Gentzen calculi for the existence predicate

Matthias Baaz and Rosalie Iemhoff[*]
Institute for Discrete Mathematics and Geometry E104,
Technical University Vienna, Wiedner Hauptstrasse 8-10, 1040 Vienna, Austria.
baaz@logic.at, iemhoff@logic.at

July 1, 2005

Abstract

We introduce Gentzen calculi for intuitionistic logic extended with an existence predicate. Such a logic was first introduced by Dana Scott, who provided a natural deduction proof system for it. We prove that the Gentzen calculus has cut elimination in so far that all cuts can be restricted to very simple ones. Applications of this logic to Skolemization, truth value logics and linear frames are also discussed.

Keywords: Intuitionistic logic, existence predicate, Gentzen calculus, cut-elimination, Skolemization, truth-value logics, Gödel logics, Scott logics, Kripke models.

1 Introduction

In this paper we introduce Gentzen calculi for so-called existence logics. These logics are extensions of intuitionistic predicate logic IQC with an existence predicate E, where the intuitive meaning of Et is that t *exists*. The motivation behind these logics is that in the context of intuitionistic logic it is natural to be able to denote whether a term exists or not.

Existence logic IQCE was first introduced by D. Scott in [11], where he presented a Hilbert style proof system for the logic. In this system both variables and terms range over arbitrary object while the quantifiers are assumed to range over existing objects only. Existence logic in which terms range over all object while quantifiers as well as variables only range over existing objects is denoted by IQCE^+ and has e.g. been used by M. Beeson in [3]. M. Unterhalt thoroughly studied the Kripke semantics of these logics and proved respectively completeness and strong completeness for the systems IQCE and IQCE^+ in [16]. Completeness results for the Gentzen calculi presented in this paper can be found in [1].

The Gentzen calculi that we introduce in this paper are called LJE and $\mathsf{LJE}(\Sigma_{\mathfrak{L}})$, which is LJE extended by axioms $\Sigma_{\mathfrak{L}}$, to be defined below. LJE corresponds to Scott's IQCE, and for a specific $\Sigma_{\mathfrak{L}}$ the calculus $\mathsf{LJE}(\Sigma_{\mathfrak{L}})$ corresponds to IQCE^+. This paper is devoted to the proof that both these systems have cut elimination in so far that cuts in proofs can be restricted to very simple ones (Theorem 4.7).

[*]Supported by the Austrian Science Fund FWF under project P17503.

1.1 Applications

Existence logic has many many applications, and sometimes leads to surprising solutions of problems that do not seem solvable in intuitionistic logic pure. We do not describe these applications in full detail here, but we try to explain the general idea and give pointers to the literature.

1.1.1 Skolemization

The foremost use of the existence predicate is in the setting of Skolemization. As is well-known, the classical Skolemization method of replacing strong quantifiers in a formula by fresh function symbols and thus obtaining a equiconsistent formula, is not complete with respect to IQC. That is, there are formulas that are underivable, but for which their Skolemized version is derivable in IQC. For example,

$$\mathsf{IQC} \nvdash \forall x(Ax \vee B) \to (\forall x Ax \vee B) \qquad \mathsf{IQC} \vdash \forall x(Ax \vee B) \to (Ac \vee B).$$

In [1] an alternative Skolemization method called *eSkolemization* is introduced and is shown to be sound and complete with respect to IQC for a large class of formulas, including all formulas in which every strong quantifier is existential or of the form $\forall x \neg\neg Ax$. This class is much larger than the class of formulas for which the standard Skolemization is sound and complete. This eSkolemization method makes use of the existence predicate. It replaces negative occurrences of existential quantifiers $\exists x Bx$ by $(Ef(\bar{y}) \wedge Bf(\bar{y}))$, and positive occurrences of universal quantifiers $\forall x Bx$ by $(Ef(\bar{y}) \to Bf(\bar{y}))$. For example, the eSkolemization of the displayed formula above is

$$\mathsf{IQCE} \nvdash \forall x(Ax \vee B) \to ((Ec \to Ac) \vee B).$$

Then it is shown in [1] that

Theorem 1.2 For each formula A in which all strong universal quantifiers $QxBx$ are of the form $\forall x \neg\neg Bx$: $\vdash_{\mathsf{LJE}(\Sigma_{\mathcal{L}})} \Rightarrow A$ if and only if $\vdash_{\mathsf{LJE}(\Sigma_{\mathcal{L}})} \Rightarrow A^s$.

The definition of $\Sigma_{\mathcal{L}}$ and LJE will be given below. We will not proceed the topic of eSkolemization here but refer the interested reader to [1] instead.

1.2.1 Truth-value logics and linear frames

Another application of the existence predicate is in the context of truth-value logics. These are logics based on truth-value sets V, i.e. closed subsets of the unit interval $[0, 1]$, also called *Gödel sets*. One can, for a given Gödel set V, interpret formulas by mapping them to elements of V. The logical symbols receive a meaning via restrictions on these interpretations, e.g. by stipulating that the interpretation of $\wedge$ is the infimum of the interpretations of the respective conjuncts, or that the interpretation of $\exists x Ax$ is the supremum of the values of Aa for all elements a in the domain. Given these interpretations, one can associate a logic with such a Gödel set V: the logic of all sentences that are mapped to 1 under any interpretation on V.

Gödel logics G_V are an example of truth value logics. Without going into the precise definition of these logics here, we only want to mention that these logics naturally correspond to the logics of linear frames. As has been shown by A. Beckmann and N. Preining this correspondence takes the following form.

Theorem 1.3 (A. Beckmann and N. Preining [2]) For every countable linear frame F there exists a Gödel set V such that

$$G_V \models A \;\Leftrightarrow\; A \text{ holds in all Kripke models on } F \text{ with constant domains,} \quad (1)$$

and vice versa: for every Gödel set V there exists a countable linear frame F such that (1).

