On interpolation in existence logics

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Abstract. In [2] Gentzen calculi for intuitionistic logic extended with an existence predicate were introduced. Such logics were first introduced by Dana Scott, who provided a proof system for it in Hilbert style. The logic seems particularly useful in settings where non constant domain Kripke models play a role. In this paper it is proved that these systems have interpolation and the Beth definability property.

Keywords: Intuitionistic logic, existence predicate, Gentzen calculus, interpolation, Beth definability, cut-elimination, Skolemization, truth-value logics, Gödel logics, Scott logics, Kripke models.

1 Introduction

In this paper we prove that existence logics have interpolation and satisfy the Beth definability property (Corollary 2 and Theorem 5). Existence logics are extensions of intuitionistic predicate logic IQC with an existence predicate E, where the intuitive meaning of Et is that t *exists*[1].

Recall that we say that a single conclusion Gentzen calculus L has *interpolation* if whenever $\mathsf{L} \vdash \Gamma_1, \Gamma_2 \Rightarrow C$, there exists an I in the common language of Γ_1 and $\Gamma_2 \cup \{C\}$ such that

$$\Gamma_1 \vdash_{\mathsf{L}} I \quad \text{and} \quad I, \Gamma_2 \vdash_{\mathsf{L}} C.$$

In the context of existence logics, the *common language* of two multisets Γ_1 and Γ_2, denoted by $\mathcal{L}(\Gamma_1, \Gamma_2)$, consists of all variables, $\top$, $\bot$ and E, and all predicates and non-variable terms that occur both in Γ_1 and Γ_2.

We say that a Gentzen calculus L satisfies the *Beth definability property* if whenever $A(R)$ is a formula with R an n-ary relation symbol in a language $\mathcal{L}$, and R', R'' are two relation symbols not in $\mathcal{L}$ such that

$$\mathsf{L} \vdash A(R') \wedge A(R'') \Rightarrow \forall \bar{x}(R'\bar{x} \leftrightarrow R''\bar{x}),$$

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[1] For a more computational view replace “t exists” by “the evaluation of t terminates”. Universal (existential) statements express consequently weak (strong) correctness properties.

then there is a formula S in $\mathcal{L}$ such that

$$\mathsf{L} \vdash \Rightarrow \forall \bar{x}(S\bar{x} \leftrightarrow R'\bar{x}).$$

Existence logic IQCE was first introduced by D. Scott in [12], where he presented a Hilbert style proof system for the logic. The motivation behind these logics is that in the context of intuitionistic logic it is natural to be able to denote whether a term exists or not. In this system both variables and terms range over arbitrary objects while the quantifiers are assumed to range over existing objects only. Existence logic in which terms range over all objects while quantifiers as well as variables only range over existing objects is denoted by IQCE^+ and has e.g. been used by M. Beeson in [4]. M. Unterhalt thoroughly studied the Kripke semantics of these logics and proved respectively completeness and strong completeness for the systems IQCE and IQCE^+ in [18]. In [2] Gentzen calculi for existence logics were introduced and proved to have cut-elimination. Completeness results for these systems are presented in [1]. Applications that use existence logic are discussed below.

The Gentzen calculi that we introduce in this paper are called LJE and $\mathsf{LJE}(\Sigma_{\mathcal{L}})$, which is LJE extended by axioms $\Sigma_{\mathcal{L}}$, to be defined below. LJE corresponds to Scott's IQCE, and for a specific $\Sigma_{\mathcal{L}}$ the calculus $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ corresponds to IQCE^+ as explained in Section 4.2.

1.1 Applications

Existence logic has many applications, and sometimes leads to surprising solutions of problems that do not seem solvable in pure intuitionistic logic. We do not describe these applications in full detail here, but we try to explain the general idea and give pointers to the literature.

Truth-value logics and linear frames One application of the existence predicate is in the context of truth-value logics. These are logics based on truth-value sets V, i.e. closed subsets of the unit interval $[0, 1]$, also called *Gödel sets*. One can, for a given Gödel set V, interpret formulas by mapping them to elements of V. The logical symbols receive a meaning via restrictions on these interpretations, e.g. by stipulating that the interpretation of $\wedge$ is the infimum of the interpretations of the respective conjuncts, or that the interpretation of $\exists x Ax$ is the supremum of the values of Aa for all elements a in the domain. Given these interpretations, one can associate a logic with such a Gödel set V: the logic of all sentences that are mapped to 1 under any interpretation on V.

Gödel logics G_V are an example of truth value logics. Without going into the precise definition of these logics here, we only want to mention that these logics naturally correspond to the logics of linear frames. As has been shown by A. Beckmann and N. Preining this correspondence takes the following form.

Theorem 1. *(A. Beckmann and N. Preining [3]) For every countable linear frame F there exists a Gödel set V such that*

$$G_V \models A \Leftrightarrow A \text{ holds in all Kripke models on } F \text{ with constant domains}, \quad (1)$$

and vice versa: for every Gödel set V there exists a countable linear frame F such that (1).

In [9] so-called *Scott logics* S_V are introduced which correspond to linear frames, but now for possibly non constant domains. That is, we have

Theorem 2. *[9] For every countable linear frame F there exists a Gödel set V such that*

$$S_V \models A \;\Leftrightarrow\; A \textit{ holds in all Kripke models based on } F, \tag{2}$$

and vice versa: for every countable Gödel set V there exists a countable linear frame F such that (2).

In the same paper it is shown that there is a natural and faithful translation from Scott logics into Gödel logics. This translation $(\cdot)^e$, that makes use of the existence predicate, allows to transfer properties about Gödel logics to Scott logics. $(\cdot)^e$ is defined as follows.

$\big(P(\bar{t})\big)^e = P(\bar{t})$ for atomic P and terms $\bar{t}$,
$(\cdot)^e$ commutes with the connectives,
$(\exists x A(x))^e = \exists x\big(Ex \wedge (A(x))^e\big)$,
$(\forall x A(x))^e = \forall x\big(Ex \to (A(x))^e\big)$.

Given this translation we then have the following theorem.

