$ is a y-rule if all elements of y occur in x.

The following eight rules are called *bad* rules: $\mathcal{R}_{\mathsf{WlRx}}$ and $\mathcal{R}_{\mathsf{LrEx}}$, where x ranges over subsets of $\{\mathsf{W}, \mathsf{L}, \mathsf{l}, \mathsf{r}, \mathsf{R}, \mathsf{E}\}$. Note that $\mathcal{R}_{\mathsf{K}}$ is not a bad rule, and neither is any rule that does not contain W or E.

As for focussed rules we fix a partition of instances of focussed modal rules, for which we have to address a small technical detail first. Given the conclusion of a focussed modal rule $S = (S_2 \cdot \Box S_1 \cdot \Box S_0)$, we assume that for every formula in S it is indicated, although we will not explicitly do so, whether it belongs to S_0, S_1 or S_2. In this way, for any partition (S^i, S^r) of S, $j = 0, 1, 2$ and $x \in \{i, r\}$, there exist unique sequents $S_j^x \subseteq S_j$ ($\subseteq$ defined in Section 2) such that $S^x = S_2^x \cdot \Box S_1^x \cdot \Box S_0^x$. Given this assumption, a partition

$$\frac{((\Box S_1 \cdot S_0)^r, (\Box S_1 \cdot S_0)^i)}{((S_2 \cdot \Box S_1 \cdot \Box S_0)^r, (S_2 \cdot \Box S_1 \cdot \Box S_0)^i)}\ \mathsf{pt}(R)$$

of an instance $R = (\Box S_1 \cdot S_0 / S_2 \cdot \Box S_1 \cdot \Box S_0)$ of a focussed modal rule is *standard* if for $x \in \{i, r\}$:

$$(\Box S_1 \cdot S_0)^x = \Box S_1^x \cdot S_0^x.$$

Lemma 12 *For any instance R of any focussed modal rule $\mathcal{R}$ with a standard partition, either R^i or R^r (or both) is an instance of $\mathcal{R}$.*

Proof In case $\mathcal{R}$ is an r-rule, either S_0^{is} or S_0^{rs} is not empty. In the first case, R^i is an instance of $\mathcal{R}$ and in the second case R^r is. The case of an l-rule is similar.

Remark 3 The fact that in focussed modal rules (3) the sequent S_2 is of the form $(\Gamma \Rightarrow \Delta)$ for two meta–multisets Γ and Δ, implies that if S/S' is an instance of a focussed modal rule, then so is $S/S' \cdot S''$ for any sequent S''.

Lemma 13 *If $R = (S_u/S)$ is an instance of a focussed modal rule $\mathcal{R}$ that is not bad, (S^r, S^i) is a partition of S such that $\mathcal{R}$ is backwards applicable to S^i and S_u^i is the premiss of R^i, then one of the following cases hold:*

1. *S_u^i/S^i is an instance of $\mathcal{R}$,*
2. *S_u^i consists of boxed formulas only,*
3. *S_u^i has an empty antecedent and $\mathcal{R}$ is an l–rule,*
4. *S_u^i has an empty succedent and $\mathcal{R}$ is an r–rule.*

Moreover, in cases (2)–(4), for any φ: $S_u^i \cdot (\varphi \Rightarrow\ )/S^i \cdot (\Box\varphi \Rightarrow\ )$ is an instance of $\mathcal{R}$ if $\mathcal{R}$ is an l–rule and $S_u^i \cdot (\ \Rightarrow \varphi)/S^i \cdot (\ \Rightarrow \Box\varphi)$ is an instance of $\mathcal{R}$ if $\mathcal{R}$ is an r–rule.

Proof We treat the case that $\mathcal{R}$ is an l–rule, the other case is analogous. Thus $\mathcal{R}$ is of the following form, where any of the $\Pi, \Gamma, \Delta, \Sigma$ may be empty.

$$\frac{\Box\Pi, \Gamma, \varphi \Rightarrow \Delta, \Box\Sigma}{\Lambda, \Box\Pi, \Box\Gamma, \Box\varphi \Rightarrow \Box\Delta, \Box\Sigma, \Theta}$$

The fact that $\mathcal{R}$ is backwards applicable to S^i implies that S^{ia} contains some $\Box\psi \in \Box\Gamma \cup \{\Box\varphi\} \cup \Box\Pi \cup \Lambda$. In case $\psi = \varphi$ we have, for some partitioning (X^i, X^r) of each of the multisets $\Lambda, \Pi, \Gamma, \Delta, \Sigma, \Theta$:

$$\begin{array}{ll} S^i = (\Lambda^i, \Box\Pi^i, \Box\Gamma^i, \Box\varphi \Rightarrow \Box\Delta^i, \Box\Sigma^i, \Theta^i) & S_u^i = (\Box\Pi^i, \Gamma^i, \varphi \Rightarrow \Delta^i, \Box\Sigma^i) \\ S^r = (\Lambda^r, \Box\Pi^r, \Box\Gamma^r \Rightarrow \Box\Delta^r, \Box\Sigma^r, \Theta^r) & S_u^r = (\Box\Pi^r, \Gamma^r \Rightarrow \Delta^r, \Box\Sigma^r). \end{array}$$

And likewise if $\psi \in \Gamma$. In these cases S_u^i/S^i is an instance of $\mathcal{R}$, thus 1. holds.