In [8] so-called *Scott logics* S_V are introduced which correspond to linear frames, but now for possibly non constant domains. That is, we have

Theorem 1.4 [8] For every countable linear frame F there exists a Gödel set V such that

$$S_V \models A \;\Leftrightarrow\; A \text{ holds in all Kripke models based on } F, \quad (2)$$

and vice versa: for every countable Gödel set V there exists a countable linear frame F such that (2).

In the same paper it is shown that there is a natural and faithful translation from Scott logics into Gödel logics. This translation $(\cdot)^e$, that makes use of the existence predicate, allows to transfer properties about Gödel logics to Scott logics. $(\cdot)^e$ is defiend as follows.

$\big(P(\bar{t})\big)^e = P(\bar{t})$ for atomic P and terms $\bar{t}$,

$(\cdot)^e$ commutes with the connectives,

$(\exists x A(x))^e = \exists x\big(Ex \wedge (A(x))^e\big)$,

$(\forall x A(x))^e = \forall x\big(Ex \to (A(x))^e\big)$.

Given this translation we then have the following theorem.

Lemma 1.5 [8] For any Gödel set V, $(\cdot)^e$ is a faithful translation of S_V into G_V, i.e. for all $\mathcal{L}$-sentences A

$$S_V \models A \;\Leftrightarrow\; G_V^e \models A^e.$$

Note the similarity between the different applications of the existence predicate: the translation $(\cdot)^e$ does a similar thing to quantifiers as eSkolemization does. Essentially, it all has to do with the fact that an existence predicate allows us in a Kripke model to name objects that do not exist in the root but come into existence only at a later stage in the model. Both [1] and [8] describe this intuition in more detail.

2 Preliminaries

We consider languages $\mathcal{L} \subseteq \mathcal{L}'$ for intuitionistic predicate logic plus the existence predicate E, without equality. For convenience we assume that $\mathcal{L}$ contains at least one constant and no variables, and that $\mathcal{L}'$ contains infinitely many variables. The reason for this has to do with the semantics for the Gentzen calculi introduced below; a topic we will not proceed here, but which is discussed in [1].

The languages contain $\bot$, and $\neg A$ is defined as $A \to \bot$. $A, B, C, D, E, ..$ range over formulas in $\mathcal{L}'$, $s, t, ..$ over terms in $\mathcal{L}'$. Γ, Δ, Π range over multisets of formulas in $\mathcal{L}'$. Sequents are expressions of the form $\Gamma \Rightarrow C$, where Γ is a finite multiset. A sequent is in $\mathcal{L}$ if all its formulas are in $\mathcal{L}$. And similarly for $\mathcal{L}'$. A formula is *closed* when it does not contain free variables. A sequent $\Gamma \Rightarrow C$ is closed if C and all formulas in Γ are closed.

In the final proof system $(\Rightarrow Et)$ will hold for the terms in $\mathcal{L}$, but not necessarily for the terms in $\mathcal{L}' \backslash \mathcal{L}$. $\mathcal{T}_{\mathcal{L}}$ denotes the set of terms in $\mathcal{L}$, $\mathcal{F}_{\mathcal{L}}$ denotes the set of formulas in $\mathcal{L}$, $\mathcal{S}_{\mathcal{L}}$ denotes the set of sequents in $\mathcal{L}$, and similarly for $\mathcal{L}'$.

In order not to drown in brackets we often write Ax for $A(x)$.

3 The proof system

In this section we define the system LJE, a conservative extension of LJ for $\mathcal{L}'$ that covers the intuition that Et means *t exists*. Such a system was first introduced by Dana Scott in [11], but then in a Hilbert style axiomatization, and called IQCE. The Gentzen calculus for this system given below is new.

Given an existence predicate, terms, including variables, typically range over existing as well as non-existing elements, while the quantifiers range over existing objects only. Proofs are assumed to be trees.

The system LJE

$Ax\ \ \Gamma, P \Rightarrow P \quad P$ atomic

$L\bot\ \ \Gamma, \bot \Rightarrow C$

$$\mathrm{L}\wedge\ \frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}$$

$$\mathrm{R}\wedge\ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\mathrm{L}\vee\ \frac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C}$$

$$\mathrm{R}\vee\ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1}\ i = 0, 1$$

$$\mathrm{L}\rightarrow\ \frac{\Gamma, A \rightarrow B \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \rightarrow B \Rightarrow C}$$

$$\mathrm{R}\rightarrow\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}$$

$$\mathrm{L}\forall\ \frac{\Gamma, \forall x Ax, At \Rightarrow C \qquad \Gamma, \forall x Ax \Rightarrow Et}{\Gamma, \forall x Ax \Rightarrow C}$$

$$\mathrm{R}\forall\ \frac{\Gamma, Ey \Rightarrow Ay}{\Gamma \Rightarrow \forall x A[x/y]}\ *$$

$$\mathrm{L}\exists\ \frac{\Gamma, Ey, Ay \Rightarrow C}{\Gamma, \exists x A[x/y] \Rightarrow C}\ *$$

$$\mathrm{R}\exists\ \frac{\Gamma \Rightarrow At \qquad \Gamma \Rightarrow Et}{\Gamma \Rightarrow \exists x Ax}$$

$$\mathrm{Cut}\ \frac{\Gamma \Rightarrow A \qquad \Gamma, A \Rightarrow C}{\Gamma \Rightarrow C}$$

Where $(*)$ denotes the condition that y does not occur free in Γ and C.

The principal formula of a rule is defined as usual. In the Cut rule the formula A is called the cutformula, and it is the principal formula of the Cut rule. The formulas Et and Ey are not principal in respectively L∀, R∃ and R∀, L∃.

We write $\mathsf{LJE} \vdash S$ if the sequent S is derivable in LJE. For a set of sequents X and a sequent S, we say that S *is derivable from* X *in* LJE, and write $X \vdash_{\mathsf{LJE}} S$, if S is derivable in the system LJE to which the sequents in X are added as initial sequents. We also denote this system by $\mathsf{LJE}(X)$.