Lemma 1. *[9] For any Gödel set V, $(\cdot)^e$ is a faithful translation of S_V into G_V, i.e. for all $\mathcal{L}$-sentences A*

$$S_V \models A \;\Leftrightarrow\; G_V^e \models A^e.$$

Skolemization Another application of the existence predicate is in the setting of Skolemization. As is well-known, the classical Skolemization method of replacing strong quantifiers in a formula by fresh function symbols and thus obtaining a equiconsistent formula, is not complete with respect to IQC. That is, there are formulas that are underivable, but for which their Skolemized version is derivable in IQC. For example,

$$\mathsf{IQC} \nvdash \forall x(Ax \vee B) \to (\forall x Ax \vee B) \qquad \mathsf{IQC} \vdash \forall x(Ax \vee B) \to (Ac \vee B).$$

In [1] an alternative Skolemization method called *eSkolemization* is introduced and is shown to be sound and complete with respect to IQC for a class of formulas larger than the class of formulas for which standard Skolemization is sound and complete. This eSkolemization method makes use of the existence predicate. It replaces negative occurrences of existential quantifiers $\exists x Bx$ by $(Ef(\bar{y}) \wedge Bf(\bar{y}))$, and positive occurrences of universal quantifiers $\forall x Bx$ by $(Ef(\bar{y}) \to Bf(\bar{y}))$. For example, the eSkolemization of the displayed formula above is

$$\mathsf{IQCE} \nvdash \forall x(Ax \vee B) \to ((Ec \to Ac) \vee B).$$

We will not proceed with the topic of eSkolemization here but refer the interested reader to [1] instead.
Note the similarity between the different applications of the existence predicate: the translation $(\cdot)^e$ does a similar thing to quantifiers as eSkolemization does. Essentially, it all has to do with the fact that an existence predicate allows us in a Kripke model to name objects that do not exist in the root but come into existence only at a later stage in the model. Both [1] and [9] describe this intuition in more detail.

2 Preliminaries

We consider languages $\mathcal{L} \subseteq \mathcal{L}'$ for intuitionistic predicate logic plus the existence predicate E, without equality. For convenience we assume that $\mathcal{L}$ contains at least one constant and no variables, and that $\mathcal{L}'$ contains infinitely many variables. The languages may or may not contain functional symbols: the results in this paper hold for all cases. The reason for this has to do with the semantics for the Gentzen calculi; a topic we will not proceed with here, but which is discussed in [1].
The languages contain $\bot$, and $\neg A$ is defined as $A \to \bot$. $A, B, C, D, E, ..$ range over formulas in $\mathcal{L}'$, $s, t, ..$ over terms in $\mathcal{L}'$. Γ, Δ, Π range over multisets of formulas in $\mathcal{L}'$. Sequents are expressions of the form $\Gamma \Rightarrow C$, where Γ is a finite multiset. A sequent is in $\mathcal{L}$ if all its formulas are in $\mathcal{L}$. And similarly for $\mathcal{L}'$. A formula is *closed* when it does not contain free variables. A sequent $\Gamma \Rightarrow C$ is closed if C and all formulas in Γ are closed. For terms t and s $A[t/s]$ denotes the result of substituting t for all occurrences (for all free occurrences if s is a variable) of s in A. If for a formula $A(x)$ we write $A(t)$, this indicates the result of replacing some, possibly not all, occurrences of x in A with t. This is a subtlety overlooked in most textbooks, but not in [14], when defining e.g. the right introduction of $\exists$ as

$$\frac{\Gamma \Rightarrow A(t)}{\Gamma \Rightarrow \exists x A[x/t]}$$

In such a system, $R(t,t) \Rightarrow \exists x R(x,t)$ is not derivable. Because of this most proofs of interpolation, although correct, seem to overlook that subtle point.
In one of the final proof systems, $(\Rightarrow Et)$ will hold for the terms in $\mathcal{L}$, but not necessarily for the terms in $\mathcal{L}' \backslash \mathcal{L}$. $\mathcal{T}_{\mathcal{L}}$ denotes the set of terms in $\mathcal{L}$, $\mathcal{F}_{\mathcal{L}}$ denotes the set of formulas in $\mathcal{L}$, $\mathcal{S}_{\mathcal{L}}$ denotes the set of sequents in $\mathcal{L}$, and similarly for $\mathcal{L}'$.
In order not to drown in brackets we often write Ax for $A(x)$.

3 The proof system

In this section we define the system LJE, a conservative extension of LJ for $\mathcal{L}'$ that covers the intuition that Et means t *exists*. Such a system was first introduced

by Dana Scott in [12], but then in a Hilbert style axiomatization, and called IQCE. The Gentzen calculus for this system was first introduced in [2].
Given an existence predicate, terms, including variables, typically range over existing as well as non-existing elements, while the quantifiers range over existing objects only. Proofs are assumed to be trees.

The system LJE

$$Ax \quad \Gamma, P \Rightarrow P \quad P \text{ atomic} \qquad\qquad L\bot \quad \Gamma, \bot \Rightarrow C$$

$$\text{L}\wedge \; \frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \qquad\qquad \text{R}\wedge \; \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\text{L}\vee \; \frac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C} \qquad\qquad \text{R}\vee \; \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1} \; i = 0, 1$$

$$\text{L}\!\to \; \frac{\Gamma, A \to B \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \to B \Rightarrow C} \qquad\qquad \text{R}\!\to \; \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

$$\text{L}\forall \; \frac{\Gamma, \forall x A x, A t \Rightarrow C \qquad \Gamma, \forall x A x \Rightarrow E t}{\Gamma, \forall x A x \Rightarrow C} \qquad\qquad \text{R}\forall \; \frac{\Gamma, E y \Rightarrow A y}{\Gamma \Rightarrow \forall x A[x/y]} \; *$$

$$\text{L}\exists \; \frac{\Gamma, E y, A y \Rightarrow C}{\Gamma, \exists x A[x/y] \Rightarrow C} \; * \qquad\qquad \text{R}\exists \; \frac{\Gamma \Rightarrow A t \qquad \Gamma \Rightarrow E t}{\Gamma \Rightarrow \exists x A x}$$

$$\text{Cut} \; \frac{\Gamma \Rightarrow A \qquad \Gamma, A \Rightarrow C}{\Gamma \Rightarrow C}$$