If this is not the case, then S^{ia} contains no formula in $\Box\Gamma \cup \{\Box\varphi\}$ but there is a $\Box\psi \in S^{ia} \cap (\Box\Pi \cup \Lambda)$. In case $\psi \in \Pi$, we have, for some partitioning (X^i, X^r) of each of the multisets $\Lambda, \Pi, \Delta, \Sigma, \Theta$ with $\psi \in \Pi^i$:

$$\begin{array}{ll} S^i = (\Lambda^i, \Box\Pi^i \Rightarrow \Box\Delta^i, \Box\Sigma^i, \Theta^i) & S_u^i = (\Box\Pi^i \Rightarrow \Delta^i, \Box\Sigma^i) \\ S^r = (\Lambda^r, \Box\Pi^r, \Box\Gamma, \Box\varphi \Rightarrow \Box\Delta^r, \Box\Sigma^r, \Theta^r) & S_u^r = (\Box\Pi^r, \Gamma, \varphi \Rightarrow \Delta^r, \Box\Sigma^r). \end{array}$$

The only way in which S_u^i does not consist of boxed formulas is if Δ^i is not empty, and whence $\mathcal{R}$ is an R–rule. Thus it is an WIR–rule, since $\Pi \neq \varnothing$. But this cannot be since $\mathcal{R}$ is not a bad rule. Hence this case cannot occur and S_u^i consists of boxed formulas only, which gives 2. It also implies that $S_u^i = (\Box\Pi^i \Rightarrow \Box\Sigma^i)$ and $S^i = (\Lambda^i, \Box\Pi^i \Rightarrow \Box\Sigma^i, \Theta^i)$. Therefore, for any φ,

$S^i_u \cdot (\varphi \Rightarrow)/S^i \cdot (\Box\varphi \Rightarrow)$ is an instance of $\mathcal{R}$, which proves that the second part of the lemma holds.

If S^{ia} contains no formula in $\Box\Gamma \cup \{\Box\varphi\} \cup \Box\Pi$ but contains a $\Box\psi \in \Lambda$, we have, for some partitioning (X^i, X^r) of each of the multisets $\Lambda, \Delta, \Sigma, \Theta$ with $\Box\psi \in \Lambda^i$:

$$\begin{array}{ll} S^i = (\Lambda^i \Rightarrow \Box\Delta^i, \Box\Sigma^i, \Theta^i) & S^i_u = (\Rightarrow \Delta^i, \Box\Sigma^i) \\ S^r = (\Lambda^r, \Box\Pi, \Box\Gamma, \Box\varphi \Rightarrow \Box\Delta^r, \Box\Sigma^r, \Theta^r) & S^r_u = (\Box\Pi, \Gamma, \varphi \Rightarrow \Delta^r, \Box\Sigma^r). \end{array}$$

Thus this is the case in which the antecedent of S^i_u is empty and for all φ, $S^i_u \cdot (\varphi \Rightarrow)/S^i \cdot (\Box\varphi \Rightarrow)$ is an instance of $\mathcal{R}$. Hence case 3 and the second part of the lemma hold.

We use the abbreviations *fc* and *fcm* for *focussed* and *focussed modal*, respectively. We will only consider terminating calculi, so that the number of disjuncts in the definitions below is finite.

$$\forall^{\mathrm{fr}}pS \sim_0 \begin{cases} \forall^{\varnothing}_p S & \text{if } S \text{ is free and contains a nonboxed formula} \\ \Box\forall pS' & \text{if } S \text{ is free, } S = \Box S' \text{ and } S^a = \varnothing \\ \neg\Box\neg\forall pS' & \text{if } S \text{ is free, } S = \Box S' \text{ and } S^s = \varnothing \\ \bot & \text{otherwise.} \end{cases}$$

$$\forall^{\mathrm{fc}}pS \sim_0 \forall^{\mathrm{at}}_p S \bigvee \{\forall^R_p S \mid R \text{ an instance of a fc rule with conclusion } S\}.$$

For an instance R of a focussed modal rule $\mathcal{R}$ backwards applicable to S, define

$$\forall^R_p S \sim_0 \begin{cases} \Box\forall pS_u & \text{if } R = S_u/S \text{ and } \mathcal{R} \text{ is a r–rule} \\ \neg\Box\neg\forall pS_u & \text{if } R = S_u/S \text{ and } \mathcal{R} \text{ is a l–rule.} \end{cases}$$

For a focussed modal l–rule $\mathcal{R}$ backwards applicable to S, define

$$\forall^{\mathcal{R}}_p S \sim_0 \bigvee \{\Box\forall pS^i_u \mid \text{for some standard } p\text{–partition } S^r_u \cdot S^i_u/S^r \cdot S^i \text{ of an instance of } \mathcal{R} \text{ with } S = S^i \text{ and } S^i_u \prec S \text{ and } \mathcal{R} \text{ not applicable to } S^i_u\}.$$

For a focussed modal r–rule $\mathcal{R}$ backwards applicable to S, define

$$\forall^{\mathcal{R}}_p S \sim_0 \bigvee \{\neg\Box\neg\forall pS^i_u \mid \text{for some standard } p\text{–partition } S^r_u \cdot S^i_u/S^r \cdot S^i \text{ of an instance of } \mathcal{R} \text{ with } S = S^i \text{ and } S^i_u \prec S \text{ and } \mathcal{R} \text{ not applicable to } S^i_u\}.$$

For a focussed modal l–rule $\mathcal{R}$ not backwards applicable to S, define

$$\forall^{\overline{\mathcal{R}}}_p S \sim_0 \bigvee \{\Box\forall pS^i_u \mid \text{for some standard } p\text{–partition } S^r_u \cdot S^i_u/S^r \cdot S^i \text{ of an instance of } \mathcal{R} \text{ with } S = S^i \text{ and } S^i_u \prec S\}.$$

For a focussed modal r–rule $\mathcal{R}$ not backwards applicable to S, define

$$\forall_p^{\overline{\mathcal{R}}} S \sim_0 \bigvee \{\neg\Box\neg\forall p S_u^i \mid \text{for some standard } p\text{–partition } S_u^r \cdot S_u^i / S^r \cdot S^i \text{ of an instance of } \mathcal{R} \text{ with } S = S^i \text{ and } S_u^i \prec S\}.$$

In case the calculus contains no WLr-rules or IRE-rules, then define

$$\begin{aligned}\forall^{\mathrm{fcm}} pS \sim_0 & \bigvee\{\forall_p^R S \mid R \text{ an instance of a fcm rule with conclusion } S\} \vee \\ & \bigvee\{\forall_p^{\mathcal{R}} S \mid \mathcal{R} \text{ a fcm rule backwards applicable to } S\} \vee \\ & \bigvee\{\forall_p^{\overline{\mathcal{R}}} S \mid \mathcal{R} \text{ a fcm rule not backwards applicable to } S\}.\end{aligned}$$