In the system LJE no existence of any term that is not a variable is assumed This implies e.g. that we cannot derive $\forall x Px \Rightarrow Pt$, but only $\forall x Px, Et \Rightarrow Pt$. Note however that the former is derivable in LJE from $(\Rightarrow Et)$. This is the reason why we consider derivations from extra axioms, especially axioms of the form $(\Rightarrow Et)$. Therefore, we define the following sets of sequents

$$\Sigma_{\mathcal{L}} \equiv_{def} \{\Gamma \Rightarrow Et \mid t \in \mathcal{T}_{\mathcal{L}}, \Gamma \text{ a multiset}\}.$$

Note that because of the assumptions on $\mathcal{L}$, $\Sigma_{\mathcal{L}}$ contains at least one sequent and for all sequents $\Gamma \Rightarrow Et$ in $\Sigma_{\mathcal{L}}$, t is a closed term. Given two languages $\mathcal{L} \subseteq \mathcal{L}'$, we write

$$\mathsf{LJE}(\Sigma_{\mathcal{L}}) \equiv_{def} \{S \in \mathcal{S}_{\mathcal{L}'} \mid \Sigma_{\mathcal{L}} \vdash_{\mathsf{LJE}} \Rightarrow S\}.$$

The $\mathcal{L}'$ is not denoted in $\mathsf{LJE}(\Sigma_{\mathcal{L}})$, but most of the time it is clear what is the "larger" language $\mathcal{L}'$ of which $\mathcal{L}$ is a subset.

We often write $\vdash_{\mathcal{L}}$ for $\vdash_{\mathsf{LJE}(\Sigma_{\mathcal{L}})}$.

Example 3.1

$$\not\vdash_{\mathsf{LJE}} \Rightarrow \exists x Ex \qquad \vdash_{\mathsf{LJE}} \Rightarrow \forall x Ex.$$

$$\vdash_{\mathsf{LJE}(\Sigma_{\mathfrak{L}})} \Rightarrow \exists x Ex \wedge \forall x Ex.$$

Lemma 3.2 For all sequents S in $\mathfrak{L}$ that do not contain E:

$$\mathsf{LJ} \vdash S \text{ implies } \mathsf{LJE}(\Sigma_{\mathfrak{L}}) \vdash S.$$

Proof Since S is a sequent in $\mathfrak{L}$, we may assume w.l.o.g. that when S is provable in LJ it has a cutfree proof in which all terms that are not eigenvariables are terms in $\mathfrak{L}$. Call this set of terms X. Clearly, $X^s = \{\Gamma \Rightarrow Et \mid t \in X\}$ is a subset of $\Sigma_{\mathfrak{L}}$. At every application of R$\exists$ or L$\forall$, add the appropriate $\Gamma \Rightarrow Et$ as the right hypothesis. At every application of R$\forall$ or L$\exists$ add the appropriate Ey to the antecedent. This gives a proof of $\Gamma \Rightarrow A$ in LJE. □

Later on, in Proposition 4.11, we will see that the converse of the above lemma holds too.

3.3 Uniqueness

Observe that given another predicate E' that satisfies the same rules of LJE as E', it follows that

$$\mathsf{LJE}(\Sigma_{\mathfrak{L}}) \vdash Et \Rightarrow E't \;\wedge\; \mathsf{LJE}(\Sigma_{\mathfrak{L}}) \vdash E't \Rightarrow Et.$$

Namely, $\mathsf{LJE}(\Sigma_{\mathfrak{L}}) \vdash (\Rightarrow (\forall x Ex \wedge \forall x E'x))$, and $\mathsf{LJE}(\Sigma_{\mathfrak{L}}) \vdash (\forall x Ex, E't \Rightarrow Et)$ and $\mathsf{LJE}(\Sigma_{\mathfrak{L}}) \vdash (\forall x E'x, Et \Rightarrow E't)$. Finally, two cuts do the trick. This shows that the existence predicate E is unique up to provable equivalence.

3.4 IQCE and IQCE$^+$

As remarked above, given an existence predicate, terms typically range over existing as well as non-existing elements, while quantifiers range over existing objects only. As to the choice of the domain for the variables, there have been different approaches. Scott in [11] introduces a system IQCE for the predicate language with the distinguished predicate E, in which variables range over all objects, like in LJE and $\mathsf{LJE}(\Sigma_{\mathfrak{L}})$. On the other hand, Beeson in [3] discusses a system in which variables range over existing objects only.

The formulation of the system IQCE in [11], where logic with an existence predicate was first introduced was in Hilbert style, where the axioms and rules for the quantifiers are the following:

$$\forall x Ax \wedge Et \to At \qquad \dfrac{\begin{array}{c}\vdots\\ B \wedge Ey \to Ay\end{array}}{B \to \forall x Ax}\,*$$

$$\dfrac{\begin{array}{c}\vdots\\ Ay \wedge Ey \to B\end{array}}{\exists x Ax \to B}\,* \qquad At \wedge Et \to \exists x Ax$$

Here $*$ are the usual side conditions on the eigenvariable y.

The following formulation of IQCE in natural deduction style was given in [14]. We call the system NDE (Natural Deduction Existence). It consists of the axioms and quantifier rules of the standard natural deduction formulation of IQC (as e.g. given in [14]), where the quantifier rules are replaced by the following rules:

$$\forall\mathrm{I}\ \frac{\begin{array}{c}[Ey]\\ \vdots\\ Ay\end{array}}{\forall x Ax}\,* \qquad \forall\mathrm{E}\ \frac{\begin{array}{cc}\vdots & \vdots\\ \forall x Ax & Et\end{array}}{At}$$

$$\exists\mathrm{I}\ \frac{\begin{array}{cc}\vdots & \vdots\\ At & Et\end{array}}{\exists x Ax} \qquad \exists\mathrm{E}\ \frac{\begin{array}{cc} & [Ay][Ey]\\ \vdots & \vdots\\ \exists x Ax & C\end{array}}{C}\,*$$

Again, the $*$ are the usual side conditions on the eigenvariable y. It is easy to see that the following holds.