Where $(*)$ denotes the condition that y does not occur free in Γ and C.
The principal formula of a rule is defined as usual. In the Cut rule the formula A is called the cut formula, and it is the principal formula of the Cut rule. The formulas Et and Ey are not principal in respectively L$\forall$, R$\exists$ and R$\forall$, L$\exists$.
We write $\mathsf{LJE} \vdash S$ if the sequent S is derivable in LJE. For a set of sequents X and a sequent S, we say that S *is derivable from* X *in* LJE, and write $X \vdash_{\mathsf{LJE}} S$, if S is derivable in the system LJE to which the sequents in X are added as initial sequents. We also denote this system by $\mathsf{LJE}(X)$.
In the system LJE no existence of any term that is not a variable is assumed. This implies e.g. that we cannot derive $\forall x P x \Rightarrow P t$, but only $\forall x P x, E t \Rightarrow P t$. Note however that the former is derivable in LJE from $(\Rightarrow Et)$. This is the reason why we consider derivations from extra axioms, especially axioms of the form $(\Rightarrow Et)$. Therefore, we define the following sets of sequents

$$\Sigma_{\mathcal{L}} \equiv_{def} \{\Gamma \Rightarrow Et \mid t \in \mathcal{T}_{\mathcal{L}}, \Gamma \text{ a multiset}\}.$$

Note that because of the assumptions on $\mathcal{L}$, $\Sigma_{\mathcal{L}}$ contains at least one sequent and for all sequents $\Gamma \Rightarrow Et$ in $\Sigma_{\mathcal{L}}$, t is a closed term. Given two languages $\mathcal{L} \subseteq \mathcal{L}'$, we write

$$\mathsf{LJE}(\Sigma_{\mathcal{L}}) \equiv_{def} \{S \in \mathcal{S}_{\mathcal{L}'} \mid \Sigma_{\mathcal{L}} \vdash_{\mathsf{LJE}} \Rightarrow S\}.$$

The $\mathcal{L}'$ is not denoted in $\mathsf{LJE}(\Sigma_{\mathcal{L}})$, but most of the time it is clear what is the "larger" language $\mathcal{L}'$ of which $\mathcal{L}$ is a subset.

Example 1.

$$\nvdash_{\mathsf{LJE}} \Rightarrow \exists x Ex \qquad \vdash_{\mathsf{LJE}} \Rightarrow \forall x Ex.$$

$$\vdash_{\mathsf{LJE}(\Sigma_{\mathcal{L}})} \Rightarrow \exists x Ex \wedge \forall x Ex.$$

Lemma 2. *For all sequents S in $\mathcal{L}$ that do not contain E:*

$$\mathsf{LJ} \vdash S \textit{ implies } \mathsf{LJE}(\Sigma_{\mathcal{L}}) \vdash S.$$

Proof. Since S is a sequent in $\mathcal{L}$, we may assume w.l.o.g. that when S is provable in LJ it has a cut free proof in which all terms that are not eigenvariables are terms in $\mathcal{L}$. Denote this set of terms by X. Clearly, $X^s = \{\Gamma \Rightarrow Et \mid t \in X\}$ is a subset of $\Sigma_{\mathcal{L}}$. At every application of R$\exists$ or L$\forall$, add the appropriate $\Gamma \Rightarrow Et$ as the right hypothesis. At every application of R$\forall$ or L$\exists$ add the appropriate Ey to the antecedent. This gives a proof of $\Gamma \Rightarrow A$ in $\mathsf{LJE}(\Sigma_{\mathcal{L}})$.

Later on, in Proposition 1, we will see that the converse of the above lemma holds too.

4 Cut elimination

In this section we recall some results from [2] that show that LJE and $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ have a restricted form of cut elimination and have weakening and contraction. Some of these results we will need later on, the others are recalled to show that the systems we consider are well-behaved. The proofs of these results are more or less straightforward, where the ECut theorem, which shows that the systems allow some partial cut-elimination, is the most involved, as usual.

Lemma 3. *(*Substitution Lemma*)*
For $\mathsf{L} \in \{\mathsf{LJE}(\Sigma_{\mathcal{L}}), \mathsf{LJE}\}$*:*
If P is a proof in L *of a sequent S in $\mathcal{L}'$ and y is a free variable in P, and t is a term in $\mathcal{L}'$ that does not contain eigenvariables or bound variables of P, then $P[t/y]$ is a proof of $S[t/y]$ in* L.
In case $\mathsf{L} = \mathsf{LJE}$ *the same holds for any term s instead of y.*

Lemma 4. *([2]) (*Weakening Lemma*)*
For $\mathsf{L} \in \{\mathsf{LJE}(\Sigma_{\mathcal{L}}), \mathsf{LJE}\}$*:* $\mathsf{L} \vdash \Gamma \Rightarrow C$ *implies* $\mathsf{L} \vdash \Gamma, A \Rightarrow C$.

Lemma 5. *([2]) (*Contraction Lemma*)*
For $\mathsf{L} \in \{\mathsf{LJE}(\Sigma_{\mathcal{L}}), \mathsf{LJE}\}$*:* $\mathsf{L} \vdash \Gamma, A, A \Rightarrow C$ *implies* $\mathsf{L} \vdash \Gamma, A \Rightarrow C$.

Theorem 3. *([2]) (*ECut theorem*)*
For $\mathsf{L} \in \{\mathsf{LJE}(\Sigma_{\mathfrak{L}}), \mathsf{LJE}\}$*: Every sequent in* $\mathfrak{L}'$ *provable in* L *has a proof in* L *in which the only cuts are instances of the ECut rule:*

$$ECut\colon \frac{\Gamma \Rightarrow Et \in \Sigma_{\mathfrak{L}} \qquad \Gamma, Et \Rightarrow C}{\Gamma \Rightarrow C}$$

In particular, LJE *has cut-elimination.*

Corollary 1. *([2])* $\mathsf{LJE}(\Sigma_{\mathfrak{L}})$ *is consistent.*

The cut elimination theorem allows us to prove the following correspondence between LJ and $\mathsf{LJE}(\Sigma_{\mathfrak{L}})$.