And otherwise define

$$\begin{aligned}\forall^{\mathrm{fcm}} pS \sim_0 & \forall^{\mathrm{em}} pS \vee \\ & \bigvee\{\forall_p^R S \mid R \text{ an instance of a fcm rule with conclusion } S\} \vee \\ & \bigvee\{\forall_p^{\mathcal{R}} S \mid \mathcal{R} \text{ a fcm rule backwards applicable to } S\} \vee \\ & \bigvee\{\forall_p^{\overline{\mathcal{R}}} S \mid \mathcal{R} \text{ a fcm rule not backwards applicable to } S\},\end{aligned}$$

where

$$\forall^{\mathrm{em}} pS \sim_0 \begin{cases} \bigvee\{\neg\Box\neg\forall p(\Gamma \Rightarrow) \mid (\Box\Gamma \Rightarrow) \subseteq S\} & \text{if the calculus contains a } \mathsf{WLr}\text{-rule.} \\ \bigvee\{\Box\forall p(\Rightarrow \Delta) \mid (\Rightarrow \Box\Delta) \subseteq S\} & \text{if the calculus contains an } \mathsf{IRE}\text{-rule.} \\ \bot & \text{otherwise.} \end{cases}$$

The *standard assignment* in this modal setting is defined as $\forall pS \sim_0 \top$ if S is an axiom, and in all other cases:

$$\forall pS \sim_0 \forall^{\mathrm{fr}} pS \vee \forall^{\mathrm{fc}} pS \vee \forall^{\mathrm{fcm}} pS.$$

Because in the case of focussed modal rules, the uniform interpolants have become more complicated, we need to reformulate the sufficient conditions for uniform interpolation in the following way. Given a calculus and an interpolant assignment, we distinguish three properties.

(IP) If S is not free and all sequents lower than S satisfy the interpolant properties, then S satisfies the independent interpolant property.

(DPB) If all sequents lower than S satisfy the interpolant properties and S has a derivation which last inference is an instance of a rule $\mathcal{R}$, then for every partition (S^r, S^i) of S such that S^i is not empty and $\mathcal{R}$ is backwards applicable to S^i, the dependent interpolant property holds.

(DPN) If all sequents lower than S satisfy the interpolant properties and S has a derivation which last inference is an instance of a rule $\mathcal{R}$, then for every partition (S^r, S^i) of S such that S^i is not empty and $\mathcal{R}$ is not backwards applicable to S^i, the dependent interpolant property holds.

Property (IP) *holds* if it holds for all sequents S, and similarly for (DPB) and (DPN). As in Lemma 2 it is not hard to show that the above properties are sufficient for having uniform interpolation:

Lemma 14 *If a logic* L *has a terminating calculus for which there exists an interpolant assignment such that (IP), (DPB), (DBN) and (FP) hold, then all sequents satisfy the interpolant properties.*

5.1 Properties of focussed modal rules

In the following lemmas,

$$\frac{\Box S_1 \cdot S_0}{S_2 \cdot \Box S_1 \cdot \Box S_0}\ R \tag{4}$$

will always be an instance of a focussed modal rule $\mathcal{R}$, and $S_u = \Box S_1 \cdot S_0$ denotes the upper sequent and $S_l = S_2 \cdot \Box S_1 \cdot \Box S_0$ the lower sequent of R.

One of the properties that we use to prove that our calculi have uniform interpolation is that of being balanced. Given a calculus, a rule $\mathcal{R}_{\mathsf{rx}}$ or $\mathcal{R}_{\mathsf{lx}}$ is *covered* if the calculus contains an Lrx-rule or an IRx-rule. A calculus is *balanced* if it does not contain bad rules, every rule is covered, and if it contains a Wr-rule or an IE-rule, that is the only modal rule it contains. Observe that this implies that if a balanced calculus contains a Wr-rule or an IE-rule, it is the unique modal rule in the calculus and a WLr–rule or an IRE–rule, respectively.

Lemma 15 *If* (S_u/S_l) *is an instance of a focussed modal* r*-rule* $\mathcal{R}$ *in a balanced calculus, then* $S_u \cdot (\varphi \Rightarrow) \vdash S_l \cdot (\Box\varphi \Rightarrow)$ *for any formula* φ*.*

Proof As the calculus is balanced, $\mathcal{R} = \mathcal{R}_{\mathsf{xr}}$ is covered, meaning that the calculus contains an Lxr-rule or an IRx-rule. If the calculus contains an Lxr-rule, then it can be applied to $S_u \cdot (\varphi \Rightarrow)$ to obtain $S_l \cdot (\Box\varphi \Rightarrow)$. If the calculus does not contain an Lxr-rule, then $\mathcal{R}$ is not an L-rule, and thus S_0^a is empty. Therefore the IRx-rule can be applied to $S_u \cdot (\varphi \Rightarrow)$ to obtain $S_l \cdot (\Box\varphi \Rightarrow)$.

Lemma 16 *If* (S_u/S_l) *is an instance of a focussed modal* l*-rule* $\mathcal{R}$ *in a balanced calculus for a logic, then* $S_u \cdot (\varphi \Rightarrow) \vdash S_l \cdot (\neg\Box\neg\varphi \Rightarrow)$ *for any formula* φ*.*

Proof As logics contain CPC by assumption, $S_u \cdot (\varphi \Rightarrow) \vdash S_u \cdot (\Rightarrow \neg\varphi)$. As the calculus is balanced, $\mathcal{R} = \mathcal{R}_{\mathsf{lx}}$ is covered, meaning that the calculus contains an IRx-rule or an Lrx-rule. In the first case, the IRx-rule can be applied to $S_u \cdot (\Rightarrow \neg\varphi)$ to obtain $S_l \cdot (\Rightarrow \Box\neg\varphi)$. This last sequent derives the desired $S_l \cdot (\neg\Box\neg\varphi \Rightarrow)$. In the second case, $\mathcal{R}$ is not an R–rule, thus S_0^s is empty. Therefore the Lrx-rule can be applied to $S_u \cdot (\Rightarrow \neg\varphi)$ to obtain $S_l \cdot (\Rightarrow \Box\neg\varphi)$.

