The eskolemization of universal quantifiers

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Abstract

This paper is a sequel to the papers [4, 6] in which an alternative skolemization method called eskolemization was introduced that, when restricted to strong existential quantifiers, is sound and complete for constructive theories. In this paper we extend the method to universal quantifiers and show that for theories satisfying the witness property it is sound and complete for all formulas. We obtain a Herbrand theorem from this, and apply the method to the intuitionistic theory of equality and the intuitionistic theory of monadic predicates.

Keywords: Skolemization, eskolemization, Herbrand's theorem, constructive theories, intuitionistic logic, decidability.

1 Introduction

Skolemization occurs in many places in mathematics and computer science. Indeed, proofs of universal statements that start with the sentence "Let c be an arbitrary element" implicitly use the fact that proving $\forall x Ax$ is equivalent to proving Ac for an arbitrary element c. In computer science, skolemization is a powerful tool when used in combination with Herbrand's theorem. Together they provide a correspondence between predicate and propositional logic, which is the reason for their important role in automated theorem proving and the investigation of the decidability of a theory.

Skolemization seems to be a method that is particularly useful in a classical setting, since for many nonclassical theories the method is no longer complete, although it is sound in many cases. That is, for A^s being the skolemization of A, we often have

$$\vdash A \ \Rightarrow \ \vdash A^s,$$

but not

$$\vdash A^s \ \Rightarrow \ \vdash A.$$

This, of course, does not exclude the possibility that there are other ways to replace the strong quantifiers in a formula and obtain an equiderivable formula in which all quantifiers are weak. In this paper we present such a method.

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In classical logic skolemization is a method that applies to formulas in prenex normal form, and since every formula has a prenex normal form, the method implicitly applies to all formulas. But in general in nonclassical theories formulas do not have a prenex normal form. This, however, is not the reason for the incompleteness of skolemization in nonclassical theories. For the absence of prenex normal forms can be overcome by skolemizing on the spot: instead of first putting a formula in prenex normal form, one directly skolemizes the strong quantifiers in the formula, that is, the positive occurrences of universal quantifiers and the negative occurrences of existential quantifiers. For classical theories, this generalization of skolemization is also sound and complete, but for many nonclassical theories it still is not.

In [4] an alternative skolemization method called *eskolemization* was introduced that, when restricted to existential quantifiers, is sound and complete for intuitionistic existence logic IQCE, which is intuitionistic logic IQC extended by an existence predicate E. In eskolemization strong existential quantifiers $\exists x Ax$ are replaced by $Ec \wedge Ac$, and strong universal quantifiers $\forall x Ax$ by $Ec \to Ac$, where c is a fresh constant not occurring in A. If the strong quantifiers occur in the scope of weak quantifiers, functions are used instead of constants, in the same way as in skolemization. This method is sound for intuitionistic existence logic, and it was shown in [4] that for strong existential quantifiers it is also complete:

$$\vdash_{\mathsf{IQCE}} A \;\Leftrightarrow \vdash_{\mathsf{IQCE}} A^{\exists},$$

where $A^{\exists}$ denotes the result of eskolemizing only the strong existential quantifiers in A. Since for formulas A not containing E we also have

$$\vdash_{\mathsf{IQC}} A \;\Leftrightarrow \vdash_{\mathsf{IQCE}} A,$$

this method can be viewed as an alternative skolemization method for pure intuitionistic logic as well, since it implies

$$\vdash_{\mathsf{IQC}} A \;\Leftrightarrow \vdash_{\mathsf{IQCE}} A^{\exists}.$$

There are many examples that show that eskolemization is not complete for universal quantifiers, such as the double negation shift, $\forall x \neg\neg Ax \to \neg\neg\forall x Ax$, for which the eskolemization $\forall x \neg\neg Ax \to \neg\neg(Ec \to Ac)$ is derivable, while the formula itself is not.

In a later paper [5], another method to remove strong quantifiers from formulas was introduced, which is sound and complete for constructive theories in the same way as eskolemization is, but for all formulas. Under this translation, $(\cdot)^o$, strong quantifiers are replaced by expressions that besides the existence predicate contain an order relation as well. The method, called *orderization*, is sound and complete for the corresponding logic IQCO, which is intuitionistic existence logic extended by an order relation:

$$\vdash_{\mathsf{IQCO}} A \;\Leftrightarrow \vdash_{\mathsf{IQCO}} A^{o}.$$

Since also for this logic derivability in IQC equals derivability in IQCO, at least for formulas not containing the new symbols, orderization could be viewed as an alternative skolemization method for IQC that applies to all formulas and all theories $\mathcal{T}$ based on intuitionistic logic:

$$\mathcal{T} \vdash_{\mathsf{IQC}} A \;\Leftrightarrow\; \mathcal{T} \vdash_{\mathsf{IQCO}} A^o.$$

In this paper we return to the eskolemization method and try to see how far it can be applied in full. We introduce a property, the *witness property*, which implies the completeness of eskolemization for all formulas. That is, for theories $\mathcal{T}$ satisfying the witness property, we show that for all formulas A:

$$\mathcal{T} \vdash_{\mathsf{IQC}} A \;\Leftrightarrow\; \mathcal{T} \vdash_{\mathsf{IQCE}} A \;\Leftrightarrow\; \mathcal{T} \vdash_{\mathsf{IQCE}} A^e,$$

where A^e denotes the eskolemization of A. As a corollary we obtain an analogue of the Herbrand theorem for universal constructive theories, and show that there exists a propositional formula A', which is the result of replacing the weak quantifiers by term instantiations, such that

$$\mathcal{T} \vdash_{\mathsf{IQC}} A \;\Leftrightarrow\; \mathcal{T} \vdash_{\mathsf{IQCE}} A^e \;\Leftrightarrow\; \mathcal{T} \vdash_{\mathsf{IQCE}} A'.$$

Thus, as for classical logic, we obtain a correspondence between a constructive theory and its propositional fragment. We apply the results to the theory of equality and the theory of monadic predicates.

There are other answers to the failure of skolemization in nonclassical settings. Several results have been obtained here, especially for modal logic, intuitionistic logic, and fuzzy logics. In modal logic, analogues of skolemization and Herbrand's theorem are presented in [12]. As in eskolemization, the language is extended and, using this extra expressive power, a method to remove strong quantifiers from formulas is introduced that is sound and complete and allows for a Herbrand-like theorem.

In the context of fuzzy logics, one of the first questions that was addressed is for which fragments skolemization is complete, and whether there is a corresponding Herbrand theorem. For intuitionistic logic, a large class of formulas belongs to this fragment, and satisfies a Herbrand theorem [15, 16, 18]. For Gödel logic, it is proved in [1, 2, 10] that this fragment at least contains all formulas in prenex normal form, and also that the Herbrand theorem holds for prenex formulas. As is shown in [8], Gödel logic is in fact the only fuzzy logic with a Herbrand theorem for its prenex fragment. For fuzzy logics for which even that does not hold, there is the notion of an approximate Herbrand theorem that could be used instead. This approach first occurred in [21], for Łukasiewicz logic, and has recently been extended to other fuzzy logics based on continuous t-norms, such as Basic logic and Product logic [9]. Thus the search for alternatives to skolemization and Herbrand theorems continues, and who knows what surprising new solutions the future has in store for us?

The paper is built up as follows. In Section 2 we introduce sequent calculi LJE and $\mathsf{LJE}_{\mathcal{L}}$ for existence logic, and in Section 2.3 we discuss theories over this logic. In Section 3 we recall the Kripke semantics for existence logic. In Section 4 we introduce the eskolemization method, which in Section 5 is shown to be sound and complete for theories satisfying the witness property. In Section 6 we prove the Herbrand theorems, and in Section 7 we apply the results to several constructive theories.