Proposition 1. *([2]) For all closed sequents* S *in* $\mathfrak{L}$ *not containing* E*:*

$$\mathsf{LJ} \vdash S \textit{ if and only if } \mathsf{LJE}(\Sigma_{\mathfrak{L}}) \vdash S.$$

4.1 Uniqueness

Observe that given another predicate E' that satisfies the same rules of LJE as E', it follows that

$$\mathsf{LJE}(\Sigma_{\mathfrak{L}}) \vdash Et \Rightarrow E't \;\wedge\; \mathsf{LJE}(\Sigma_{\mathfrak{L}}) \vdash E't \Rightarrow Et.$$

Namely, $\mathsf{LJE}(\Sigma_{\mathfrak{L}}) \vdash (\Rightarrow (\forall x Ex \wedge \forall x E'x))$, and $\mathsf{LJE}(\Sigma_{\mathfrak{L}}) \vdash (\forall x Ex, E't \Rightarrow Et)$ and $\mathsf{LJE}(\Sigma_{\mathfrak{L}}) \vdash (\forall x E'x, Et \Rightarrow E't)$. Finally, two cuts do the trick. This shows that the existence predicate E is unique up to provable equivalence.

4.2 IQCE and IQCE^+

As remarked above, given an existence predicate, terms typically range over existing as well as non-existing elements, while quantifiers range over existing objects only. As to the choice of the domain for the variables, there have been different approaches. Scott in [12] introduces a system IQCE for the predicate language with the distinguished predicate E, in which variables range over all objects, like in LJE and $\mathsf{LJE}(\Sigma_{\mathfrak{L}})$. On the other hand, Beeson in [4] discusses a system in which variables range over existing objects only.
The formulation of the system IQCE in [12], where logic with an existence predicate was first introduced was in Hilbert style, where the axioms and rules for the quantifiers are the following:

$$\forall x Ax \wedge Et \to At \qquad \frac{\begin{array}{c}\vdots\\ B \wedge Ey \to Ay\end{array}}{B \to \forall x Ax}\,*$$

$$\frac{\begin{array}{c}\vdots\\ Ay \wedge Ey \to B\end{array}}{\exists x Ax \to B}\,* \qquad At \wedge Et \to \exists x Ax$$

Here $*$ are the usual side conditions on the eigenvariable y.

The following formulation of IQCE in natural deduction style was given in [16]. We call the system NDE (Natural Deduction Existence). It consists of the axioms and quantifier rules of the standard natural deduction formulation of IQC (as e.g. given in [16]), where the quantifier rules are replaced by the following rules:

$$\forall\mathrm{I}\ \frac{\begin{array}{c}[Ey]\\ \vdots\\ Ay\end{array}}{\forall x Ax}\ * \qquad \forall\mathrm{E}\ \frac{\begin{array}{c}\vdots\\ \forall x Ax\end{array}\quad \begin{array}{c}\vdots\\ Et\end{array}}{At}$$

$$\exists\mathrm{I}\ \frac{\begin{array}{c}\vdots\\ At\end{array}\quad \begin{array}{c}\vdots\\ Et\end{array}}{\exists x Ax} \qquad \exists\mathrm{E}\ \frac{\begin{array}{c}\vdots\\ \exists x Ax\end{array}\quad \begin{array}{c}[Ay][Ey]\\ \vdots\\ C\end{array}}{C}\ *$$

Again, the $*$ are the usual side conditions on the eigenvariable y. It is easy to see that the following holds.

Fact $\forall A \in \mathcal{F}_{\mathcal{L}'}$: $\vdash_{\mathsf{IQCE}} A$ if and only if $\vdash_{\mathsf{NDE}} A$ if and only if $\vdash_{\mathsf{LJE}} \Rightarrow A$.

Existence logic in which terms range over all objects while quantifiers and variables only range over existing objects is denoted by IQCE^+ and has e.g. been used by M. Beeson in [4]. The logic is the result of leaving out Ey in the two rules for the quantifiers in IQCE given above and adding Ex as axioms for all variables x. A formulation in natural deduction style is obtained from NDE by replacing the $\forall$I and $\exists$E by their standard formulations for IQC and adding Ex as axioms for all variables x. We call the system NDE^+. In this case we have the following correspondence.

Fact $\forall A \in \mathcal{F}_{\mathcal{L}'}$:
$\vdash_{\mathsf{IQCE}^+} A$ iff $\vdash_{\mathsf{NDE}^+} A$ iff $\{\Gamma \Rightarrow Ex \mid x$ a variable, Γ a multiset$\} \vdash_{\mathsf{LJE}(\Sigma_{\mathcal{L}})} \Rightarrow A$.

M. Unterhalt in [18] thoroughly studied the Kripke semantics of these logics and proved respectively completeness and strong completeness for the systems IQCE and IQCE^+. Similar results for the Gentzen calculi presented here can be found in [1].

5 Interpolation

In this section we prove that the calculus LJE and $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ have interpolation. To this end we use a calculus LJE' that is equivalent to LJE but in which the structural rules are not hidden.

The system LJE'

$$Ax \quad P \Rightarrow P \qquad P \text{ atomic} \qquad\qquad L\bot \quad \bot \Rightarrow C$$

$$\text{LW}\ \frac{\Gamma \Rightarrow C}{\Gamma, A \Rightarrow C} \qquad\qquad \text{LC}\ \frac{\Gamma, A, A \Rightarrow C}{\Gamma, A \Rightarrow C}$$

$$\text{L}\wedge\ \frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \qquad\qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\text{L}\vee\ \frac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C} \qquad\qquad \text{R}\vee\ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1}\ i = 0, 1$$

$$\text{L}\!\rightarrow\ \frac{\Gamma \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \rightarrow B \Rightarrow C} \qquad\qquad \text{R}\!\rightarrow\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}$$

$$\text{L}\forall\ \frac{\Gamma, At \Rightarrow C \qquad \Gamma \Rightarrow Et}{\Gamma, \forall x Ax \Rightarrow C} \qquad\qquad \text{R}\forall\ \frac{\Gamma, Ey \Rightarrow Ay}{\Gamma \Rightarrow \forall x A[x/y]}\ *$$

$$\text{L}\exists\ \frac{\Gamma, Ay, Ey \Rightarrow C}{\Gamma, \exists x A[x/y] \Rightarrow C}\ * \qquad\qquad \text{R}\exists\ \frac{\Gamma \Rightarrow At \qquad \Gamma \Rightarrow Et}{\Gamma \Rightarrow \exists x Ax}$$

$$\text{ECut:}\ \frac{\Gamma \Rightarrow Et \in \Sigma_{\mathfrak{L}} \qquad \Gamma, Et \Rightarrow C}{\Gamma \Rightarrow C}$$

The calculus $\mathsf{LJE}'(\Sigma_{\mathfrak{L}})$ is the system LJE' extended by the axioms $\Sigma_{\mathfrak{L}}$ (Section 3).