Lemma 17 *In any balanced terminating calculus consisting of focussed and focussed modal rules (IP) holds.*

Proof If S is an axiom, it is derivable and has to be focussed, and is therefore closed under weakening at the left. The independent interpolant property follows from this. Therefore assume S is not an axiom and not free. We have to show, under the assumption that all sequents $\prec$–lower than S satisfy the interpolant properties, that

$$\vdash S^a, \forall^{\mathrm{fr}} pS \Rightarrow S^s \quad \vdash S^a, \forall^{\mathrm{fc}} pS \Rightarrow S^s \quad \vdash S^a, \forall^{\mathrm{fcm}} pS \Rightarrow S^s.$$

This means that for every disjunct χ of $\forall^{\mathrm{fr}} pS$, $\forall^{\mathrm{fc}} pS$ and $\forall^{\mathrm{fcm}} pS$ we have to prove that

$$\vdash S^a, \chi \Rightarrow S^s. \tag{5}$$

First consider a disjunct χ of $\forall^{\mathrm{fc}} pS$. Thus $\chi = \forall^{\mathrm{at}}_p S$ or $\chi = \forall^R_p S$ for an instance R of a focussed rule. The first case is treated as in the proof of Lemma 2, and the second case as in the proof of Lemma 7, using the assumption that the premisses of R satisfy the interpolant properties.

Next suppose χ is a disjunct of $\forall^{\mathrm{fr}} p$. Since S is not free, $\chi = \bot$ and (5) trivially holds. The case that χ is a disjunct of $\forall^{\mathrm{fcm}} pS$ remains. We distinguish the following four cases.

First, we consider disjuncts χ of $\forall^{\mathrm{em}} pS$. The case that $\chi = \bot$ is trivial. We treat the case that $\chi = \neg\Box\neg\forall p(\Gamma \Rightarrow)$, where $(\Box\Gamma \Rightarrow) \subseteq S$ and the calculus contains a WLr-rule $\mathcal{R}$. The disjuncts of $\forall^{\mathrm{em}} pS$ of the form $\Box\forall p(\Rightarrow \Delta)$ can be treated in a similar way. As $\vdash \Gamma \Rightarrow \neg\forall p(\Gamma \Rightarrow)$ and $\mathcal{R}$ is an Lr-rule, it can be applied to $\Gamma \Rightarrow \neg\forall p(\Gamma \Rightarrow)$ to obtain $\vdash S^a \Rightarrow \Box\neg\forall p(\Gamma \Rightarrow), S^s$. This gives $\vdash S^a, \chi \Rightarrow S^s$which is what had to be shown.

Second, suppose $\chi = \Box\forall pS_u$, $S = S_l$ and S_u/S_l is an instance of a fcm r–rule. Thus $\vdash S_u \cdot (\forall pS_u \Rightarrow)$ by assumption. Lemma 15 implies (5). The case $\chi = \neg\Box\neg\forall pS_u$, where $S = S_l$ and S_u/S_l is an instance of a fcm l–rule, follows in a similar way from Lemma 16.

Third, $\chi = \neg\Box\neg\forall pS_u$, where $S_u \prec S$ and for some fcm r–rule $\mathcal{R}$ that is backwards applicable to S there are sequents S', S'_u such that $S'_u \cdot S_u/S' \cdot S$ is a standard p–partition of an instance of $\mathcal{R}$ and $\mathcal{R}$ is not applicable to S_u. This implies that $\vdash S_u \cdot (\forall pS_u \Rightarrow)$, and thus $\vdash S_u \cdot (\Rightarrow \neg\forall pS_u)$. As S_u/S is not an instance of $\mathcal{R}$, $\vdash S \cdot (\Rightarrow \Box\neg\forall pS_u)$ holds by Lemma 13, which is what had to be shown. The case that χ is of the form $\Box\forall pS_u$ and $\mathcal{R}$ is a fcm l–rule backwards applicable to S is similar.

In the fourth case, $\chi = \Box\forall pS_u$, where $S_u \prec S$ and for some fcm l–rule $\mathcal{R}$ that is not backwards applicable to S there are sequents S'_u, S' such that $S'_u \cdot S_u/S' \cdot S$ is a standard p–partition of an instance of $\mathcal{R}$. We have $\vdash S_u \cdot (\forall pS_u \Rightarrow)$ because (IP) holds for S_u. As S_u/S is not an instance of $\mathcal{R}$, it follows from Lemma 13 that $\vdash S \cdot (\Box\forall pS_u \Rightarrow)$, which gives (5). The case that χ is of the form $\neg\Box\neg\forall pS_u$ and $\mathcal{R}$ is a fcm r–rule not backwards applicable to S is similar.

Lemma 18 *In any balanced terminating calculus consisting of focussed and focussed modal rules (DPB) holds.*

Proof Since axioms receive $\top$ as interpolant assignment, (DPB) holds for axioms. Therefore consider a sequent S that is not an axiom, that has a derivation which last inference is an instance R of a rule $\mathcal{R}$. Let (S^r, S^i) be a p–partition of S such that S^i is not empty and $\mathcal{R}$ is backwards applicable to S^i. We have to show that

$$\vdash S^r \cdot (\Rightarrow \forall p S^i). \tag{6}$$

For focussed rules the proof of (6) is similar to the proof of Lemma 8. Therefore suppose $\mathcal{R}$ is a focussed modal rule, $R = S_u/S$, and $\mathcal{R}$ is backwards applicable to S^i and consider the standard p–partition of $\mathcal{R}$. By the assumption that all sequents lower than S satisfy the interpolant properties, $\vdash S^r_u \cdot (\Rightarrow \forall p S^i_u)$.