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2 The proof system

We work with two languages, $\mathcal{L}$ and $\mathcal{L}_e$. $\mathcal{L}$ can be any language for predicate logic not containing E that contains at least one constant. $\mathcal{L}_e$ can be any language for predicate logic that contains $\mathcal{L}$ and a unary predicate E, the *existence predicate*, and, for every arity, infinitely many functions of that arity. Unless explicitly stated otherwise, formulas and theories are in $\mathcal{L}_e$, where it is assumed that there are always infinitely many functions of every arity that do not occur in the axioms of a theory, so that there are enough functions available to use as skolem functions. As we will see in the definition of existence logic, given the existence predicate, terms, including variables, typically range over existing as well as non-existing objects, while the quantifiers range over existing objects only.

Sequents are expressions of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ range over finite sets of formulas. They are interpreted as $I(\Gamma \Rightarrow \Delta) = (\bigwedge \Gamma \to \bigvee \Delta)$.

Positive and *negative* occurrences of formulas in sequents are inductively defined as follows. Given a sequent $S = (\Gamma \Rightarrow \Delta)$, all formulas in Δ occur positively in S, and all formulas in Γ occur negatively in S. If $A \wedge B$, $A \vee B$, $\forall x Ax$ or $\exists x Ax$ occur positively (negatively) in S, then A occurs positively (negatively) in S. If $A \to B$ occurs positively (negatively) is S, then B occurs positively (negatively) in S and A occurs negatively (positively) in S. The *strong quantifiers* in a sequent are the positive occurrences of universal quantifiers and the negative occurrences of existential quantifiers. The *weak quantifiers* are the quantifiers that are not strong.

2.1 The calculus LJE

The sequent calculus LJE (Figure 1) is an analogue of LJ that includes the existence predicate E and formalizes the intuition that Et means *t exists*. A single-succedent version of the calculus has been introduced in [3]. The system has no rules for weakening and contraction, but these are admissible. A proof system for existence logic was first introduced by Scott in [22], but then in a Hilbert-style formulation.

We let $\mathsf{LJE}^{\mathsf{ex}}$ and $\mathsf{LJ}^{\mathsf{dec}}$ be, respectively, the systems LJE and LJ extended by the following rules, where P ranges over atomic formulas different from E ("ex" standing for both *ex*istence and *ex*cluded middle, and "dec" for decidability):

$$\frac{\Gamma, P \Rightarrow \Delta}{\Gamma \Rightarrow \neg P, \Delta} \qquad \frac{\Gamma \Rightarrow P, \Delta}{\Gamma, \neg P \Rightarrow \Delta}$$

$$\text{Ax}\quad \Gamma, P \Rightarrow P, \Delta \quad (P \text{ atomic}) \qquad\qquad \text{L}\bot\quad \Gamma, \bot \Rightarrow \Delta$$

$$\text{L}\wedge\ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad\qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$\text{L}\vee\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad\qquad \text{R}\vee\ \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}$$

$$\text{L}\rightarrow\ \frac{\Gamma, A \rightarrow B \Rightarrow A, \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta} \qquad\qquad \text{R}\rightarrow\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B, \Delta}$$

$$\text{L}\forall\ \frac{\Gamma, \forall x A x, At \Rightarrow \Delta \qquad \Gamma, \forall x A x \Rightarrow Et, \Delta}{\Gamma, \forall x A x \Rightarrow \Delta} \qquad\qquad \text{R}\forall\ \frac{\Gamma, Ey \Rightarrow Ay}{\Gamma \Rightarrow \forall x A[x/y], \Delta}$$

$$\text{L}\exists\ \frac{\Gamma, Ay, Ey \Rightarrow \Delta}{\Gamma, \exists x A[x/y] \Rightarrow \Delta} \qquad\qquad \text{Cut}\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$\text{R}\exists\ \frac{\Gamma \Rightarrow At, \exists x A x, \Delta \qquad \Gamma \Rightarrow Et, \exists x A x, \Delta}{\Gamma \Rightarrow \exists x A x, \Delta}$$

Figure 1: The sequent calculus LJE. (In L$\exists$ and R$\forall$ y does not occur free in Γ and Δ.)

2.2 The calculus $\mathsf{LJE}_{\mathcal{L}}$

In the calculus LJE, for no term is it assumed that is exists. This implies that one cannot derive formulas such as $\Rightarrow \exists x Ex$ or $\forall x Px \Rightarrow Pt$, although one can derive $\forall x Px, Et \Rightarrow Pt$. This, of course, is undesirable, but as we will see, it is crucial in eskolemization that not all terms do exist, that is, that not for all terms t, Et is derivable. This is the reason for working with two languages: all terms of the language $\mathcal{L}$ exist, while the terms in $\mathcal{L}_e \backslash \mathcal{L}$ do not. That is, we add the following set of axioms to LJE:

$$Ax_{\mathcal{L}} \equiv_{def} \{\Gamma \Rightarrow Et, \Delta \mid t \text{ is a closed term in } \mathcal{L} \text{ and } \Gamma \text{ and } \Delta \text{ are multisets}\}.$$

$\mathsf{LJE}_{\mathcal{L}}$ is LJE extended by $Ax_{\mathcal{L}}$, and $\mathsf{LJE}^{\mathsf{ex}}_{\mathcal{L}}$ is defined similarly. We write $\vdash$, $\vdash^{\mathsf{ex}}$, $\vdash_{\mathsf{LJ}}$, and $\vdash^{\mathsf{dec}}$ for derivability in respectively $\mathsf{LJE}_{\mathcal{L}}$, $\mathsf{LJE}^{\mathsf{ex}}_{\mathcal{L}}$, LJ and $\mathsf{LJ}^{\mathsf{dec}}$.
Recall that $\mathcal{L}$ contains at least one constant. Therefore $Ax_{\mathcal{L}}$ contains at least one sequent, which implies that $(\Rightarrow \exists x Ex \wedge \forall x Ex)$ is derivable. In [3] single-succedent versions of LJE and $\mathsf{LJE}_{\mathcal{L}}$ have been introduced that satisfy a similar kind of cut-elimination as LJE and $\mathsf{LJE}_{\mathcal{L}}$, Theorem 1. Also, these systems are well-behaved in the sense that they have interpolation and the Beth property, and a decidable quantifier-free fragment.

2.3 Theories

The theories we consider are in $\mathcal{L}_e$ and defined over the logic $\mathsf{LJE}_{\mathcal{L}}$, unless explicitly stated otherwise. If a theory is said to be in $\mathcal{L}$ we consider it a theory over LJ. We assume that every theory is axiomatized over one of the logics by a set of sequents, that is, the theories do not contain additional rules. Since every theory is equivalent to such a theory, this does not exclude any theories, but just facilitates the arguments below. All theories that we will consider are closed in the sense that the free variables in the axioms are considered to be universally quantified, or equivalently, that we may substitute any term for them. Of course, in the context of LJ these terms belong to $\mathcal{L}$, while in the context of LJE they belong to $\mathcal{L}_e$ and have to exist, as quantifiers range over existing objects only. This implies that we have to change the axioms slightly if we consider a theory over LJ a theory over LJE. We explain how.

Given a theory $\mathcal{T}$ in $\mathcal{L}$, $\mathcal{T}^{dec}$ is the theory in which the logic LJ is replaced by $\mathsf{LJ}^{\mathsf{dec}}$, and $\mathcal{T}^e$ ($\mathcal{T}^{ex}$) is the theory in which the logic LJ is replaced by $\mathsf{LJE}_{\mathcal{L}}$ ($\mathsf{LJE}_{\mathcal{L}}^{\mathsf{ex}}$), and the axioms $\Gamma \Rightarrow \Delta$ of $\mathcal{T}$ that are not part of the underlying logic by $E\bar{x}, \Gamma \Rightarrow \Delta$, where $\bar{x}$ are all the free variables in $\Gamma \Rightarrow \Delta$. Note that $\mathcal{T}^e \vdash^{\mathsf{ex}}$ equals $\mathcal{T}^{ex} \vdash$.