Lemma 6. *For all formulas A in $\mathfrak{L}'$:*

$$\mathsf{LJE} \vdash A \ \Leftrightarrow\ \mathsf{LJE}' \vdash A \qquad \mathsf{LJE}(\Sigma_{\mathfrak{L}}) \vdash A \ \Leftrightarrow\ \mathsf{LJE}'(\Sigma_{\mathfrak{L}}) \vdash A.$$

Proof. Use Theorem 3 and Lemmas 4 and 5.

Recall that we write $\mathfrak{L}(\Gamma_1, \Gamma_2)$ for the common language of Γ_1 and Γ_2, i.e. the language consisting of the predicates and non-variable terms that occur both in Γ_1 and Γ_2, plus $\top$, $\bot$ and E and the variables.

Theorem 4. LJE' *and* $\mathsf{LJE}'(\Sigma_{\mathfrak{L}})$ *have interpolation.*

Proof. We first prove the theorem for LJE' and then for $\mathsf{LJE}'(\Sigma_{\mathfrak{L}})$ by showing how this case can be reduced to the LJE' case. We write $\vdash$ for $\vdash_{\mathsf{LJE}'}$ in this proof. Assume $\vdash \Gamma_1, \Gamma_2 \Rightarrow C$. We look for a formula I in the common language $\mathfrak{L}(\Gamma_1, \Gamma_2 \cup \{C\})$ of Γ_1 and $\Gamma_2 \cup \{C\}$ such that

$$\vdash \Gamma_1 \Rightarrow I \quad \vdash I, \Gamma_2 \Rightarrow C. \tag{3}$$

We prove the theorem with induction to the depth d of P. Recall that the depth of a sequent in a proof is inductively defined as the sum of the depths of its upper sequents plus 1. Thus axioms have depth 1. The depth of a proof is the depth of its endsequent.
$d = 1$: P is an instance of an axiom. When the axiom is Ax we have $\Gamma_1\Gamma_2, Q \Rightarrow Q$, where Q is an atomic formula. There are two cases: we look for interpolants I and J such that

$$\vdash \Gamma_1, Q \Rightarrow I \quad \vdash I, \Gamma_2 \Rightarrow Q \qquad \text{and} \qquad \vdash \Gamma_1 \Rightarrow J \quad \vdash J, Q, \Gamma_2 \Rightarrow Q.$$

This case is trivial: take $I = Q$ and $J = \top$. The case that P is an instance of L$\bot$ is equally simple: again there are two possibilities, like above, and the interpolants are $\top$ and $\bot$.
$d > 1$. We distinguish by cases according to the last rule applied in P. If it is a LC, the last lines of P look as follows.

$$\frac{\Gamma_1\Gamma_2, A, A \Rightarrow C}{\Gamma_1\Gamma_2, A \Rightarrow C}$$

Again there are several cases: we look for interpolants

$$\vdash \Gamma_1, A \Rightarrow I \quad \vdash I, \Gamma_2 \Rightarrow C \qquad \text{and} \qquad \vdash \Gamma_1 \Rightarrow J \quad \vdash J, A, \Gamma_2 \Rightarrow C.$$

By the induction hypothesis there are interpolants I' and J' such that the sequents $\Gamma_1, A, A \Rightarrow I'$ and $I', \Gamma_2 \Rightarrow C$, and $\Gamma_1 \Rightarrow J'$ and $J', A, A, \Gamma_2 \Rightarrow C$ are derivable. Moreover, I' is in $\mathcal{L}(\Gamma_1 \cup \{A\}, \Gamma_2 \cup \{C\})$, and J' is in $\mathcal{L}(\Gamma_1, \Gamma_2 \cup \{A, C\})$. Hence taking $I = I'$ and $J = J'$ and applying contraction gives the desired result. The case LW is equally trivial.
The connective cases are equal to their treatment in proofs of interpolation for LJ. For completeness sake we sketch the proof for the case that the last rule is L$\rightarrow$. Then the last lines of the proof look as follows.

$$\frac{\Gamma_1\Gamma_2 \Rightarrow A \qquad \Gamma_1\Gamma_2, B \Rightarrow C}{\Gamma_1\Gamma_2, A \rightarrow B \Rightarrow C}$$

We have to find I in $\mathcal{L}(\Gamma_1 \cup \{A \rightarrow B\}, \Gamma_2 \cup \{C\})$ and J in $\mathcal{L}(\Gamma_1, \Gamma_2 \cup \{A \rightarrow B, C\})$ such that

$$\vdash \Gamma_1, A \rightarrow B \Rightarrow I \quad \vdash I, \Gamma_2 \Rightarrow C \qquad \text{and} \qquad \vdash \Gamma_1 \Rightarrow J \quad \vdash J, A \rightarrow B, \Gamma_2 \Rightarrow C.$$

We teat the case I and leave J to the reader. For I, note that by the induction hypothesis there are $I' \in \mathcal{L}(\Gamma_2, \Gamma_1 \cup \{A\})$ and $I'' \in \mathcal{L}(\Gamma_1 \cup \{B\}, \Gamma_2 \cup \{C\})$ such that $\vdash \Gamma_2 \Rightarrow I'$, $\vdash I', \Gamma_1 \Rightarrow A$, $\vdash \Gamma_1, B \Rightarrow I''$ and $\vdash I'', \Gamma_2 \Rightarrow C$. Hence we can take $I = I' \rightarrow I''$.
The case of the existential quantifier is more or less similar to the corresponding case for LJ. Suppose the last rule is L$\exists$. Then the last two lines of the proof are

$$\frac{\Gamma_1\Gamma_2, Ey, Ay \Rightarrow C}{\Gamma_1\Gamma_2, \exists x A[x/y] \Rightarrow C}$$

We write $\exists x Ax$ for $\exists x A[x/y]$. Note that y is not free in $\Gamma_1\Gamma_2$ and C. We have to find $I \in \mathcal{L}(\Gamma_1 \cup \{\exists x Ax\}, \Gamma_2 \cup \{C\})$ and $J \in \mathcal{L}(\Gamma_1, \Gamma_2 \cup \{\exists x Ax, C\})$ such that

$$\vdash \Gamma_1, \exists x Ax \Rightarrow I \quad \vdash I, \Gamma_2 \Rightarrow C \qquad \text{and} \qquad \vdash \Gamma_1 \Rightarrow J \quad \vdash J, \exists x Ax, \Gamma_2 \Rightarrow C.$$