Fact 3.5 $\forall A \in \mathcal{F}_{\mathcal{L}'}$: $\vdash_{\mathsf{IQCE}} A$ if and only if $\vdash_{\mathsf{NDE}} A$ if and only if $\vdash_{\mathsf{LJE}} \Rightarrow A$.

Existence logic in which terms range over all object while quantifiers and variables only range over existing objects is denoted by IQCE^+ and has e.g. been used by M. Beeson in [3]. The logic is the result of leaving out Ey in the two rules for the quantifiers in IQCE given above and adding Ex as axioms for all variables x. A formulation in natural deduction style is obtained from NDE by replacing the $\forall$I and $\exists$E by their standard formulations for IQC and adding Ex as axioms for all variables x. We call the system NDE^+. In this case we have the following correspondence.

Fact 3.6 $\forall A \in \mathcal{F}_{\mathcal{L}'}$:
$\vdash_{\mathsf{IQCE}^+} A$ iff $\vdash_{\mathsf{NDE}^+} A$ iff $\{\Gamma \Rightarrow Ex \mid x$ a variable, Γ a multiset$\} \vdash_{\mathsf{LJE}(\Sigma_{\mathcal{L}})} \Rightarrow A$.

M. Unterhalt in [16] thoroughly studied the Kripke semantics of these logics and proved respectively completeness and strong completeness for the systems IQCE and IQCE^+. Similar results for the Gentzen calculi presented here can be found in [1].

4 Cut elimination

We assume eigenvariables, free and bound variables to be three distinct sets of variables. The variable y in L$\exists$ and R$\forall$ is called an eigenvariable. The depth of a sequent in a proof is inductively defined as the sum of the depths of its upper sequents plus 1. Thus axioms have depth 1. The complexity $|C|$ of a formula is the number of occurrences of connectives and quantifiers in C. The rank of a cut is 1 + the complexity of the cut formula. The level of a cut is the sum of the depths of its two hypotheses. The cutrank *cr(P)* of a proof P is the maximal rank of cuts in P. The depth of a proof, *dp(P)*, is the depth of its endsequent. We write $\mathsf{LJE} \vdash_d S$ when S has a proof of depth $\leq d$ in LJE, We write $\mathsf{LJE} \vdash^c S$ when S has a proof of cutrank $\leq c$. Similarly for $\mathsf{LJE}(\Sigma_{\mathcal{L}})$. For a proof P, $P[t/y]$ denotes the result of substituting t for y everywhere in P.

4.1 Substitution, Weakening and Contraction

We start with the substitution lemma.

Lemma 4.2 For $\mathsf{L} \in \{\mathsf{LJE}(\Sigma_{\mathcal{L}}), \mathsf{LJE}\}$:
If P is a proof in L of a sequent S in $\mathcal{L}'$ in which y occurs free, and if t is a term in $\mathcal{L}'$ that does not contain eigenvariables or bound variables of P, then $P[t/y]$ is a proof of $S[t/y]$ in L. Moreover, $cr(P[t/y]) \leq cr(P)$ and $dp(P[t/y]) \leq dp(P)$.

Proof We treat the case $\mathsf{L} = \mathsf{LJE}(\Sigma_{\mathcal{L}})$. We use induction to the depth d of P. Let $P' = P[t/y]$, $S' = S[t/y]$. First $d = 1$, the case that P is an instance of an axiom. The axioms Ax, L$\bot$ in P are replaced by instances of the same axioms in P', so these will not be violated under the transformation. For axioms $\Pi \Rightarrow Es$ in $\Sigma_{\mathcal{L}}$ it follows that s is a closed term in $\mathcal{L}$. Hence the sequent that results from the substitution, $(\Pi[t/y] \Rightarrow Es)$, belongs to $\Sigma_{\mathcal{L}}$ too. This completes the case $d = 1$.

Suppose $d > 1$. First note that because eigenvariables are distinct from free variables in a proof, y cannot be an eigenvariable in P. We distinguish by cases acoording to the last rule in P. The connective rules and cuts in P are replaced by instances of the same rules in P', so these will not be violated under the transformation. Thus the quantifier rules remain.

Suppose the last inference in P is a quantifier rule. In the case of L$\forall$ and R$\exists$ there are no side conditions, whence these rules will not be violated in going from P to P'. We treat R$\forall$, the case L$\exists$ is similar. Consider an application of R$\forall$ in P:

$$\dfrac{\begin{array}{c}P_1\\ \Pi, Ez \Rightarrow Bz\end{array}}{\Pi \Rightarrow \forall u Bu}$$

Thus z is not free in Π, and $z \neq y$ and $u \neq y$, since y is no eigenvariable or bound variable. By assumption on t, u does not occur in t. Under the transformation this will become

$$\dfrac{\begin{array}{c}P_1[t/y]\\ \Pi[t/y], Ez \Rightarrow Bz[t/y]\end{array}}{\Pi[t/y] \Rightarrow \forall u Bu[t/y]}$$

To see that this a valid application of R$\forall$, it suffices to see that z is not free in $\Pi[t/y]$, which is clear from the assumption on t.

To check that $cr(P') \leq cr(P)$ and $dp(P') \leq dp(P)$ is left to the reader. □

Lemma 4.3 For $\mathsf{L} \in \{\mathsf{LJE}(\Sigma_{\mathcal{L}}), \mathsf{LJE}\}$: $\mathsf{L} \vdash^c_d \Gamma \Rightarrow C$ implies $\mathsf{L} \vdash^c_d \Gamma, A \Rightarrow C$.

Proof Left to the reader. For the quantifier rules, use Lemma 4.2 to repair variable clashes. □

Lemma 4.4 For $\mathsf{L} \in \{\mathsf{LJE}(\Sigma_{\mathcal{L}}), \mathsf{LJE}\}$: L has contraction. In fact:

$$\mathsf{L} \vdash^c_d \Gamma, A, A \Rightarrow C \ \text{ implies } \ \mathsf{L} \vdash^c_d \Gamma, A \Rightarrow C \tag{3}$$

Proof To show that the system has contraction we need the following claim.