Thus under these conventions, in going from $\mathcal{T}$ to $\mathcal{T}^e$ or $\mathcal{T}^{ex}$, an axiom of the form $\Rightarrow Bx$ is replaced by $Ex \Rightarrow Bx$, and stands for $\Rightarrow \forall x Bx$. This is the reason for adding $E\bar{x}$ to the antecedents of the axioms: the quantifier $\forall$ ranges over existing objects, and if we did not add $E\bar{x}$, we could derive Bt also for terms t that do not exist.

A theory is *atomic* if it is axiomatized by sequents in which only atomic formulas occur. A *strong quantifier* theory is axiomatized by sequents without weak quantifiers.

It is easy to see that the following lemma holds.

Lemma 1 [3] If a theory $\mathcal{T}$ and a closed sequent S are in $\mathcal{L}$, then

$$\mathcal{T} \vdash_{\mathsf{LJ}} S \text{ if and only if } \mathcal{T}^e \vdash S \qquad \mathcal{T} \vdash^{\mathsf{dec}} S \text{ if and only if } \mathcal{T}^e \vdash^{\mathsf{ex}} S.$$

2.4 Fragments

The $\mathcal{L}$-fragment is the set of sequents that are in $\mathcal{L}$. The *quantifier-free* fragment of a theory consists of all quantifier-free sequents in the language of the theory. In the *strong quantifier* fragment (sq) the sequents do not contain weak quantifiers. In the *strong existential weak quantifier* fragment (sewq) the sequents do not contain strong universal quantifiers. The *strong existential quantifier* fragment (seq) is the intersection of the sq and the sewq fragment. In the *no nesting of strong quantifiers in the scope of weak quantifiers* fragment (nnswq) the sequents do not contain strong quantifiers that are in the scope of weak quantifiers.

2.5 Cut-elimination

The *cut-hull* of a theory is the set of all sequents that have a derivation in $\mathcal{T}$ in which all inferences are cuts or axioms of $\mathcal{T}$ (including the axioms of $\mathsf{LJE}_{\mathcal{L}}$). It is not difficult to prove the following theorem, but we do not need it in what follows, and have therefore omitted the proof. Note that it implies that the quantifier-free fragment of atomic theories is decidable.

Theorem 1 For every atomic theory $\mathcal{T}$, every sequent derivable in $\mathcal{T}$ has a proof in $\mathcal{T}$ in which the conclusion of every cut belongs to the cut-hull of $\mathcal{T}$.

3 Models

In the completeness proof below, Kripke models for the logic $\mathsf{LJE}_{\mathcal{L}}$ are used, and in this section we describe these models. They are close to regular Kripke models, the only difference being that the existence predicate is used in the forcing of quantifiers. Because of the existence predicate, we can without loss of generality assume that the models have constant domains: since quantifiers are assumed to range over existing objects, $k \Vdash Ed$ will replace $d \in D_k$.

A *classical existence model* is a classical model for $\mathcal{L}_e$ defined in the usual way, with the additional requirement that the interpretation of the existence predicate is nonempty. To fix the notation we spell out the definition. The model consists of a pair (D, I), where D is a set and I a map on $\mathcal{L}_e$ such that $I(E)$ is a nonempty unary predicate on D, for every n-ary predicate P in $\mathcal{L}_e$, $I(P)$ is an n-ary predicate on D, and for every n-ary function f in $\mathcal{L}_e$, $I(f)$ is an n-ary function from D^n to D (constants are 0-ary functions). I is extended to the interpretation of formulas in the standard way. For terms t_i, $I(t_1, \dots, t_n)$ is short for $I(t_1), \dots, I(t_n)$. $\bar{d} \in D$ means that $d_i \in D$ for all d_i in the sequence $\bar{d}$.

A *Kripke existence model* is a quadruple $K = (W, \preccurlyeq, D, I)$, where $(W, \preccurlyeq)$ is a rooted frame, D a nonempty set, the *domain*, and I a collection $\{I_k \mid k \in W\}$ such that the (D, I_k) are classical existence models satisfying the persistency requirements, which means that for terms $\bar{t}(\bar{x})$ we have

$$
\begin{aligned}
k \preccurlyeq l \;&\Rightarrow\; \forall \bar{d} \in D : (D, I_k) \models P(\bar{d}) \;\Rightarrow\; (D, I_l) \models P(\bar{d}) \\
k \preccurlyeq l \;&\Rightarrow\; \forall \bar{d} \in D : I_k(\bar{t}(\bar{d})) = I_l(\bar{t}(\bar{d})).
\end{aligned}
$$

In particular, $I_k(t) = I_l(t)$ for all closed terms t, since frames are rooted. When it is clear from the context that we work in $\mathcal{L}_e$ and not in $\mathcal{L}$ we talk about (Kripke) models instead of Kripke existence models.

Given a Kripke existence model $K = (W, \preccurlyeq, D, I)$, the *forcing relation* is defined as follows. For predicates $P(\bar{t}(\bar{x}))$ in $\mathcal{L}_e$ (including E), where $\bar{x}$ are the free variables in the terms $\bar{t}$:

$$\forall \bar{d} \in D : K, k \Vdash P(\bar{t}(\bar{d})) \;\equiv_{def} (D, I_k) \models P(\bar{t}(\bar{d})).$$

We define $\Vdash$ in the usual way for connectives, but differently for the quantifiers:

$$\begin{array}{lcl} k \Vdash \exists x A(x) & \Leftrightarrow & \exists d \in D \; k \Vdash Ed \wedge A(d) \\ k \Vdash \forall x A(x) & \Leftrightarrow & \forall d \in D : \; k \Vdash Ed \rightarrow A(d). \end{array}$$

Note that

$$k \Vdash \forall x A(x) \;\Leftrightarrow\; \forall l \succcurlyeq k \forall d \in D \, l \Vdash Ed \rightarrow Ad.$$

A formula $A(\bar{x})$ is *forced* in K, $K \Vdash A(\bar{x})$, if for all $\bar{a} \in D$, $A(\bar{a})$ is forced at all nodes. A sequent $S = (\Gamma \Rightarrow \Delta)$ is forced, when $I(S)$ is forced. K is an $\mathcal{L}$*-model* when it forces all sequents in $Ax_{\mathcal{L}}$. K is a *tree* if its frame is a tree. It is *well-founded* if its frame has no infinite descending chains, and *conversely well-founded* if its frame has no infinite ascending chains. Finite models are obviously conversely well-founded and well-founded.

Theorem 2 [4] For all theories $\mathcal{T}$ and all closed sequents S: $\mathcal{T} \vdash S$ if and only if $K \Vdash S$ for all $\mathcal{L}$-models K that are well-founded trees and force $\mathcal{T}$.

Since $\mathcal{T}^{ex}$ can be viewed as a theory over LJE containing the axioms $\Rightarrow \forall \bar{x}(P(\bar{x}) \vee \neg P(\bar{x}))$, for all atomic formulas $P(\bar{x})$, the previous theorem implies the following theorem.

Theorem 3 For all theories $\mathcal{T}$ and all closed sequents S: $\mathcal{T} \vdash^{\mathsf{ex}} S$ if and only if $K \Vdash S$ for all $\mathcal{L}$-models K that are well-founded trees and force $\mathcal{T}^{ex}$.

3.1 Correspondence

There is a natural correspondence between Kripke models K in the usual sense, for $\mathcal{L}$, and Kripke existence models K^e for $\mathcal{L}_e$. K and K^e only differ in their domains and the language in which they are a model: if the D_k are domains of K, then the domain of K^e is $\bigcup D_k$, and the existence predicate and the domains of K are connected in the following way:

$$K^e, k \Vdash Ed \;\Leftrightarrow\; d \in D_k.$$

The interpretations of K^e are extensions of the interpretations of K to $\mathcal{L}_e$ that interpret all functions in $\mathcal{L}_e \backslash \mathcal{L}$ as the identity on D, and all predicates in $\mathcal{L}_e \backslash \mathcal{L}$, except E, as empty. (in fact, one could interpret them by arbitrary functions and predicates on D). The following lemma is easy to prove.