For I, use that the induction hypothesis gives a I' such that

$$\vdash \Gamma_1, Ey, Ay \Rightarrow I' \quad \vdash I', \Gamma_2 \Rightarrow C.$$

Observe that we have

$$\vdash \Gamma_1, Ey, Ay \Rightarrow \exists z I'[z/y] \quad \vdash \exists z I'[z/y], \Gamma_2 \Rightarrow C,$$

because we also have $\vdash I'(y), \Gamma_2, Ey \Rightarrow C$ by the weakening lemma. An application of L$\exists$ to $\Gamma_1, Ey, Ay \Rightarrow \exists z I'[z/y]$ shows that we can take $I = \exists z I'[z/y]$ as interpolant. Of course, if y is not free in I' we can take $I = I'$ as well.
For J, use that the induction hypothesis gives a J' such that

$$\vdash \Gamma_1 \Rightarrow J' \quad \vdash J', \Gamma_2, Ey, Ay \Rightarrow C.$$

Observe that whe have

$$\vdash \Gamma_1 \Rightarrow \forall z J'[z/y] \quad \vdash \forall z J'[z/y], \Gamma_2, Ey, Ay \Rightarrow C,$$

because we also have

$$\vdash \Gamma_1, Ey \Rightarrow J' \quad \forall z J'[z/y], J', \Gamma_2, Ey, Ay \Rightarrow C \quad \forall z J'[z/y], \Gamma_2, Ey, Ay \Rightarrow Ey$$

by the weakening lemma. An application of L$\exists$ to $\forall z J'[z/y], \Gamma_2, Ey, Ay \Rightarrow C$ shows that we can take $J = \forall z J'[z/y]$ as interpolant. Of course, if y is not free in J' we can take $J = J'$ as well.
Suppose the last rule is R$\exists$:

$$\dfrac{\begin{matrix}\vdots \\ \Gamma_1\Gamma_2 \Rightarrow At\end{matrix} \qquad \begin{matrix}\vdots \\ \Gamma_1\Gamma_2 \Rightarrow Et\end{matrix}}{\Gamma_1\Gamma_2 \Rightarrow \exists x Ax}$$

We have to find $I \in \mathcal{L}(\Gamma_1, \Gamma_2 \cup \{ExAx\})$ such that $\vdash \Gamma_1 \Rightarrow I$ and $\vdash I \Rightarrow ExAx$. By the induction hypothesis there are I_1 and I_2 such that

$$\vdash \Gamma_1 \Rightarrow I_1 \quad \vdash I_1, \Gamma_2 \Rightarrow At \quad \vdash \Gamma_1 \Rightarrow I_2 \quad \vdash I_2, \Gamma_2 \Rightarrow Et.$$

Thus we can take $I = I_1 \wedge I_2$ as interpolant.
Finally, we treat the universal quantifier, the most complicated case. Suppose the last rule is R$\forall$:

$$\dfrac{\begin{matrix}\vdots \\ \Gamma_1\Gamma_2, Ey \Rightarrow A(y)\end{matrix}}{\Gamma_1\Gamma_2 \Rightarrow \forall x A[x/y]}$$

By the induction hypothesis there is a interpolant $I \in \mathcal{L}(\Gamma_1, \Gamma_2 \cup \{Ey, A(y)\})$ for the upper sequent: $\vdash \Gamma_1 \Rightarrow I$ and $\vdash I, Ey, \Gamma_2 \Rightarrow A(y)$. In case y is not free in I the sequent $I, \Gamma_2 \Rightarrow \forall x A[x/y]$ is derivable too. Hence we can take I as an interpolant of the lower sequent and are done. Therefore, suppose y occurs free in I. By the side conditions y is not free in $\Gamma_1\Gamma_2$. Hence we have the following derivation:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma_1, Ey \Rightarrow I\end{array}}{\Gamma_1 \Rightarrow \forall z I[z/y]}$$

Thus the following derivation shows that $\forall z I[z/y]$ is an interpolant for the lower sequent:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ I, Ey, \Gamma_2 \Rightarrow A(y)\end{array} \qquad Ey, \Gamma_2 \Rightarrow Ey}{\forall z I[z/y], Ey, \Gamma_2 \Rightarrow A(y)}}{\forall z I[z/y], \Gamma_2 \Rightarrow \forall x A[x/y]}$$

Finally, we treat L$\forall$, when the last lines of the proof are:

$$\dfrac{\Gamma_1\Gamma_2, A(t) \Rightarrow C \qquad \Gamma_1\Gamma_2 \Rightarrow Et}{\Gamma_1\Gamma_2, \forall x A(x) \Rightarrow C}$$

We have to find $I \in \mathcal{L}(\Gamma_1 \cup \{\forall x A(x)\}, \Gamma_2 \cup \{C\})$ and $J \in \mathcal{L}(\Gamma_1, \Gamma_2 \cup \{\forall x A(x), C\})$ such that

$$\vdash \Gamma_1, \forall x A(x) \Rightarrow I \quad \vdash I, \Gamma_2 \Rightarrow C \qquad \text{and} \qquad \vdash \Gamma_1 \Rightarrow J \quad \vdash J, \forall x A(x), \Gamma_2 \Rightarrow C.$$

First we treat the case J. Note that by the induction hypothesis there are three formulas $I' \in \mathcal{L}(\Gamma_1, \Gamma_2 \cup \{A(t), C\})$, $J' \in \mathcal{L}(\Gamma_1, \Gamma_2 \cup \{Et\})$ and $H' \in \mathcal{L}(\Gamma_2, \Gamma_1 \cup \{Et\})$ such that

$$\vdash \Gamma_1 \Rightarrow I' \quad \vdash I', A(t), \Gamma_2 \Rightarrow C \qquad \text{and} \qquad \vdash \Gamma_1 \Rightarrow J' \quad \vdash J', \Gamma_2 \Rightarrow Et \tag{4}$$

$$\vdash \Gamma_2 \Rightarrow H' \quad \vdash H', \Gamma_1 \Rightarrow Et.$$

Note that I', J' and H' may contain t. If t does not occur in I' and J' or it occurs in $\mathcal{L}(\Gamma_1, \Gamma_2 \cup \{\forall x A(x), C\})$, then $I', J' \in \mathcal{L}(\Gamma_1, \Gamma_2 \cup \{\forall x A(x), C\})$. Moreover, (4) implies

$$\vdash \Gamma_1 \Rightarrow I' \wedge J' \quad \vdash I' \wedge J', \forall x A(x), \Gamma_2 \Rightarrow C.$$