First, suppose S^i_u/S^i is an instance of $\mathcal{R}$. Thus $\Box\forall p S^i_u$ is a disjunct of $\forall p S^i$ if $\mathcal{R}$ is an r-rule and $\neg\Box\neg\forall p S^i_u$ is a disjunct of $\forall p S^i$ if $\mathcal{R}$ is an l-rule. Since $\vdash S^r_u \cdot (\Rightarrow \forall p S^i_u)$, for (6) it therefore suffices to show that

$$\begin{array}{ll} S^r_u \cdot (\Rightarrow \forall p S^i_u) \vdash S^r \cdot (\Rightarrow \Box\forall p S^i_u) & \text{if } \mathcal{R} \text{ is an } \mathsf{r}\text{-rule,} \\ S^r_u \cdot (\Rightarrow \forall p S^i_u) \vdash S^r \cdot (\Rightarrow \neg\Box\neg\forall p S^i_u) & \text{if } \mathcal{R} \text{ is an } \mathsf{l}\text{-rule.} \end{array}$$

If $\mathcal{R}$ is an r-rule, there are three possibilities: (1) $\mathcal{R}$ is an R-rule, (2) $\mathcal{R}$ is not an R-rule but it is a E–rule and S^{rs}_u consists of boxed formulas only, (3) $\mathcal{R}$ is not an R-rule and not a E–rule and S^{rs}_u is empty. In all three cases $\mathcal{R}$ can be applied to $S^r_u \cdot (\Rightarrow \forall p S^i_u)$ to obtain $S^r \cdot (\Rightarrow \Box\forall p S^i_u)$.

We turn to the case that S^i_u/S^i is not an instance of $\mathcal{R}$. We distinguish according to cases 2–4 in Lemma 13, one of which has to hold, and use the notation from the proof of that lemma.

2. We start with the case that S^i_u consists of boxed formulas only. We only treat the case that $\mathcal{R}$ is an l–rule, say $\mathcal{R} = \mathcal{R}_{\mathsf{lx}}$, the other case being similar. Thus we have

$$\begin{array}{ll} S^i = (\Lambda^i, \Box\Pi^i \Rightarrow \Box\Sigma^i, \Theta^i) & S^i_u = (\Box\Pi^i \Rightarrow \Box\Sigma^i) \\ S^r = (\Lambda^r, \Box\Pi^r, \Box\Gamma, \Box\varphi \Rightarrow \Box\Delta, \Box\Sigma^r, \Theta^r) & S^r_u = (\Box\Pi^r, \Gamma, \varphi \Rightarrow \Delta, \Box\Sigma^r). \end{array}$$

If $\Pi \neq \varnothing$, then $\mathcal{R}$ is a WI–rule, and as every rule is covered the calculus contains a WIR–rule or a WLr–rule. Since the first rule is bad, the second rule has to be present. But that contradicts the fact that if a WLr–rule is present it is the unique fcm rule. This proves that $\Pi = \varnothing$, and thus $S^{ia}_u = \varnothing$. As S^i_u is not empty, $\mathcal{R}$ is an IE–rule, and thus the only modal rule, because the calculus is balanced. Observe that whence all disjuncts of $\forall p S^i_u$ are boxed formulas or $\bot$, so $\forall p S^i_u$ is equivalent to a formula $\bigvee_{i=1}^n \Box\varphi_i$. Hence $\vdash S^r_u \cdot (\Rightarrow \Box\varphi_1, \ldots, \Box\varphi_n)$. An application of $\mathcal{R}$ to that sequent gives $\vdash S^r \cdot (\Rightarrow \Box\varphi_1, \ldots, \Box\varphi_n)$, which implies $\vdash S^r \cdot (\Rightarrow \forall p S^i_u)$.

Therefore, to prove (6) it suffices to show that $\forall p S^i_u$ implies $\forall p S^i$, which means, that any disjunct of $\forall p S^i_u$ implies $\forall p S^i$. Since $\mathcal{R}$ is the only modal rule the only disjuncts that need to be considered are $\forall^{\mathrm{em}} p S^i_u$ and the disjuncts of $\forall^{\overline{\mathcal{R}}}_p S^i_u$. For disjuncts of $\forall^{\mathrm{em}} p S^i_u$ it is clear that they are disjuncts of $\forall^{\mathrm{em}} p S^i$ since $S^i_u \subseteq S^i$.

Therefore consider a disjunct $\Box\forall p S_2$ of $\forall^{\overline{\mathcal{R}}}_p S^i_u$. Thus there is a standard p–partition $S_3 \cdot S_2/S_1 \cdot S^i_u$ of an instance of $\mathcal{R}$ with $S_2 \prec S^i_u$ and $S^a_2 = S^{ia}_u = \varnothing$.

Note that $S_3 \cdot S_2/S_1 \cdot S^i$ is the standard p–partition of an instance of $\mathcal{R}$ as well. And as $\mathcal{R}$ is backwards applicable to S^i and is not applicable to S_2, $\Box\forall pS_2$ is a disjunct of $\forall^{\mathcal{R}}_p S^i$ and thus of $\forall pS^i$.

This completes the proof that any disjunct $\forall pS^i_u$ is a disjunct of $\forall pS^i$, and thereby finishes case 2 of Lemma 13.

3. Suppose S^i_u/S^i is not an instance of $\mathcal{R}$, S^i_u contains nonboxed formulas, $S^{ia}_u = \varnothing$, and $\mathcal{R}$ is an I–rule. By Lemma 13 we have:

$$S^i = (\Lambda^i \Rightarrow \Box\Delta^i, \Box\Sigma^i, \Theta^i) \qquad S^i_u = (\Rightarrow \Delta^i, \Box\Sigma^i)$$
$$S^r = (\Lambda^r, \Box\Pi, \Box\Gamma, \Box\varphi \Rightarrow \Box\Delta^r, \Box\Sigma^r, \Theta^r) \quad S^r_u = (\Box\Pi, \Gamma, \varphi \Rightarrow \Delta^r, \Box\Sigma^r).$$

As Δ^i is not empty, $\mathcal{R}$ is an IR–rule. Thus we can apply $\mathcal{R}$ to $S^r_u \cdot (\Rightarrow \forall pS^i_u)$, which is derivable, and establish that $\vdash S^r \cdot (\Rightarrow \Box\forall pS^i_u)$. Since $\mathcal{R}$ is not applicable to S^i_u but is backwards applicable to S^i, $\Box\forall pS^i_u$ is a disjunct of $\forall^{\mathcal{R}}_p S^i$ and thereby of $\forall pS^i$, which implies $\vdash S^r \cdot (\Rightarrow \forall pS^i)$.