Lemma 2 For all closed sequents S in $\mathcal{L}$: $K, k \Vdash S \;\Leftrightarrow\; K^e, k \Vdash S$.

Proof It suffices to show by induction that $K, k \Vdash \Gamma \Rightarrow \Delta$ if and only if $K^e, k \Vdash E\bar{t}, \Gamma \Rightarrow \Delta$, where $\bar{t}$ are all terms that occur in $\Gamma \Rightarrow \Delta$. ♡

A similar correspondence between Kripke models with and without constant domains can be found in the paper [13] by Dick de Jongh.

3.2 The witness property

In this section we introduce a semantical property that is a sufficient condition for the completeness of eskolemization for a theory.

Given a formula Ax, an existence Kripke model has the A-*witness property* if it is a well-founded tree and the following holds:

$$k \not\Vdash \forall xAx \;\;\Rightarrow\;\; \exists d \exists l \succcurlyeq k\big(l \not\Vdash Ad \text{ and } l \Vdash Ed \wedge (Ad \to \forall xAx)\big).$$

If the model satisfies this property for all formulas A it has the witness property. A theory has the (A-)witness property if it is sound and complete with respect to a class of models that satisfy the (A-)witness property.

The name of the property corresponds to the fact that Ad functions as a witness of $\forall xAx$ along any branch through l: $Ed \to Ad$ is forced exactly where $\forall xAx$ is forced. The well-foundedness implies that there is a witness for formulas $\exists xAx$ too: if it is forced along a branch, there is a lowest node where it is forced, say $Ed \wedge Ad$ is forced there. Then along that branch, $\exists xAx$ is forced exactly where $Ed \wedge Ad$ is forced.

Below are examples of models with and without the witness property: the model in Figure 2 has the witness property and the two models in Figure 3 do not. Note that in the left model in Figure 3 $\neg\forall xA(x)$ is forced, and in the right model $\neg\neg\forall xA(x)$.

$$\begin{tikzcd}
k_2 \Vdash Ed \wedge Ad \wedge \forall xAx & & k_3 \Vdash Ed \wedge \neg Ad \wedge \neg\forall xAx \\
& k_1 \Vdash Ed \wedge (Ad \to \forall xAx) \arrow[ul] \arrow[ur] & \\
& k_0 \not\Vdash \forall xAx \arrow[u] &
\end{tikzcd}$$

Figure 2: A model that has the witness property

In the introduction we saw that the *double negation shift*, $\forall x\neg\neg Ax \Rightarrow \neg\neg\forall xAx$, is a counter example to the completeness of eskolemization, since it is not derivable in intuitionistic existence logic while its eskolemized version, $\forall x\neg\neg Ax \Rightarrow \neg\neg(Ec \to Ac)$, clearly is. As we will see, eskolemization is complete for theories with the witness property. Therefore such theories should prove the double negation shift, which indeed they do:

Lemma 3 Every theory with the witness property derives the double negation shift.

Proof It suffices to show that every model K satisfying the witness property is a model of the double negation shift. We therefore assume that k in K forces $\forall x\neg\neg Ax$, and show that for all $l \succcurlyeq k$ there exists a node $m \succcurlyeq l$ that forces $\forall xAx$.

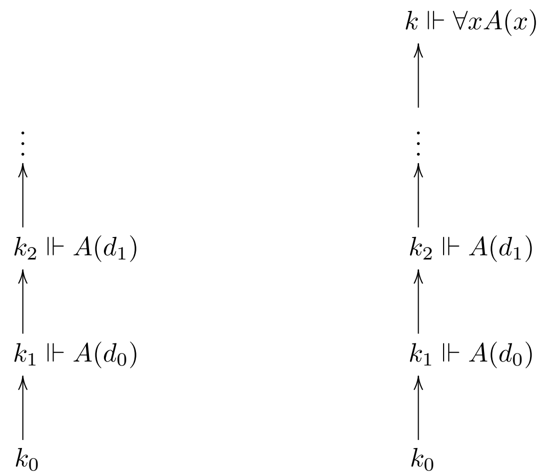

Figure 3: Two models that do not have the witness property. Their domain is $\{d_0, d_1, \dots\}$, Ed_i is forced at all nodes, and $\Vdash A(d_i)$ is written at the first node where it is forced.

Let $l \succcurlyeq k$. If l forces $\forall x Ax$ we are done. If not, the witness property implies that there exists a node $m \succcurlyeq l$ that, for some d, forces $Ed \wedge (Ad \to \forall x Ax)$ and not Ad. Since k forces $\forall x \neg\neg Ax$, it follows that m forces $\neg\neg Ad$, and hence also $\neg\neg\forall x Ax$. Thus there is a node above m, and hence above l, that forces $\forall x Ax$, which is what we had to show. ♡

That the converse of this lemma does not hold is illustrated by the rightmost model in Figure 3, which is a model of the double negation shift, but does not satisfy the witness property. The following theorem shows that many models have the witness property.

Lemma 4 Every tree model that is well-founded and conversely well-founded has the witness property. In particular every finite model does. Thus every theory with the finite model property satisfies the witness property.

4 Eskolemization

In this section we recall the eskolemization procedure introduced in [4]. The *eskolem sequence* of a formula A is a sequence of formulas $A = A_1, \dots, A_n = A^e$ such that A_n does not contain any strong quantifiers and A_{i+1} is the result of replacing the first strong quantifier $QxB(x)$ in A_i (when reading A_i from left to right) by

$$
\begin{array}{ll}
Ef(y_1,\dots,y_n) \to B\big(f(y_1,\dots,y_n)\big) & \text{if } Q = \forall \text{, and by} \\
Ef(y_1,\dots,y_n) \wedge B\big(f(y_1,\dots,y_n)\big) & \text{if } Q = \exists,
\end{array}
$$

where $f \in \mathcal{L}_e \backslash \mathcal{L}$ does not occur in A_i, and the weak quantifiers in the scope of which $QxB(x)$ occurs are exactly $Qy_1, \ldots, Qy_n$. If we work in the context of a theory $\mathcal{T}$, it is also assumed that the skolem functions f do not occur in the axioms of $\mathcal{T}$. The notion is extended to sequents in a straightforward way: if $S = (\Gamma \Rightarrow \Delta)$ and $\big(I(\Gamma \Rightarrow \Delta)\big)^e = I(\Gamma' \Rightarrow \Delta')$, then $S^e \;\equiv_{def} (\Gamma' \Rightarrow \Delta')$. This transformation $(\cdot)^e$ on formulas and sequents is called *existence skolemization*, or *eskolemization* for short.

Note that if $QxB(x)$ is not in the scope of weak quantifiers, then f is a constant, and that given S, S^e is unique up to renaming of the skolem functions. Therefore we speak of *the* eskolemization of a sequent.

Observe that classical skolemization is existence skolemization without the existence predicate, that is, without "$Ef(y_1, \ldots, y_n) \to$" and "$Ef(y_1, \ldots, y_n) \wedge$". Clearly, $\vdash_{\mathsf{LJE}} A \Rightarrow A^e$. Hence also

$$\vdash S \;\Rightarrow \vdash S^e.$$

Here follow some examples of eskolemization (P and Q are unary predicates):

$$\begin{array}{ll} S = \exists x Px \Rightarrow \forall x Qx & S^e = Ec \wedge Pc \Rightarrow Ed \to Qd \\ S = \forall x \exists y R(x,y) \Rightarrow & S^e = \forall x (Ef(x) \wedge R(x, f(x))) \Rightarrow \end{array}$$

Using the completeness result in [4] it can be shown that

$$\begin{array}{rl} \nvdash \forall x(Ax \vee B) \Rightarrow (\forall x Ax \vee B) & \nvdash \forall x(Ax \vee B) \Rightarrow ((Ec \to Ac) \vee B) \\ \nvdash \neg\neg\exists x Ax \to \exists x \neg\neg Ax & \nvdash \neg\neg(Ec \to Ac) \to \exists x \neg\neg Ax. \end{array}$$

Thus although these sequents are counterexamples to the completeness of skolemization, since IQC derives $\forall x(Ax \vee B) \Rightarrow (Ac \vee B)$ and $\neg\neg Ac \to \exists x \neg\neg Ax$, they are no longer so for eskolemization. That eskolemization is still not complete with respect to all formulas is illustrated by the double negation shift, which was discussed in the section on the witness property. As mentioned in the introduction, an alternative skolemization method was developed in [5] that applies to all constructive theories, and therefore covers more theories than the ones discussed in this paper.