Claim 4.5 For $d > 0$, it holds that

$$
\begin{array}{lll}
\mathsf{L} \vdash^c_d \Gamma, A \wedge B \Rightarrow C & \text{implies} & \mathsf{L} \vdash^c_d \Gamma, A, B \Rightarrow C \\
\mathsf{L} \vdash^c_d \Gamma, A \vee B \Rightarrow C & \text{implies} & \mathsf{L} \vdash^c_d \Gamma, A \Rightarrow C \text{ and } \mathsf{L} \vdash_d \Gamma, B \Rightarrow C \\
\mathsf{L} \vdash^c_d \Gamma \Rightarrow A \to B & \text{implies} & \mathsf{L} \vdash^c_d \Gamma, A \Rightarrow B \\
\mathsf{L} \vdash^c_d \Gamma, \exists x A x \Rightarrow C & \text{implies} & \mathsf{L} \vdash^c_d \Gamma, Ey, Ay \Rightarrow C, \text{ for all } y.
\end{array}
$$

Proof of Claim The only detail here is the possibility of variable clashes. We only treat the case of the existential quantifier, with induction to d. If $\Gamma, \exists x Ax \Rightarrow C$ is an axiom, then so is $\Gamma, Ey, Ay \Rightarrow C$. Suppose it it not an axiom. If in the last inference in the proof of $\Gamma, \exists x Ax \Rightarrow C$, $\exists x Ax$ is not principal, then the induction hypothesis applies: for the rules without eigenvariables this is immediate. For the rules with eigenvariables, if the eigenvariable is y, we just replace it by a fresh eigenvariable not occuring in the proof, and then using the induction hypothesis we obtain a proof of $\Gamma, Ey, Ay \Rightarrow C$ of same rank and depth. If $\exists x Ax$ is principal in the last rule, the result follows immediately. This proofs the claim.

Using this claim we prove (3) with induction to the depth d of the proof of $\Gamma, A, A \Rightarrow C$ in L. If $d = 1$, the sequent is an axiom, and so $\Gamma, A \Rightarrow C$ clearly is an axiom too (also in the case of $\Sigma_{\mathcal{L}}$). Consider the case $d + 1$. If the last rule in the proof is a right rule or the principal formula is in Γ, then the induction hypothesis applies. Therefore, suppose it is a left rule and the principal formula is not in Γ. We distinguish by cases. We treat L$\wedge$ and leave the other cases to the reader. In this case the last part of the proof then looks as follows.

$$
\dfrac{\begin{array}{c}\vdots \\ \Gamma, A \wedge B, A, B \Rightarrow C\end{array}}{\Gamma, A \wedge B, A \wedge B \Rightarrow C}
$$

Assume the cutrank of the proof is n. Let P be the proof of $\Gamma, A \wedge B, A, B \Rightarrow C$. Note that P has depth d. Thus we can apply the claim and obtain a proof of $\Gamma, A, B, A, B \Rightarrow C$ of depth $\leq d$ and cutrank $\leq n$. Then we apply the induction hypothesis, first to A and then to B, and obtain a proof of $\Gamma, A, B \Rightarrow C$ of depth $\leq d$ and cutrank $\leq n$. An application of L$\wedge$ provides a proof of $\Gamma, A \wedge B \Rightarrow C$ of depth $\leq d + 1$ and cutrank $\leq n$, as desired. □

4.6 Restriction to Ecuts

Theorem 4.7 For $\mathsf{L} \in \{\mathsf{LJE}(\Sigma_{\mathcal{L}}), \mathsf{LJE}\}$:
Every sequent in $\mathcal{L}'$ provable in L has a proof in L in which the only cuts are instances of the ECut rule:

$$
\text{ECut:} \ \dfrac{\Gamma \Rightarrow Et \in \Sigma_{\mathcal{L}} \qquad \Gamma, Et \Rightarrow C}{\Gamma \Rightarrow C}
$$

In particular, LJE has cut-elimination.

Proof For a smooth induction it is convenient to replace the Cut rule in LJE by the following generalization of it, the so-called *Mix rule*:

$$\text{Mix}\ \frac{\Gamma \Rightarrow A \qquad \Gamma', A \Rightarrow C}{\Gamma\Gamma' \Rightarrow C}$$

In the Mix rule A is called the cutformula. When we speak about cuts in a proof, we refer to instances of the Cut or the Mix rule. The notions of cutrank are extended to proofs with the Mix rule in the obvious way. To prove the theorem we then show that applications of Mix can be removed from a proof, unless they are instances of EMix, which is

$$\text{EMix:}\ \frac{\Gamma \Rightarrow Et \in \Sigma_{\mathcal{L}} \qquad \Gamma', Et \Rightarrow C}{\Gamma\Gamma' \Rightarrow C}$$

Note that this indeed implies that all provable sequents have a proof in which the only cuts are instances of ECut: $\Gamma \Rightarrow Et \in \Sigma_{\mathcal{L}}$ implies $\Gamma' \Rightarrow Et \in \Sigma_{\mathcal{L}}$ for all Γ', and thus the conclusion of the EMix as above can be obtained also via the ECut

$$\frac{\Gamma\Gamma' \Rightarrow Et \in \Sigma_{\mathcal{L}} \qquad \Gamma\Gamma', Et \Rightarrow C}{\Gamma\Gamma' \Rightarrow C}$$

For now, we call a proof *ecutfree* if all applications of Mix are instances of EMix, and we call it *cutfree* when it contains no cuts at all. Recall that the cutrank *cr(P)* of a proof P is 1 + the maximal complexity of cutformulas in P.

The proof of the theorem consists of two claims. The first shows how to remove cuts of rank > 1 from a proof, and the second shows how cuts of rank 1 that are not instances of EMix can be removed from a proof. These two claims together imply the theorem.

Claim 4.8 For $\mathsf{L} \in \{\mathsf{LJE}(\Sigma_{\mathcal{L}}), \mathsf{LJE}\}$: Every sequent in $\mathcal{L}'$ provable in L has a proof in L in which all cuts have rank 1.