Thus in this case we can take $J = I' \wedge J'$.
On the other hand, if t does occur in I' or J' and not in $\mathcal{L}(\Gamma_1, \Gamma_2 \cup \{\forall x A(x), C\}$ we proceed as follows. Either t does not occur in Γ_1 or t does not occur in $\Gamma_2 \cup \{\forall x A(x), C\}$. In the first case, it follows that t does not occur in I' and not in J', contradicting our assumptions. Thus t occurs in Γ_1 but not in $\Gamma_2 \cup \{\forall x A(x), C\}$. Hence t does not occur in H'. Note that we have a derivation

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ H', \Gamma_1 \Rightarrow I' \wedge J'\end{array} \qquad \begin{array}{c}\vdots\\ H', \Gamma_1 \Rightarrow Et\end{array}}{H', \Gamma_1 \Rightarrow \exists x (I' \wedge J')[x/t]}}{\Gamma_1 \Rightarrow \big(H' \to \exists x (I' \wedge J')[x/t]\big)}$$

Now note something important: because t does not occur in $\forall x A(x)$, this implies that $\forall x A(x) = \forall x A[x/t]$ (for the difference between $A(x)$ and $A[x/t]$ see the preliminaries, Section 2). Thus also $\forall x(A[y/t])[x/y] = \forall x A(x)$. And because t does not occur in Γ_2 or C, by the substitution lemma, Lemma 3, we also have a derivation for a variable y not occurring in P of

$$\dfrac{\dfrac{\vdots}{\Gamma_2 \Rightarrow H'} \qquad \dfrac{\dfrac{\dfrac{\vdots}{(I' \wedge J')[y/t], Ey, A[y/t], \Gamma_2 \Rightarrow C} \qquad \dfrac{\vdots}{Ey, (I' \wedge J')[y/t], \Gamma_2 \Rightarrow Ey}}{Ey, (I' \wedge J')[y/t], \forall x A(x), \Gamma_2 \Rightarrow C}}{\exists x(I' \wedge J')[x/t], \forall x A(x), \Gamma_2 \Rightarrow C}}{\big(H' \to \exists x(I' \wedge J')[x/t]\big), \forall x A(x), \Gamma_2 \Rightarrow C}$$

Hence we can take $J = \big(H' \to \exists x(I' \wedge J')[x/t]\big)$ and we are done.
The last case we have to treat is the one where we look for the interpolant $I \in \mathcal{L}(\Gamma_1 \cup \{\forall x A(x)\}, \Gamma_2 \cup \{C\})$ such that

$$\vdash \Gamma_1, \forall x A(x) \Rightarrow I \quad \vdash I, \Gamma_2 \Rightarrow C. \tag{5}$$

Note that by the induction hypothesis there are $I' \in \mathcal{L}(\Gamma_1 \cup \{A(t)\}, \Gamma_2 \cup \{C\})$, $J' \in \mathcal{L}(\Gamma_2, \Gamma_1 \cup \{Et\})$ and $H' \in \mathcal{L}(\Gamma_2, \Gamma_1 \cup \{Et\})$ such that

$$\vdash \Gamma_1, A(t) \Rightarrow I' \quad \vdash I', \Gamma_2 \Rightarrow C \qquad \text{and} \qquad \vdash \Gamma_2 \Rightarrow J' \quad \vdash J', \Gamma_1 \Rightarrow Et$$

$$\vdash \Gamma_1 \Rightarrow H' \quad \vdash H', \Gamma_2 \Rightarrow Et.$$

Observe that whence we have $\vdash (J' \to I'), \Gamma_2 \Rightarrow C$. Furthermore, we have a derivation

$$\dfrac{\dfrac{\dfrac{\vdots}{J', A(t), \Gamma_1 \Rightarrow I'} \qquad \dfrac{\vdots}{J', \Gamma_1 \Rightarrow Et}}{J', \forall x A(x), \Gamma_1 \Rightarrow I'}}{\forall x A(x), \Gamma_1 \Rightarrow J' \to I'}$$

Thus, in case t belongs to the common language $\mathcal{L}(\Gamma_1 \cup \{\forall x A(x)\}, \Gamma_2 \cup \{C\})$ we can take $I = (J' \to I')$ and we are done. Therefore, assume t does not belong to the common language. In case it does not belong to $\Gamma_2 \cup \{C\}$, it follows that both I' and J' cannot contain t and we can again take $I = (J' \to I')$. Therefore, assume t does not belong to $\Gamma_1 \cup \{\forall x A(x)\}$. Hence H' does not contain t. But then we can infer, by Lemma 3, for a fresh variable y, from $\vdash \forall x A(x), \Gamma_1 \Rightarrow J' \to I'$ above, that we have the following derivation

$$\dfrac{\dfrac{\dfrac{\vdots}{\forall x A(x), Ey, \Gamma_1 \Rightarrow (J' \to I')[y/t]}}{\forall x A(x), \Gamma_1 \Rightarrow \forall z(J' \to I')[z/t]} \qquad \dfrac{\vdots}{\Gamma_1 \Rightarrow H'}}{\forall x A(x), \Gamma_1 \Rightarrow \forall z(J' \to I')[z/t] \wedge H'}$$

On the other hand we also have

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ H', J' \rightarrow I', \Gamma_2 \Rightarrow C\end{matrix} \qquad \begin{matrix}\vdots \\ H', J' \rightarrow I', \Gamma_2 \Rightarrow Et\end{matrix}}{H', \forall z(J' \rightarrow I')[z/t], \Gamma_2 \Rightarrow C}}{H' \wedge \forall z(J' \rightarrow I')[z/t], \Gamma_2 \Rightarrow C}$$