5 Completeness

In this section we prove the completeness of eskolemization with respect to theories that satisfy the witness property. We treat the existential and universal quantifiers separately, in Lemma 5 and 6. They state that for S' being the result of replacing a strong existential quantifier $\exists x A(x)$ by $Ef(\bar{y}) \wedge A(f(\bar{y}))$, or a strong universal quantifier $\forall x A(x)$ by $Ef(\bar{y}) \to A(f(\bar{y}))$, it holds that

$$\mathcal{T} \vdash S \;\Leftrightarrow\; \mathcal{T} \vdash S'. \tag{1}$$

Lemma 5, treating the existential quantifier, has been proved before, both semantically and syntactically [4, 6]. Here we present a somewhat different semantical proof, because in this form it resembles the universal case. Also, the proof for the existential quantifier is simpler, and might help the reader to better understand the proof for the universal quantifier.

We first sketch the idea of the proof before we proceed with the technical details. The direction from left to right of (1) is straightforward. For the other direction we restrict ourselves to the case that f is a constant c; the general case will be treated in the proofs. We consider a countermodel $K = (W, \preccurlyeq, I, D)$ to S, and from this construct a countermodel $K' = (W, \preccurlyeq, I', D')$ to S'. D' consists of all closed terms in $D \cup \mathcal{L}_e$, and terms are interpreted as themselves in K'. To make K' into a countermodel to S' we define the forcing in K' in such a way that $Ec \wedge Ac$ is forced in K' at exactly those nodes where $\exists x Ax$ is forced in K, or in the universal case, $Ec \rightarrow Ac$ is forced in K' at exactly those nodes where $\forall x Ax$ is forced in K. If the forcing of other formulas remains unchanged, K' will indeed be a countermodel to S'.

To define the forcing in K', we choose for every node k an element c_k in $D \cup \{c\}$, which will correspond to c in the forcing at k in K'. In the case of the existential quantifier we consider the lowest nodes k where $\exists x A(x)$ is forced, and pick an element $e \in D$ such that $Ee \wedge A(e)$ is forced at k, and put $c_l = e$ for all nodes $l \succcurlyeq k$. In the case of the universal quantifier we consider the lowest nodes k where, for some $e \in D$, Ae is not forced while Ee and $(Ae \rightarrow \forall x Ax)$ are, and put $c_l = e$ for all nodes $l \succcurlyeq k$. In both cases, for all nodes l not yet treated, we put $c_l = c$. Note that in the latter case $c_l \notin D$, while in the former case $c_l \in D$. That such nodes k and elements e exist follows from the fact that the models satisfy the witness property.

When we treat all branches in this way, we have defined c_k for all k in K. Given a term d in $D \cup \mathcal{L}_e$, d_k denotes the term in which c is replaced by c_k, and $\bar{d}_k$ is short for $(d_1)_k, \ldots, (d_n)_k$. The forcing of atomic formulas is defined as follows.

$$\forall \bar{d} \in D' : \begin{cases} K', k \Vdash P(\bar{d}) \;\Leftrightarrow\; K, k \Vdash P(\bar{d}_k) & \text{if } \bar{d}_k \in D \\ K', k \nVdash P(\bar{d}) & \text{otherwise.} \end{cases}$$

Thus at the nodes where $c_k = c$, all atomic formulas containing c are not forced at that node. At the other nodes, the forcing is inherited from K, where c is replaced by c_k. It will be shown in the completeness proofs that c has the desired properties: $Ec \wedge A(c)$ or $Ec \rightarrow A(c)$ is forced in K' exactly where $\exists x A(x)$ or $\forall x A(x)$ is forced in K.

For the case in which we deal with an n-ary skolem function f instead of a constant, we have to choose elements corresponding to $f(\bar{d})$ at every node in the model. We therefore construct a map $w : W \times (D')^n \rightarrow D'$ and let $f(\bar{d})$ correspond to $w\langle k, \bar{d}\rangle$ in the forcing at k in K'. This completes the sketch of the proof, and we continue with the technical details.

5.1 Companions

Since the construction of the model K' does not depend on the form of the quantifier we are considering, we treat this construction separately in this section. Suppose an n-ary function f, a model $K = (W, \preccurlyeq, I, D)$, and a map $w : W \times (D')^n \to D'$ are given. The set of closed terms in $(D \cup \mathcal{L}_e)\backslash\{f\}$ is denoted by $\mathcal{C}$. The model $K' = (W, \preccurlyeq, I', D')$ we are going to define is called the *f-companion* of K. In the completeness proof we will construct w is such a way that

$$k \preccurlyeq l \wedge w\langle k, \bar{d}\rangle \in \mathcal{C} \;\Rightarrow\; w\langle k, \bar{d}\rangle = w\langle l, \bar{d}\rangle. \tag{2}$$

The domain D' of K' is the set of closed terms in $D \cup \mathcal{L}_e$, and terms are interpreted as themselves. To define the forcing of atomic formulas we inductively define for every $k \in W$ the following translation d_k on D'.

$$d_k = \begin{cases} d & \text{if } d \in D \\ I_k(d) & \text{if } d \text{ is a constant in } \mathcal{L}_e \\ I_k(g(\bar{e}_k)) & \text{if } d = g(\bar{e}),\ \bar{e}_k \in \mathcal{C}, \text{ and } g \neq f \\ w\langle k, \bar{e}_k\rangle & \text{if } d = f(\bar{e}) \text{ and } \bar{e}_k \in \mathcal{C} \\ f(\bar{c}) & \text{if } d = g(\bar{e}) \text{ for some } g \in \mathcal{L}_e, \text{ and } \bar{e}_k \notin \mathcal{C}. \end{cases}$$

Here $\bar{c}$ denotes some fixed sequence of n elements in D'. Recall that $\bar{d}_k$ denotes $(d_1)_k, \ldots, (d_n)_k$. Observe that if d does not contain f, $d_k \in \mathcal{C}$. The forcing of atomic formulas $P(\bar{x})$, including E, is defined in the following way.

$$\forall \bar{d} \in D' : \begin{cases} K', k \Vdash P(\bar{d}) \;\Leftrightarrow\; K, k \Vdash P(\bar{d}_k) & \text{if } \bar{d}_k \in \mathcal{C} \\ K', k \nVdash P(\bar{d}) & \text{otherwise.} \end{cases}$$

The upwards persistency requirement for atomic formulas, and hence for all formulas, is satisfied, because (2) implies

$$k \preccurlyeq l \wedge d_k \in \mathcal{C} \;\Rightarrow\; d_k = d_l, \tag{3}$$

That the upwards persistency requirement is also satisfied for terms follows from the fact that terms are interpreted as themselves in K'. Also note that

$$K', k \Vdash Ed \;\Leftrightarrow\; d_k \in \mathcal{C} \wedge K, k \Vdash Ed_k. \tag{4}$$

This model K', the f-companion of K, is the main ingredient in the following two lemmas, which together form the completeness proof.