Proof of Claim We treat the case $\mathsf{LJE}(\Sigma_{\mathcal{L}})$, the case LJE is similar. It suffices to show that a proof P ending in a cut

$$\frac{\begin{array}{c}P_1\\ \Gamma \Rightarrow A\end{array} \qquad \begin{array}{c}P_2\\ \Gamma', A \Rightarrow C\end{array}}{\Gamma\Gamma' \Rightarrow C}$$

with $|A| > 0$ and with $cr(P_1), cr(P_2) \leq |A|$, can be transformed into a proof P' of $\Gamma\Gamma' \Rightarrow C$ such that $cr(P') < cr(P)$. Note that $cr(P) = |A| + 1 > 1$. We prove this by induction on the cutrank of P with a subinduction to the level of the lowest cut of maximal rank in P (the level of a cut is the sum of the depths of its two hypotheses). We call $\Gamma \Rightarrow A$ and $\Gamma', A \Rightarrow C$ the hypotheses of the cut and $\Gamma\Gamma' \Rightarrow C$ the conclusion. Since $|A| > 0$, A cannot be principal in an axiom, including $\Sigma_{\mathcal{L}}$. Note also that A cannot be of the form Et. Therefore, we only have to distinguish the following two cases:
(a) the cutformula is not principal in one of the hypotheses,
(b) cutformula is principal in both hypotheses, which are not axioms.

(a) Suppose the cut formula is not principal in one of the hypotheses. If this hypothesis is an instance of axioms Ax or L$\bot$, then so is the conclusion of the cut, and whence we have a cutfree proof of it. If this hypothesis is an instance of an axiom $\Gamma \Rightarrow Et$ in $\Sigma_{\mathcal{L}}$, then since $|A| > 0$ it has to be the right hypothesis. Observe that $(\Gamma \Rightarrow Et) \in \Sigma_{\mathcal{L}}$, implies that $(\Pi \Rightarrow Et) \in \Sigma_{\mathcal{L}}$ for all Π. Hence the conclusion of the cut is a sequent in $\Sigma_{\mathcal{L}}$, in which case we have a cutfree proof of it.

Next suppose that the hypothesis in which A is not principal is the lower sequent of an application of one of the rules. In this case we can cut higher up. That is, suppose the cutformula is not principal in the left hypothesis, and assume this is a two hypotheses rule R, say R$\vee$. Then P looks as follows.

$$\dfrac{\mathrm{R}\vee\ \dfrac{\begin{matrix}P_1\\ \Gamma_1 \Rightarrow A\end{matrix} \qquad \begin{matrix}P_2\\ \Gamma_2 \Rightarrow A\end{matrix}}{\Gamma \Rightarrow A} \qquad \begin{matrix}P_3\\ \Gamma', A \Rightarrow C\end{matrix}}{\Gamma\Gamma' \Rightarrow C}$$

Note that by assumption $cr(P_i) < cr(P)$ for $i = 1, 2, 3$. Then we transform the proof into a proof P' as follows.

$$\mathrm{R}\ \dfrac{\dfrac{\begin{matrix}P_1\\ \Gamma_1 \Rightarrow A\end{matrix} \qquad \begin{matrix}P_3\\ \Gamma', A \Rightarrow C\end{matrix}}{\Gamma_1\Gamma' \Rightarrow C} \qquad \dfrac{\begin{matrix}P_2\\ \Gamma_2 \Rightarrow A\end{matrix} \qquad \begin{matrix}P_3\\ \Gamma', A \Rightarrow C\end{matrix}}{\Gamma_2\Gamma' \Rightarrow C}}{\Gamma\Gamma' \Rightarrow C}$$

Now we have two cuts on A, but the level of the lowest cut of maximal rank in P' is one of these cuts. Thus $cr(P') = cr(P)$, but the level of the lowest cut of maximal rank in P' is smaller than the level of the lowest cut of maximal rank in P. Therefore, we can apply the induction hypothesis and are done. The other cases are similar. Note that in the case that R is a cut, it is by assumption a cut of rank $< |A| + 1$. Hence also in this case the induction hypothesis applies to P'.

(b) In this case the cut is principal in both hypotheses, and both hypotheses are not axioms. We distinguish by cases according to the outermost logical symbol in A: the cases $\wedge$, $\vee$, $\rightarrow$ are treated in the same way as in the case of LJ, see e.g. [15]. We treat the quantifiers.

$\forall$: then P looks as follows:

$$\dfrac{\dfrac{\begin{matrix}P_1\\ \Gamma, Ey \Rightarrow Ay\end{matrix}}{\Gamma \Rightarrow \forall x Ax}\,d_1 \qquad \dfrac{\begin{matrix}P_2\\ \Gamma', \forall x Ax, At \Rightarrow C\end{matrix} \qquad \begin{matrix}P_3\\ \Gamma', \forall x Ax \Rightarrow Et\end{matrix}}{\Gamma', \forall x Ax \Rightarrow C}\,d_2}{\Gamma\Gamma' \Rightarrow C}$$

Note that y is not free in Γ because of the conditions on $R\forall$, and y is not free in Γ', C and t because of the conditions on eigenvariables in a proof. By assmptions on variables, t does not contain eigenvariables or bound variables in P.