Hence we take $I = H' \wedge \forall z(J' \rightarrow I')[z/t]$ as the interpolant.
It is interesting to note that (5) also holds for $I = (Et \rightarrow I') \wedge H'$. But in this case I does not in general belong to the common language.
Finally, we show that $\mathsf{LJE}'(\Sigma_{\mathfrak{L}})$ has interpolation too, by reducing this case to the case LJE' in the following way. Given a proof P of $\Gamma_1\Gamma_2 \Rightarrow C$ in $\mathsf{LJE}'(\Sigma_{\mathfrak{L}})$ we consider all axioms of the form $\Pi \Rightarrow Et \in \Sigma_{\mathfrak{L}}$ that occur in P. Suppose there are n of them: $\Pi_1 \Rightarrow Et_1, \ldots, \Pi_n \Rightarrow Et_n$. Note that all t_i have to be closed. Clearly, there is a proof of $Et_1, \ldots, Et_n, \Gamma_1\Gamma_2 \Rightarrow C$ in LJE' by replacing the axioms $\Pi_i \Rightarrow Et_i$ by the logical axioms $\Pi_i, Et_i \Rightarrow Et_i$. Now we consider the following partition $\Gamma_1'\Gamma_2' \Rightarrow C$ of $Et_1, \ldots, Et_n, \Gamma_1\Gamma_2 \Rightarrow C$:

$$\Gamma_1' = \Gamma_1 \cup \{Et_j \mid j \leq n, \;\; t_j \text{ occurs in } \Gamma_1 \text{ or not in } \Gamma_1 \cup \Gamma_2\}.$$

$$\Gamma_2' = \Gamma_2 \cup \{Et_j \mid j \leq n, \;\; t_j \text{ occurs in } \Gamma_2\}.$$

By the interpolation theorem for LJE' there exists an interpolant I such that $\vdash_{\mathsf{LJE}'} \Gamma_1' \Rightarrow I$ and $\vdash_{\mathsf{LJE}'} I, \Gamma_2' \Rightarrow C$ where I is in the common language of Γ_1' and $\Gamma_2' \cup \{C\}$. It is not difficult to see that whence I is in the common language of Γ_1 and $\Gamma_2 \cup \{C\}$ too. By cutting on the Et_i's we obtain

$$\vdash_{\mathsf{LJE}'(\Sigma_{\mathfrak{L}})} \Gamma_1 \Rightarrow I \qquad \vdash_{\mathsf{LJE}'(\Sigma_{\mathfrak{L}})} I, \Gamma_2 \Rightarrow C.$$

This proves that $\mathsf{LJE}'(\Sigma_{\mathfrak{L}})$ has interpolation too.

Corollary 2. LJE *and* $\mathsf{LJE}(\Sigma_{\mathfrak{L}})$ *have interpolation.*

5.1 Interpolation of fragments

We say that a Gentzen calculus L interpolates for the fragment $\mathfrak{F}$ where $\mathfrak{F}$ is a set of formulas, if whenever $\mathsf{L} \vdash \Gamma_1, \Gamma_2 \Rightarrow C$, there exists an $I \in \mathfrak{F}$ in the common language of Γ_1 and $\Gamma_2 \cup \{C\}$ such that

$$\vdash_{\mathsf{L}} \Gamma_1 \Rightarrow I \quad \vdash_{\mathsf{L}} I, \Gamma_2 \Rightarrow C.$$

As is well-known, the fragment consisting of formulas containing no other logical symbols as $\wedge, \vee, \forall, \exists$, interpolates for LJ. We conjecture that LJE and $\mathsf{LJE}(\Sigma_{\mathfrak{L}})$ do not interpolate for this fragment because of the L$\forall$ case in the proof of the interpolation theorem above.

5.2 Beth's theorem

Following standard proofs for the Beth definability property of LJ, it is easy to prove the following theorem.

Theorem 5. LJE *and* LJE($\Sigma_{\mathcal{L}}$) *satisfy the Beth definability property.*

References

1. Baaz, M. and Iemhoff, R., Skolemization in intuitionistic logic, *Manuscript*, 2005.
2. Baaz, M. and Iemhoff, R., Gentzen calculi for the existence predicate, *Studia Logica*, to appear.
3. Beckmann, A. and Preining, N., Linear Kripke Frames and Gödel Logics, *Submitted*, 2005.
4. Beeson, M., Foundations of Constructive Mathematics, Springer, Berlin, 1985.
5. Corsi, G., A cut-free calculus for Dummett's LC quantified, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 35, pp. 289-301, 1989.
6. Corsi, G., A logic characterized by the class of connected models with nested domains, *Studia Logica* 48(1), pp. 15-22, 1989.
7. Corsi, G., Completeness theorem for Dummett's LC quantified and some of its extensions, *Studia Logica* 51(2), pp. 317-335, 1992.
8. Heyting, A., Die formalen Regeln der intuitionistische Mathematik II, *Sitzungsberichte der Preussischen Akademie von Wissenschaften. Physikalisch-mathematische Klasse*, pp. 57-71, 1930.
9. Iemhoff, R., A note on linear Kripke models, *Journal of Logic and Computation*, to appear.
10. Mints, G.E., Axiomatization of a Skolem function in intuitionistic logic, *Formalizing the dynamics of information*, Faller, M. (ed.) et al., CSLI Lect. Notes 91, pp. 105-114, 2000.
11. Preining, N., *Complete Recursive Axiomatizability of Gödel Logics*, PhD-thesis, Technical University Vienna, 2003.
12. Scott, D.S., Identity and existence in intuitionistic logic, *Applications of sheaves, Proc. Res. Symp. Durham 1977*, Fourman (ed.) et al., Lect. Notes Math. 753, pp. 660-696, 1979.
13. Takano, M., Another proof of the strong completeness of the intuitionistic fuzzy logic, *Tsukuba J. Math.* 11(1), pp. 101-105, 1987.
14. Takeuti, G. , *Proof Theory*, North-Holland Publishing Company, 1979.
15. Takeuti, G. AND Titani, M., Intuitionistic fuzzy logic and intuitionistic fuzzy set theory, *Journal of Symbolic Logic* 49, pp. 851-866, 1984.
16. Troelstra, A.S., and van Dalen, D., *Constructivism in Mathematics*, vol. I North-Holland, 1988.
17. Troelstra, A.S., and Schwichtenberg, H.,, *Basic Proof Theory*, Cambridge Tracts in Theoretical Computer Science 43, Cambridge University Press, 1996.
18. Unterhalt, M., *Kripke-Semantik mit partieller Existenz*, PhD-thesis, University of Münster, 1986.