5.2 The completeness proof

Lemma 5 If $\mathcal{T}$ is a theory, S a closed sequent, $\exists x Ax$ is an occurrence of a strong existential quantifier in S, and S' is the result of replacing this occurrence by $Ef(\bar{y}) \wedge Af(\bar{y})$, where $\bar{y}$ are the variables of all the weak quantifiers in the scope of which $\exists x Ax$ occurs, and $f \in \mathcal{L}_e \backslash \mathcal{L}$ does not occur in S, then

$$\mathcal{T} \vdash S \;\Leftrightarrow\; \mathcal{T} \vdash S'.$$

Proof The non-trivial part is to show that $\mathcal{T} \nvdash S$ implies $\mathcal{T} \nvdash S'$. Since this is a semantical proof, it is more convenient to consider sentences rather than sequents. Therefore let $C = I(S)$, and $C' = I(S')$, and suppose there is an $\mathcal{L}$-model K of $\mathcal{T}$ that refutes C. By Theorem 2 we can assume that K is a well-founded tree. We will define a map $w : W \times (D')^n \to D'$ such that the corresponding f-companion K' refutes C'. We assume that $\bar{y}$ consists of one variable, the general case being similar. Thus $A = A(x, y)$. The set of closed terms in $(D \cup \mathcal{L}_e) \backslash \{f\}$ is denoted by $\mathcal{C}$.
w will be defined in stages, $w_i : W \times D'_i \to D'$, where D'_i are the terms of depth i in D', and $w = \bigcup w_i$, that is, for $d \in D'_i$, $w\langle k, d\rangle = w_i\langle k, d\rangle$. For $d \in D'_i$, we define d^i_k as in the definition of the f-companion, but then relativized to w_i. Thus for d a constant in $D \cup \mathcal{L}_e$, we define

$$d^0_k = \begin{cases} d & \text{if } d \in D \\ I_k(d) & \text{if } d \text{ is a constant in } \mathcal{L}_e. \end{cases}$$

And given w_i and $d \in D'_{i+1}$, d^{i+1}_k is defined as follows, where $\bar{d}^j_k$ is short for $(d_1)^j_k, \ldots, (d_n)^j_k$.

$$d^{i+1}_k = \begin{cases} I_k(g(\bar{e}^i_k)) & \text{if } d = g(\bar{e}),\ \bar{e}^i_k \in \mathcal{C}, \text{ and } g \neq f \\ w_i\langle k, e\rangle & \text{if } d = f(e) \text{ and } e^i_k \in \mathcal{C} \\ f(a) & \text{if } d = g(\bar{e}) \text{ for some } g \in \mathcal{L}_e, \text{ and } \bar{e}^i_k \notin \mathcal{C}. \end{cases}$$

Here a denotes some fixed element in D', it does not matter which one. Note that for all $d \in D'_0$, d^0_k is defined, and if w_i is defined, then so is d^{i+1}_k for all $d \in D'_{i+1}$. This implies that the following inductive definition of the w_i is well-defined. For $i \geq 0$ and $d \in D'_i$, w_i is defined as follows.

(a) Consider the lowest nodes k in K for which $d^i_k \in \mathcal{C}$ and $\exists x A(x, d^i_k)$ is forced at k in K. This means that for no node l below one of these k's, $d^i_l \in \mathcal{C}$ and l forces $\exists x A(x, d^i_l)$. For all these lowest nodes k we pick an element $c^k \in D$ for which k forces $Ec^k \wedge A(c^k, d^i_k)$ and put $w_i\langle l, d\rangle = c^k$ for all $l \succcurlyeq k$. Note that because K is well-founded, such a node k exists along every branch unless for all nodes l along the branch either $d^i_l \notin \mathcal{C}$ or $l \nVdash \exists x A(x, d^i_l)$.

(b) For all k and $d \in D'_i$ for which $w_i\langle k, d\rangle$ is not defined in (a), put $w_i\langle k, d\rangle = f(d)$.

Note that w_i is indeed a map: for all k and $d \in D'_i$, $w_i\langle k, d\rangle$ is not defined twice, as K is a tree.
The case that f has larger arity than 1 is similar to the case we consider here. For the case that f is a constant, the definition of w_0 has to be changed accordingly. This was explained in the proof sketch above. It is easy to show with induction on i that for $d \in D'_i$, d_k, as defined in the definition of f-companion, equals d^i_k.

In the following observations we use that in the definition of w_i, in (a) we have $w_i\langle k, d\rangle \in D \subseteq \mathcal{C}$, and in (b) we have $w_i\langle k, d\rangle \notin \mathcal{C}$. It is easy to prove by induction on w_i that

$$k \preccurlyeq l \wedge w\langle k, \bar{d}\rangle \in \mathcal{C} \ \Rightarrow\ w\langle k, \bar{d}\rangle = w\langle l, \bar{d}\rangle.$$

Hence (3). Recall that (4) holds too.

To complete the theorem it suffices to show that $K', k \Vdash C' \Leftrightarrow K, k \Vdash C$ and that K' is a model of $\mathcal{T}$. We first show that for all formulas B,

$$\forall \bar{d}_k \in \mathcal{C}:\ K', k \Vdash B(\bar{d}) \ \Leftrightarrow\ K, k \Vdash B(\bar{d}_k). \tag{5}$$

We prove this by induction on the complexity of B. If B is a predicate, the definition of the forcing of atomic formulas in K' applies. Conjunction, disjunction, and implication are straightforward. Note that for implication we use (3). We treat the quantifiers, where we suppress $\bar{d}$.

$\forall$ $\Rightarrow$: If $K, k \nVdash \forall z B(z)$, then there is some $l \succcurlyeq k$ and $e \in D$ such that $K, l \Vdash Ee$ and $K, l \nVdash B(e)$. Since $e \in D$, $e_l = e$ and thus $e_l \in \mathcal{C}$. Therefore $K', l \Vdash Ee$ and $K', l \nVdash B(e)$ by the induction hypothesis. Hence $K', k \nVdash \forall z B(x)$.

$\Leftarrow$: If $K', k \nVdash \forall z B(z)$, then there is some $l \succcurlyeq k$ and $e \in D'$ such that $K', l \Vdash Ee$ and $K', l \nVdash B(e)$. Hence $e_l \in \mathcal{C}$ by (4). Thus $K, l \Vdash Ee_l$ and $K, l \nVdash B(e_l)$ by the induction hypothesis. Hence $K, k \nVdash \forall z B(x)$.

$\exists$ This follows from the induction hypothesis in the same way as for the universal quantifier. This proves (5). From this it follows that K' is a model of $\mathcal{T}$.

It remains to show that

$$\forall e_k \in \mathcal{C}:\ K', k \Vdash Ef(e) \wedge A(f(e), e) \ \Leftrightarrow\ K, k \Vdash \exists x A(x, e_k). \tag{6}$$

For together with (5) a straightforward induction on subformulas of C that are not subformulas of $A(x, y)$, shows that $K', k \Vdash C' \Leftrightarrow K, k \Vdash C$. The proof of (6) runs as follows.

$\Rightarrow$: Suppose $K', k \Vdash Ef(e) \wedge A(f(e), e)$. $K', k \Vdash Ef(e)$ implies $f(e)_k \in \mathcal{C}$ by (4). Thus by (5) $K, k \Vdash Ef(e)_k \wedge A(f(e)_k, e_k)$, which implies that $K, k \Vdash \exists x A(x, e_k)$.