We can transform the above proof into the following proof P':

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{\overset{P_1}{\Gamma, Ey \Rightarrow Ay}}{\Gamma \Rightarrow \forall x Ax} \qquad \overset{P_3}{\Gamma', \forall x Ax \Rightarrow Et}}{\Gamma\Gamma' \Rightarrow Et}
\qquad \overset{P_1[t/y]}{\Gamma, Et \Rightarrow At}}{\Gamma\Gamma\Gamma' \Rightarrow At}
\qquad
\dfrac{\dfrac{\overset{P_1}{\Gamma, Ey \Rightarrow Ay}}{\Gamma \Rightarrow \forall x Ax} \qquad \overset{P_2}{\Gamma', \forall x Ax, At \Rightarrow C}}{\Gamma\Gamma', At \Rightarrow C}
}{\Gamma\Gamma\Gamma\Gamma'\Gamma' \Rightarrow C}
$$

Note that the endsequent of $P_1[t/y]$ indeed is $\Gamma, Et \Rightarrow At$ as y is not free Γ. By Lemma 4.2, $P_1[t/y]$ is a proof of $(\Gamma, Et \Rightarrow At)$ in $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ such that $cr(P_1[t/y]) \leq cr(P_1) < cr(P)$. The cuts on $\forall x Ax$ both have a lower level and the same rank as in P. Therefore, we can apply the induction hypothesis and obtain proofs of their conclusions of cutrank $< cr(P)$. Whence there is a proof of $\Gamma\Gamma\Gamma\Gamma'\Gamma' \Rightarrow C$ of cutrank $< cr(P)$. Application of some contractions, Lemma 4.4, gives a proof of $\Gamma\Gamma' \Rightarrow C$ of cutrank $< cr(P)$. This proves the case $\forall$.

$\exists$: Similar. Here P looks as follows:

$$
\dfrac{\dfrac{\overset{P_1}{\Gamma \Rightarrow At} \qquad \overset{P_2}{\Gamma \Rightarrow Et}}{\Gamma \Rightarrow \exists x Ax} \qquad \dfrac{\overset{P_3}{\Gamma', Ey, Ay \Rightarrow C}}{\Gamma', \exists x Ax \Rightarrow C}}{\Gamma\Gamma' \Rightarrow C}
$$

Because of the side condition that y is not free in Γ' and C we can transform this proof into the following proof P':

$$
\dfrac{\overset{P_1}{\Gamma \Rightarrow At} \qquad \dfrac{\overset{P_2}{\Gamma \Rightarrow Et} \qquad \overset{P_3[t/y]}{\Gamma', Et, At \Rightarrow C}}{\Gamma\Gamma', At \Rightarrow C}}{\Gamma\Gamma\Gamma' \Rightarrow C}
$$

By Lemma 4.2, $cr(P_3[t/y]) \leq cr(P_3)$. Thus $cr(P') < cr(P)$. This completes (b) and thereby the proof of the claim. □

Claim 4.9 For $\mathsf{L} \in \{\mathsf{LJE}(\Sigma_{\mathcal{L}}), \mathsf{LJE}\}$: Every sequent in $\mathcal{L}'$ that has a proof in L of cutrank 1, has a proof in L in which all cuts are instances of EMix.

Proof of Claim We treat the case $\mathsf{LJE}(\Sigma_{\mathcal{L}})$. We use induction to the depth d of a proof P of cutrank ≤ 1 of a sequent S. The case $d = 1$ is trivial, as then P consists of an axiom only. Suppose $d > 1$. If the last inference in P is not a cut or it is an application of EMix, we can apply the induction hypothesis and are done. Therefore, suppose P ends in a cut that is not an instance of EMix:

$$
\dfrac{\overset{P_1}{\Gamma \Rightarrow A} \qquad \overset{P_2}{\Gamma', A \Rightarrow C}}{\Gamma\Gamma' \Rightarrow C}\ d
$$

Thus by the induction hypothesis P_1 and P_2 are ecutfree, i.e. all cuts they contain are instances of EMix. And as P has cutrank ≤ 1, A is atomic or $\bot$ or of the form Et. Denote $\Gamma\Gamma' \Rightarrow C$ by S. We distinguish the following cases:
(c) the cutformula is principal in the rigth hypothesis,
(d) the cutformula is not principal in the right hypothesis.

(c) Assume the cutformula is principal in the right hypothesis. The form of A implies that whence the right hypothesis $\Gamma', A \Rightarrow C$ has to be an axiom. Since A is principal in it, $C = A$ or $A = \bot$. In the former case we can obtain a ecutfree proof of S by weakening the sequent $\Gamma \Rightarrow A$. If $A = \bot$, then it follows that either $\bot \in \Gamma$ or A is not principal in the left hypothesis. In the former case S is an instance of L$\bot$ and we are done. In the latter case, since A is not principal in it, $\Gamma \Rightarrow \bot$ is the conclusion of a rule R in which $\bot$ is not principal. In this case one can cut higher up, like in case (b) in the proof of the first claim: we treat the case that R is an EMix, and leave the other cases to the reader. In this case P looks as follows.

$$\dfrac{\dfrac{\Gamma \Rightarrow Et \ \in \ \Sigma_{\mathcal{L}} \qquad \begin{array}{c} P_1 \\ \Gamma', Et \Rightarrow \bot \end{array}}{\Gamma\Gamma' \Rightarrow \bot} \qquad \Gamma'', \bot \Rightarrow C}{\Gamma\Gamma'\Gamma'' \Rightarrow C}$$

We transform this proof into the proof P':

$$\dfrac{\Gamma \Rightarrow Et \ \in \ \Sigma_{\mathcal{L}} \qquad \dfrac{\begin{array}{c} P_1 \\ \Gamma', Et \Rightarrow \bot \end{array} \qquad \Gamma'', \bot \Rightarrow C}{\Gamma'\Gamma'', Et \Rightarrow \bot}}{\Gamma\Gamma'\Gamma'' \Rightarrow \bot}$$

We apply the induction hypothesis to P' and are done.