4. Similar to case 3 and therefore left to the reader.

Lemma 19 *In any balanced terminating calculus consisting of focussed and focussed modal rules (DPN) holds.*

Proof Consider a p–partition (S^r, S^i) of a derivable sequent S, where the last rule in the derivation is $\mathcal{R}$, which is not backwards applicable to S^i (whence $S^i \neq S$) and S^i is not empty. We have to show that under the assumption that all sequents lower than S satisfy the interpolant properties,

$$\vdash S^r \cdot (\Rightarrow \forall pS^i). \tag{7}$$

For $\mathcal{R}$ a focussed rule (or axiom), the proof of (7) is analogous to the proof of Lemma 10. It remains to consider the case that $\mathcal{R}$ is a focussed modal rule and the last step of the derivation of S consists of instance $R = S_u/S$, which has standard p–partition $S^r_u \cdot S^i_u/S^r \cdot S^i$. We distinguish the case that $S^i_u \prec S^i$ and that $S^i_u = S^i$. In the first case, $\Box\forall pS^i_u$ is a disjunct of $\forall^{\overline{\mathcal{R}}}_p S^i$ and thus of $\forall pS^i$ if $\mathcal{R}$ is an I-rule and $\neg\Box\neg\forall pS^i_u$ is a disjunct of $\forall pS^i$ if $\mathcal{R}$ is an r-rule. We only treat the last case, where $\mathcal{R} = \mathcal{R}_{\mathsf{xr}}$. For (7) it suffices to show that $\vdash S^r \cdot (\Rightarrow \neg\Box\neg\forall pS^i_u)$.

We have $\vdash S^r_u \cdot (\Rightarrow \forall pS^i_u)$ as S_u is derivable, and thus $S^r_u \cdot (\neg\forall pS^i_u \Rightarrow)$ is derivable as well. As the calculus is balanced, it contains an Lxr–rule or an IRx–rule. In the first case, that rule can be applied to $S^r_u \cdot (\neg\forall pS^i_u \Rightarrow)$. It yields $\vdash S^r \cdot (\Box\neg\forall pS^i_u \Rightarrow)$, from which $\vdash S^r \cdot (\Rightarrow \neg\Box\neg\forall pS^i_u)$ follows. In the second case, $\mathcal{R}$ is not an L–rule, and thus the IRx–rule can be applied to $S^r_u \cdot (\neg\forall pS^i_u \Rightarrow)$, to obtain $\vdash S^r \cdot (\Box\neg\forall pS^i_u \Rightarrow)$.

The case that $S^i_u = S^i$ remains. We only treat the case that $\mathcal{R}$ is an r–rule. Given that $\mathcal{R}$ is of the form (4), this implies that S^i is part of $\Box S_1$. As $\mathcal{R}$ is not backwards applicable to S^i, $S^i = S^i_u = (\Box\Gamma \Rightarrow)$ and $\mathcal{R}$ is a Wr–rule. Because the calculus is balanced, it is the only modal rule and a WLr–rule. This implies that $\forall pS^i = \forall pS^i_u$ is a nonempty disjunction of the form $\bigvee_{i=1}^n \neg\Box\neg\varphi_i$. Since $\vdash S^r_u \cdot (\Rightarrow \forall pS^i)$, also $S^r_u \cdot (\Box\neg\varphi_1, \ldots, \Box\neg\varphi_n \Rightarrow)$ is derivable. As $\mathcal{R}$ is an W-rule, it can be applied to this sequent to obtain $\vdash S^r \cdot (\Box\neg\varphi_1, \ldots, \Box\neg\varphi_n \Rightarrow)$, which implies $\vdash S^r \cdot (\Rightarrow \forall pS^i)$.

Lemma 20 *If a terminating calculus consists of focussed and focussed modal rules only, then (FP) holds.*

Proof Let S be a free sequent and assume that all sequents $\prec$-lower than S satisfy the interpolant properties. We have to show that S satisfies the independent interpolant property, that is, that $S^a, \forall pS \Rightarrow S^s$ is derivable. It suffices to show that for all disjuncts χ of $\forall^{\mathrm{fr}}pS, \forall^{\mathrm{fc}}pS, \forall^{\mathrm{fcm}}pS$:

$$\vdash S^a, \chi \Rightarrow S^s. \tag{8}$$

If χ is a disjunct of $\forall^{\mathrm{fc}}pS$ or a disjunct of $\forall^{\mathrm{fr}}pS$ of the form $\forall^{\varnothing}pS$, then the proof is similar to the proof of Lemma 11. If it is a disjunct of $\forall^{\mathrm{fr}}pS$ of the form $\chi = \neg\Box\neg\forall pS'$ for some $S' = (\Gamma \Rightarrow)$, then we use that $\Gamma, \forall pS' \Rightarrow$ is derivable by assumption. As $S = \Box S'$ is free, the calculus contains no I-rules, and because it is balanced it therefore contains an Lr-rule (if it contains no modal rules, the calculus is just $\mathsf{G3p}$). This rule can be applied to obtain $\vdash \Box\Gamma \Rightarrow \Box\neg\forall pS'$, and thereby $\vdash (\Box\Gamma, \chi \Rightarrow)$. The case that $\chi = \Box\forall pS'$ for some $S' = (\Rightarrow \Delta)$ is analogous.