$\Leftarrow$: Suppose $K, k \Vdash \exists x A(x, e_k)$. By the definition of w there exists a lowest node $l \preccurlyeq k$ for which $e_l \in \mathcal{C}$, and for which for some $c \in D$, $K, l \Vdash Ec \wedge A(c, e_l)$, and $w\langle m, e\rangle = c$ for all $m \succcurlyeq l$. Note that since $e_l \in \mathcal{C}$ and $l \preccurlyeq k$, $e_k = e_l$. Hence $K, k \Vdash Ec \wedge A(c, e_k)$. Since $e_l \in \mathcal{C}$ and $l \preccurlyeq k$, we have $f(e)_k = f(e)_l = w\langle l, e\rangle = c$, and thus $K', k \Vdash Ef(e) \wedge A(f(e), e)$ by (5). ♡

Lemma 6 If a theory $\mathcal{T}$ satisfies the A-witness property, S is a closed sequent, $\forall x A x$ is an occurrence of a strong universal quantifier in S, and S' is the result of replacing this occurrence by $Ef(\bar{y}) \rightarrow Af(\bar{y})$, where $\bar{y}$ are the variables of all the weak quantifiers in the scope of which $\forall x A x$ occurs, and $f \in \mathcal{L}_e \backslash \mathcal{L}$ does not occur in S, then

$$\mathcal{T} \vdash S \ \Leftrightarrow\ \mathcal{T} \vdash S'.$$

Proof The proof is similar to the proof of the previous lemma, except that the countermodel K that we consider now is a model that has the A-witness property. Recall that this implies that it is a well-founded tree. Again we assume that f is a unary function. The only difference lies in the definition of w_i and the proof of (6). In the definition of w_i only the case (a) differs, which is replaced by the following:

(a) Consider the lowest nodes k in K for which $d^i_k \in \mathcal{C}$, and for some $c \in D$, k forces Ec and $A(c, d^i_k) \to \forall x A(x, d^i_k)$ but not $A(c, d^i_k)$. This means that for no node l below one of the k's there is an $e \in D$ such that l forces Ee and $A(e, d^i_l) \to \forall x A(x, d^i_l)$ but not $A(e, d^i_l)$. For all these lowest nodes k we pick an element $c^k \in D$ such that k forces Ec^k and $A(c^k, d^i_k) \to \forall x A(x, d^i_k)$ but not $A(c^k, d^i_k)$, and put $w_i\langle l, d\rangle = c^k$ for all $l \succcurlyeq k$.

That w_i is indeed a map, that is, for all k and $d \in D'_i$, $w_i\langle k, d\rangle$ is not defined twice, is not difficult to see. It is easy to show by induction on i that for $d \in D'_i$, d_k, as defined in the definition of f-companion, equals d^i_k, and that

$$k \preccurlyeq l \wedge w\langle k, \bar{d}\rangle \in \mathcal{C} \;\Rightarrow\; w\langle k, \bar{d}\rangle = w\langle l, \bar{d}\rangle.$$

To complete the theorem it suffices to show that $K', k \Vdash C' \Leftrightarrow K, k \Vdash C$ and that K' is a model of $\mathcal{T}$. As in the proof of the existential quantifier, it suffices to show that

$$\forall \bar{d}_k \in \mathcal{C}:\; K', k \Vdash B(\bar{d}) \;\Leftrightarrow\; K, k \Vdash B(\bar{d}_k), \tag{7}$$

and that

$$\forall e_k \in \mathcal{C}:\; K', k \nVdash Ef(e) \to A(f(e), e) \;\Leftrightarrow\; K, k \nVdash \forall x A(x, e_k). \tag{8}$$

The proof of (7) is the same as the proof of (5) in the previous lemma. As in the existential case, (7) implies that K' is a model of $\mathcal{T}$, and together with (8) it implies $K', k \Vdash C' \Leftrightarrow K, k \Vdash C$.

Thus it remains to show (8).

$\Rightarrow$: Let $l \succcurlyeq k$ be such that $K', l \Vdash Ef(e)$ and $K', l \nVdash A(f(e), e)$. Since $e_k \in \mathcal{C}$, $e_l = e_k$ by (3). Also, $l \Vdash Ef(e)$ implies $f(e)_l \in \mathcal{C}$ by (4). Thus by the induction hypothesis $K, l \Vdash Ef(e)_l$ and $K, l \nVdash A(f(e)_l, e_l)$, which implies that $K, k \nVdash \forall x A(x, e_k)$.

$\Leftarrow$: Suppose $K, k \nVdash \forall x A(x, e_k)$. By the witness property there exists a node $m \succcurlyeq k$ such that for some $b \in D$, m forces Eb and $A(b, e_k) \to \forall x A(x, e_k)$, but not $A(b, e_k)$. Note that $e_k = e_m \in \mathcal{C}$. Because K is a well-founded tree, there is a smallest such node $l \preccurlyeq m$, for which $e_l \in \mathcal{C}$, and which forces Ec and $A(c, e_l) \to \forall x A(x, e_l)$, but not $A(c, e_l)$, for some $c \in D$. The definition of w implies that for some c with this property, $w\langle n, e\rangle = c$ for all $n \succcurlyeq l$. Thus $f(e)_l = c \in \mathcal{C}$. Hence by (7), $K', l \Vdash Ef(e)$ and $K', l \nVdash A(f(e), e)$. Thus $K', l \nVdash Ef(e) \to A(f(e), e)$. We have to show that $K', k \nVdash Ef(e) \to A(f(e), e)$. We distinguish the cases $k \preccurlyeq l$ and $l \prec k$. The first case is immediate. If $l \prec k$, then $K', k \Vdash Ef(e)$. From $e_l = e_k$ it follows that $K, k \Vdash A(c, e_k) \to \forall x A(x, e_k)$. Since $K, k \nVdash \forall x A(x, e_k)$, also $K, k \not\Vdash A(c, e_k)$. Since $f(e)_k = f(e)_l = c$, $K', k \not\Vdash A(f(e), e)$ by (7). Hence $K', k \not\Vdash Ef(e) \to A(f(e), e)$. ♡

Lemmas 1, 4, 5, and 6 imply the following theorems. Note that the theories in the theorems include theories of the form $\mathcal{T}^{ex}$ or $\mathcal{T}^{dec}$, that is, which logic is $\mathsf{LJE}^{\mathsf{ex}}_{\mathcal{L}}$ or $\mathsf{LJ}^{\mathsf{dec}}$.

Theorem 4 For every theory $\mathcal{T}$ with the witness property, and every closed sequent S:

$$\mathcal{T} \vdash S \;\Leftrightarrow\; \mathcal{T} \vdash S^e.$$

Theorem 5 For every theory $\mathcal{T}$ and every closed sequent S in the sewq fragment:

$$\mathcal{T} \vdash S \;\Leftrightarrow\; \mathcal{T} \vdash S^e.$$

Corollary 1 For every theory $\mathcal{T}$ with the finite model property, and every closed sequent S:

$$\mathcal{T} \vdash S \;\Leftrightarrow\; \mathcal{T} \vdash S^e.$$

Corollary 2 For every theory $\mathcal{T}$ in $\mathcal{L}$ for which $\mathcal{T}^e$ has the witness property, and every closed sequent S in $\mathcal{L}$:

$$\mathcal{T} \vdash_{\mathsf{LJ}} S \;\Leftrightarrow\; \mathcal{T}^e \vdash S \;\Leftrightarrow\; \mathcal{T}^e \vdash S^e.$$

Corollary 3 The sq fragment of every theory with a decidable quantifier-free fragment and the witness property is decidable.

Corollary 4 The seq fragment of every theory with a decidable quantifier-free fragment is decidable. This also holds for theories in $\mathcal{L}$.

Mints proved in [17] that the sq fragment of LJ is decidable. The above theorem holds in particular for empty $\mathcal{T}$, and therefore implies a part of Mints's result, namely that the seq fragment of LJ is decidable. The same holds for LJE and $\mathsf{LJE}_{\mathcal{L}}$.

Note that it follows from Lemma 1 and Corollary 3 that for a theory $\mathcal{T}$ in $\mathcal{L}$ with a decidable quantifier-free fragment, and for which $\mathcal{T}^e$ has the witness property, the sq $\mathcal{L}$-fragment is decidable.