(d) Assume the cutformula is not principal in the right hypothesis. If $\Gamma', A \Rightarrow C$ is an axiom, then $\bot \in \Gamma'$, $C \in \Gamma'$ or $C = Et$ for some $t \in \mathcal{T}_{\mathcal{L}}$. In all cases S is an instance of the same axiom. If the right hypothesis is an application of a rule R we proceed as follows. We treat the cases that R is a two hypothesis rule that is not a cut, and the case that it is a cut, and leave the other cases to the reader. First, suppose R is not a cut. Then P looks as follows.

$$\dfrac{\begin{array}{c} P_1 \\ \Gamma \Rightarrow A \end{array} \qquad \dfrac{\begin{array}{c} P_2 \\ \Gamma_1, A \Rightarrow C_1 \end{array} \qquad \begin{array}{c} P_3 \\ \Gamma_2, A \Rightarrow C_2 \end{array}}{\Gamma', A \Rightarrow C}\,\mathrm{R}}{\Gamma\Gamma' \Rightarrow C}$$

Note that by the induction hypothesis the P_i are ecutfree. Then we transform the proof into a proof P' as follows.

$$\dfrac{\dfrac{\begin{array}{c} P_1 \\ \Gamma \Rightarrow A \end{array} \qquad \begin{array}{c} P_2 \\ \Gamma_1, A \Rightarrow C_1 \end{array}}{\Gamma\Gamma_1 \Rightarrow C_1} \qquad \dfrac{\begin{array}{c} P_1 \\ \Gamma \Rightarrow A \end{array} \qquad \begin{array}{c} P_3 \\ \Gamma_2, A \Rightarrow C_2 \end{array}}{\Gamma, \Gamma_2 \Rightarrow C_2}}{\Gamma\Gamma' \Rightarrow C}\,\mathrm{R}$$

Since R is not a cut we can apply the induction hypothesis to P' and are done. Finally, we treat the case that R is a cut. By the induction hypothesis it is an instance of EMix. Hence P looks like this:

$$\dfrac{\begin{matrix} P_1 \\ \Gamma \Rightarrow A \end{matrix} \qquad \dfrac{\Gamma', A \Rightarrow Et \ \in \ \Sigma_{\mathcal{L}} \qquad \begin{matrix} P_2 \\ \Gamma'', Et, A \Rightarrow C \end{matrix}}{\Gamma'\Gamma'', A \Rightarrow C}}{\Gamma\Gamma'\Gamma'' \Rightarrow C}$$

Then we transform the proof into a proof P' as follows:

$$\dfrac{\Gamma' \Rightarrow Et \ \in \ \Sigma_{\mathcal{L}} \qquad \dfrac{\begin{matrix} P_1 \\ \Gamma \Rightarrow A \end{matrix} \qquad \begin{matrix} P_2 \\ \Gamma'', Et, A \Rightarrow C \end{matrix}}{\Gamma\Gamma'', Et \Rightarrow C}}{\Gamma\Gamma'\Gamma'' \Rightarrow C}$$

To see that this is indeed a proof, note that $(\Gamma', A \Rightarrow Et) \in \Sigma_{\mathcal{L}}$ implies $t \in \mathcal{T}_{\mathcal{L}}$, which implies $(\Gamma' \Rightarrow Et) \in \Sigma_{\mathcal{L}}$. Now the induction hypothesis applies to P', and we are done. This proves the second claim. □

As explained above, the two claims imply the theorem. □

Corollary 4.10 $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ is consistent.

The cut elimination theorem allows us to proof the following correspondence between LJ and $\mathsf{LJE}(\Sigma_{\mathcal{L}})$, one direction of which has already been proved above.

Proposition 4.11 For every sequent S in $\mathcal{L}$ not containing E:

$$\mathsf{LJ} \vdash S \text{ if and only if } \mathsf{LJE}(\Sigma_{\mathcal{L}}) \vdash S.$$

Proof For the direction from left to right see Proposition 3.2. The direction from right to left: show with induction to the depth of the proof that for Γ and A not containing E, if $Et_1, \ldots, Et_n, \Gamma \Rightarrow A$ is derivable in $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ by a proof in which all cuts are instances of ECut, then $\Gamma \Rightarrow A$ is derivable in LJ. □

References

[1] Baaz, M. and Iemhoff, R., Skolemization in intuitionistic logic, *Manuscript*, 2005.

[2] Beckmann, A. and Preining, N., Linear Kripke Frames and Gödel Logics, *Submitted*, 2005.

[3] Beeson, M., Foundations of Constructive Mathematics, Springer, Berlin, 1985.

[4] Corsi, G., A cut-free calculus for Dummett's LC quantified, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 35, pp. 289-301, 1989.

[5] Corsi, G., A logic characterized by the class of connected models with nested domains, *Studia Logica* 48(1), pp. 15-22, 1989.

[6] Corsi, G., Completeness theorem for Dummett's LC quantified and some of its extensions, *Studia Logica* 51(2), pp. 317-335, 1992.

[7] Heyting, A., Die formalen regeln der intuitionistische Mathematik II, *Sitzungsberichte der Preussischen Akademie von Wissenschaften. Physikalisch-mathematische Klasse*, pp. 57-71, 1930.

[8] Iemhoff, R., A note on linear Kripke models, *Journal of Logic and Computation*, 2005, to appear.

[9] Mints, G.E., Axiomatization of a Skolem function in intuitionistic logic, *Formalizing the dynamics of information*, Faller, M. (ed.) et al., CSLI Lect. Notes 91, pp. 105-114, 2000.

[10] Preining, N., *Complete Recursive Axiomatizability of Gödel Logics*, PhD-thesis, Technical University Vienna, 2003.

[11] Scott, D.S., Identity and existence in intuitionistic logic, *Applications of sheaves, Proc. Res. Symp. Durham 1977*, Fourman (ed.) et al., Lect. Notes Math. 753, pp. 660-696, 1979.

[12] Takano, M., Another proof of the strong completeness of the intuitionistic fuzzy logic, *Tsukuba J. Math.* 11(1), pp. 101-105, 1987.

[13] Takeuti, G. AND Titani, M., Intuitionistic fuzzy logic and intuitionistic fuzzy set theory, *Journal of Symbolic Logic* 49, pp. 851-866, 1984.

[14] Troelstra, A.S., and van Dalen, D., *Constructivism in Mathematics*, vol. I North-Holland, 1988.

[15] Troelstra, A.S., and Schwichtenberg, H.,, *Basic Proof Theory*, Cambridge Tracts in Theoretical Computer Science 43, Cambridge University Press, 1996.

[16] Unterhalt, M., *Kripke-Semantik mit partieller Existenz*, PhD-thesis, University of Münster, 1986.