Thus $\forall^{\mathrm{fcm}}pS$ remains, where χ has to be a disjunct of $\forall^{\overline{\mathcal{R}}}pS$ for some rule $\mathcal{R}$ such that $S^r_u \cdot S^i_u / S^r \cdot S^i$ is a standard p–partition of an instance of it, where $S = S^i$. We distinguish the case that $S^i_u \prec S^i$ and that $S^i_u = S^i$. In the first case we treat the case that $\mathcal{R}$ is an r–rule, the case that it is an I–rule is similar. Thus $\mathcal{R} = \mathcal{R}_{\mathsf{xr}}$ and $\chi = \neg\Box\neg\forall pS^i_u$, and S^{is}_u is empty as $\mathcal{R}$ is not backwards applicable to S^i. Since all sequents $\prec$-lower than S satisfy the interpolant properties, $S^i_u \cdot (\forall pS^i_u \Rightarrow)$ is derivable. Because $\mathcal{R}$ is an r–rule applicable to $S^r_u \cdot S^i_u$ and S^{is}_u is empty, it is applicable to $S^i_u \cdot (\Rightarrow \neg\forall pS^i_u)$. This yields $\vdash S^i \cdot (\Rightarrow \Box\neg\forall pS^i_u)$, from which $\vdash S^i \cdot (\neg\Box\neg\forall pS^i_u \Rightarrow)$ follows.

The case that $S^i_u = S^i$ remains. We only treat the case that $\mathcal{R}$ is an I–rule, the other case is treated in the last case of Lemma 19, to which this case is analogous. Given that $\mathcal{R}$ is of the form (4), this implies that S^i is part of $\Box S_1$. And as $\mathcal{R}$ is not backwards applicable to S^i, $S^i = S^i_u = (\Rightarrow \Box\Delta)$ and $\mathcal{R}$ is an E–rule. Since the calculus is balanced, it therefore is the only modal rule and of the form $\mathcal{R}_{\mathsf{IREx}}$. This implies that $\forall pS^i = \forall pS^i_u$ is a nonempty disjunction of the form $\bigvee_{i=1}^n \Box\varphi_i$. Since $\vdash S^r_u \cdot (\Rightarrow \forall pS^i)$, also $S^r_u \cdot (\Rightarrow \Box\varphi_1, \ldots, \Box\varphi_n)$ is derivable. As $\mathcal{R}$ is an E-rule, it can be applied to this sequent to obtain $\vdash S^r \cdot (\Rightarrow \Box\varphi_1, \ldots, \Box\varphi_n)$, which implies $\vdash S^r \cdot (\Rightarrow \forall pS^i)$.

Since any calculus that consists of focussed and focussed modal rules is closed under weakening, the previous lemmas imply the following two theorems.

Theorem 4 *For any balanced terminating calculus that consists of focussed and focussed modal rules, (IP), (DPB), (DBN) and (FP) hold for the standard interpolant assignment.*

Theorem 5 *A logic* $\mathsf{L} \supseteq \mathsf{CPC}$ *with a balanced terminating calculus that consists of focussed and focussed modal rules has uniform interpolation.*

6 Applications

In order to apply Theorem 5 it first has to be established that many standard calculi for modal logics are terminating. For this we use the following *weight* function on modal formula from [1]: $\varphi \prec \psi \equiv_{def} w(\varphi) < w(\psi)$, where

$$\begin{array}{l} w(p) = w(\bot) = 1 \\ w(\varphi \circ \psi) = w(\varphi) + w(\psi) + 1 \quad \circ \in \{\wedge, \vee, \rightarrow\} \\ w(\Box\varphi) = w(\neg\varphi) = w(\varphi) + 1. \end{array}$$

The weight $w(\Gamma)$ of a multiset Γ is the sum of the weights of the formula occurrences in Γ, and the weight $w(S)$ of a sequent is $w(S^a \cup S^s)$. It is easy to see that calculi consisting of G3p plus some focussed modal rules are terminating with respect to this order.

Next, we want to guarantee that our calculi G3K and G3KD are indeed calculi for the modal logics K and KD. To that end we prove that the restricted focussed modal rules $\mathcal{R}$ are in fact not weaker than their full version $\mathcal{R}^+$, which are defined in the first part of Section 5.

Lemma 21 *For any extension* G *of* G3p *that consists of focussed and focussed modal rules, the calculus* G *extended by* $\mathcal{R}$ *is equal to* G *extended by* $\mathcal{R}^+$.

Proof We have to show that for any calculus G that consists of focussed rules and focussed modal rules $\mathcal{R}$ for which $\mathcal{R}^+ = \mathcal{R}$, for any number of rules $\mathcal{R}_1, \ldots, \mathcal{R}_n$ such that $\mathcal{R}_i \neq \mathcal{R}_i^+$ for all i, G extended by $\mathcal{R}_1, \ldots, \mathcal{R}_n$ is equal to G extended by $\mathcal{R}_1^+, \ldots, \mathcal{R}_n^+$. Let $\vdash$ denote derivability in G extended by $\mathcal{R}_1, \ldots, \mathcal{R}_n$. It suffices to prove with induction to the length d of derivations that if $\vdash S$, then $\vdash S \cdot (\Rightarrow \varphi)$ and $\vdash S \cdot (\varphi \Rightarrow)$ for any sequent S and any formula φ. We prove both cases simultaneously and use a subinduction to φ, but we only treat the case $(\Rightarrow \varphi)$ explicitly. Let $S' = (\Rightarrow \varphi)$.

If S is an instance of an axiom, then because the axiom has to be focussed, $S \cdot S'$ is an instance of that axiom too. Suppose that S has a proof of depth $d > 1$. If the last rule is not one of $\mathcal{R}_i$, then the derivability of $S \cdot S'$ follows from the induction hypothesis and the fact that the rules of G are closed under weakening. Therefore suppose it is an instance of one of the $\mathcal{R}_h$, where $S = S_2 \cdot \Box S_1 \cdot \Box S_0$ and the premiss is $\Box S_1 \cdot S_0$. Thus for $i = 0$ or $i = 1$ (or both): S_i^a or S_i^s contains a meta–multiset. We show that $\vdash S \cdot S'$ with induction to $w(\varphi)$.

If φ is an atom, then $\vdash S \cdot S'$ follows by an application of $\mathcal{R}_h$ to $\Box S_1 \cdot S_0$. Therefore suppose that φ is not an atom and that $\vdash S \cdot S''$ has been shown for all $S'' = (\Rightarrow \psi)$ and $S'' = (\psi \Rightarrow)$ with $w(\psi) < w(\varphi)$. If φ is a conjunction $\psi \wedge \chi \in S^a$, then $\vdash S \cdot (S'^a \backslash \{\chi\}, \psi, \chi \Rightarrow S'^s)$ by the induction hypothesis. Thus an application of the left conjunction rule implies that $\vdash S \cdot S'$. Likewise for $\psi \wedge \chi \in S^s$ and for all other connectives. Finally, suppose $\varphi = \Box\psi$. Since $\mathcal{R}_h$ is not equal to $\mathcal{R}_h^+$, there is an $x \in \{a, s\}$ such that S_2^x ranges over multisets of atoms, and either S_1^x or S_0^x contains a multiset. Thus $\mathcal{R}$ is an R–rule or an E–rule. We treat the case that $x = s$. By the induction hypothesis the following

two sequents are derivable.

$$\Box S_1 \cdot S_0 \cdot (\Rightarrow \psi) \qquad \Box S_1 \cdot S_0 \cdot (\Rightarrow \Box\psi)$$

In case $\mathcal{R}$ is an R–rule we apply it to the first sequent to establish $\vdash S \cdot S'$ and if $\mathcal{R}$ is an E–rule we apply it to the second sequent.

Despite the above lemma, not all calculi consisting of focussed and focussed modal rules correspond to a logic, because the cut rule is not always admissible in them. An example is the logic $\mathsf{G3p} + \mathcal{R}_{\mathsf{WLr}}$. Below it is explained that contraction is not admissible in it. Hence neither is the cut rule.

For the application of Theorem 5 the next issue that needs to be addressed is which of the calculi that consist of $\mathsf{G3p}$ (for the modal language) plus some focussed modal rules are balanced. There are 32 calculi that contain exactly one focussed modal rule, for which calculi it is easy to establish whether they are balanced.

Theorem 6 *Every calculus that consists of* $\mathsf{G3p}$ *plus a single rule* $\mathcal{R}_{\mathsf{x}}$*, where* x *contains* Lr *or* IR *but not* LrE *and not* WIR*, has uniform interpolation.*

For calculi that contain more than one focussed modal rule the situation is more complex. Here we restrict ourselves to calculi with two focussed modal rules where one is an extension of $\mathcal{R}_{\mathsf{K}}$. The following theorem follows immediately from the properties of balanced calculi.

Theorem 7 *If* $\mathcal{R}_1 = \mathcal{R}_{\mathsf{Lrx}}$ *and* $\mathcal{R}_2$ *are two different fcm rules and* $\mathsf{G3p}$ *plus these two rules is a balanced caculus, then either* x *is empty or equal to* R *and* $\mathcal{R}_2$ *is one of* $\mathcal{R}_{\mathsf{l}}, \mathcal{R}_{\mathsf{lR}}, \mathcal{R}_{\mathsf{r}}$*. Thus for these rules the calculus has uniform interpolation.*

Similar results can be obtained for extensions of IR-rules, as well as for calculi with more than two focussed modal rules.

The terminating calculus $\mathsf{G3K}$ for the modal logic K consists of $\mathsf{G3p}$ (for the modal language) extended by the rule $\mathcal{R}_{\mathsf{K}} = \mathcal{R}_{\mathsf{Lr}}$ below, and similarly for the calculus $\mathsf{G3D}$. $\mathsf{G3KD}$ stands for the calculus $\mathsf{G3K}$ extended by $\mathcal{R}_{\mathsf{D}} = \mathcal{R}_{\mathsf{Ll}}$.

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma', \Box\Gamma \Rightarrow \Box\varphi, \Delta}\ \mathcal{R}_{\mathsf{K}} \qquad \frac{\Gamma, \varphi \Rightarrow}{\Gamma', \Box\Gamma, \Box\varphi \Rightarrow \Delta}\ \mathcal{R}_{\mathsf{D}}$$

The versions $\mathcal{R}_{\mathsf{K}}^-$ and $\mathcal{R}_{\mathsf{D}}^-$ of the above rules in which Γ' ranges over multisets of atoms are focussed modal rules, and in Lemma 21 it is shown that $\mathsf{G3p}+\mathcal{R}_{\mathsf{K}}^-$ and $\mathsf{G3p} + \mathcal{R}_{\mathsf{K}}^- + \mathcal{R}_{\mathsf{D}}^-$ are equivalent to $\mathsf{G3K}$ and $\mathsf{G3KD}$, respectively. As a corollary of Theorem 6 and Theorem 7 we therefore obtain the following results, where the result about K was first proved by Bílková [1].

Theorem 8 K *and* KD *have uniform interpolation.*

$\mathsf{S4}$ is known to have no uniform interpolation [4], and the same holds for $\mathsf{K4}$ [1]. This leads to the following observation.

Theorem 9 *Neither* K4 *nor* S4 *is sound and complete with respect to a balanced terminating extension of* G3p *by focussed and focussed modal rules.*

The theorem implies that the calculus $\mathsf{G3p} + \mathcal{R}_{\mathsf{WLr}}$, which is G3p extended by rule

$$\frac{\Box\Pi, \Gamma \Rightarrow \varphi}{\Lambda, \Box\Pi, \Box\Gamma \Rightarrow \Box\varphi, \Theta}\ \mathcal{R}_{\mathsf{WLr}},$$

cannot be a calculus for K4. And because $\mathsf{G3p} + \mathcal{R}_{\mathsf{WLr}}$ would be equivalent to one of the standard calculi for K4, were it closed under contraction, the calculus $\mathsf{G3p} + \mathcal{R}_{\mathsf{WLr}}$ cannot be closed under contraction. Thus it is a calculus for a substructural logic. This in contrast to rules that are not WL–rules or RE–rules: extending G3p by such rules preserves the admissibility of contraction and weakening.

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