6 Herbrand's Theorem

In the context of intuitionistic logic there is a natural analogue of Herbrand's theorem. Following [11], we define an analogue of the notion of $\wedge\vee$-expansion for the setting of existence logic. Given a theory $\mathcal{T}$ and a sequent S, let $\mathcal{H}(\mathcal{T}, S)$ be the *Herbrand universe* of $(\mathcal{T}, S)$, which consists of all terms generated by the constants and functions occurring in S or in (the axioms of) $\mathcal{T}$, that is, $\mathcal{H}(\mathcal{T},S) = \bigcup \mathcal{H}_i(\mathcal{T},S)$, where

$$
\begin{array}{lll}
\mathcal{H}_0(\mathcal{T},S) & \equiv_{def} & \{t \mid \ t \text{ is a constant in } S \text{ or } \mathcal{T}\} \\
\mathcal{H}_{i+1}(\mathcal{T},S) & & \mathcal{H}_i(\mathcal{T},S) \cup \{f(\bar{t}) \mid \bar{t} \in \mathcal{H}_i(\mathcal{T},S) \text{ and } f \text{ in } S \text{ or in } \mathcal{T}\}.
\end{array}
$$

Note that terms in $\mathcal{T}$ include all terms in $\mathcal{L}$, as theories contain the logic $\mathsf{LJE}_{\mathcal{L}}$, in which axioms all closed terms in $\mathcal{L}$ occur. A sequent S' is an $\wedge\vee$-expansion of a sequent S if every positive occurrence of an existential quantifier $QxA(x)$ in S is replaced by $\bigvee_{i=1}^n Es_i \wedge A(s_i)$ for some terms $s_i \in \mathcal{H}(\mathcal{T},S)$, and every negative occurrence of a universal quantifier $QxA(x)$ is replaced by $\bigwedge_{i=1}^n (Et_i \to A(t_i))$ for some terms $t_i \in \mathcal{H}(\mathcal{T},S)$. It is not difficult to prove the following analogues of Herbrand's theorem. Note that these theorems include theories of the form $\mathcal{T}^{ex}$.

Theorem 6 For every strong quantifier theory $\mathcal{T}$ and for every sequent S there exists an $\wedge\vee$-expansion S' of S such that

$$\mathcal{T} \vdash S \ \Leftrightarrow \ \mathcal{T} \vdash S'.$$

Theorem 7 For every strong quantifier theory $\mathcal{T}$ that has the witness property and for every S, there exists an $\wedge\vee$-expansion S' of S^e such that

$$\mathcal{T} \vdash S \ \Leftrightarrow \ \mathcal{T} \vdash S^e \ \Leftrightarrow \ \mathcal{T} \vdash S'.$$

Theorem 8 For every strong quantifier theory $\mathcal{T}$ and for every S in the sewq fragment, there exists an $\wedge\vee$-expansion S' of S^e such that

$$\mathcal{T} \vdash S \ \Leftrightarrow \ \mathcal{T} \vdash S^e \ \Leftrightarrow \ \mathcal{T} \vdash S'.$$

Corollary 5 For every strong quantifier theory $\mathcal{T}$ and for every S in $\mathcal{L}$ in the sewq fragment, there exists an $\wedge\vee$-expansion S' of S^e such that

$$\mathcal{T} \vdash_{\mathsf{LJ}} S \ \Leftrightarrow \ \mathcal{T}^e \vdash S^e \ \Leftrightarrow \ \mathcal{T}^e \vdash S'.$$

If $\mathcal{T}^e$ also has the witness property this holds for all sequents S in $\mathcal{L}$.

7 Applications

Theorem 8 and Corollary 5 apply to many constructive theories, such as the theory of groups and the theory of apartness as given in [26], and several order theories discussed in [20], and Theorem 7 obviously applies to all strong quantifier theories with the finite model property. Of course, there are many theories without the witness property, but even for some of these the results above can be obtained. We conclude the paper by discussing some typical examples of such theories.

7.1 Equality

Let iEq be the theory of intuitionistic equality without functions given by the following axioms over the logic LJ:

$$\begin{array}{lll} Ax_{eq} & \equiv_{def} & \Rightarrow t = t, \\ & & t = s \Rightarrow s = t, \\ & & r = s, s = t \Rightarrow r = t. \end{array}$$

Thus $\mathsf{iEq}^{\mathsf{e}}$ is $\mathsf{LJE}_{\mathcal{L}}$ extended by the following axioms:

$$\begin{array}{lll} Ax_{eq} & \equiv_{def} & \Gamma, Et \Rightarrow t = t, \Delta \\ & & \Gamma, Et, Es, t = s \Rightarrow s = t, \Delta \\ & & \Gamma, Et, Es, Er, r = s, s = t \Rightarrow r = t, \Delta. \end{array}$$

Because the theory $\mathsf{iEq}^{\mathsf{e}}$ contains the predicate E it should also contain the axiom $Et, Es, t = s \Rightarrow Es$, which is the translation of the axiom $t = s, Pt \Rightarrow Ps$ that holds in equality logic in the presence of predicates P. Since, however, this sequent is already derivable in $\mathsf{LJE}_{\mathcal{L}}$ we do not have to include it in the axioms. We have to add the side formulas Γ and Δ because LJE does not contain weakening.
Theorem 4 and Corollary 8 imply the following.

Theorem 9 For every S in the sewq $\mathcal{L}$-fragment, there exists an $\wedge\vee$-expansion S' of S^e such that

$$\mathsf{iEq} \vdash_{\mathsf{LJ}} S \;\Leftrightarrow\; \mathsf{iEq}^{\mathsf{e}} \vdash S \;\Leftrightarrow\; \mathsf{iEq}^{\mathsf{e}} \vdash S^e \;\Leftrightarrow\; \mathsf{iEq}^{\mathsf{e}} \vdash S'.$$

Thus the seq fragment of iEq is decidable.

For $\mathsf{iEq}^{\mathsf{dec}}$ and $\mathsf{iEq}^{\mathsf{ex}}$ we obtain a full version of Herbrand's theorem by using the following theorem by Craig Smoryński that shows that every formula is equivalent to a formula in the nnswq fragment that contains no strong universal quantifiers. Note that in the eskolemization of such formulas no functions occur.

Theorem 10 [24] In $\mathsf{iEq}^{\mathsf{dec}}$ every sequent S in $\mathcal{L}$ is equivalent to a sequent of the form $\Rightarrow \bigvee_{i=1}^n A_i \wedge B_i$, where the A_i are conjunctions of atomic formulas and their negations, and the B_i are propositional combinations of the formula $\exists x(x = x)$, denoted by E_1, and the formulas

$$E_n \qquad \exists x_1 \dots x_n \bigwedge_{i \neq j} x_i \neq x_j \qquad (n > 1).$$

The sequent $\Rightarrow \bigvee_{i=1}^n A_i \wedge B_i$ is the *normal form* of S and denoted by S_{nf}.

Corollary 6 For every S in $\mathcal{L}$ there exists an $\wedge\vee$-expansion S' of S^e_{nf} such that

$$\mathsf{iEq}^{\mathsf{dec}} \vdash S \;\Leftrightarrow\; \mathsf{iEq}^{\mathsf{ex}} \vdash S \;\Leftrightarrow\; \mathsf{iEq}^{\mathsf{ex}} \vdash S^e \;\Leftrightarrow\; \mathsf{iEq}^{\mathsf{ex}} \vdash S'.$$

7.2 Monadic predicates

In the same way as for equality we can derive Herbrand theorems for the intuitionistic theory of monadic predicates without functions, iMP, again using a theorem by Smoryński. Let P_i range over the predicates in the language.

Theorem 11 For every S in the sewq fragment and in $\mathcal{L}$, there exists an $\wedge\vee$-expansion S' of S^e such that

$$\mathsf{iMP} \vdash S \;\Leftrightarrow\; \mathsf{iMP^e} \vdash S \;\Leftrightarrow\; \mathsf{iMP^e} \vdash S^e \;\Leftrightarrow\; \mathsf{iMP^e} \vdash S'.$$

Thus the seq fragment of iMP is decidable.

Theorem 12 [24] In $\mathsf{iMP^{dec}}$ every sequent S in $\mathcal{L}$ is equivalent to a sequent $\Rightarrow \bigvee_{i=1}^n A_i \wedge B_i$, where the A_i are conjunctions of atomic formulas and their negations, and the B_i are propositional combinations of the formulas

$$\exists x(\bigwedge_{i=1}^m P_i(x) \wedge \bigwedge_{j=1}^n \neg P_j(x).$$

The sequent $\Rightarrow \bigvee_{i=1}^n A_i \wedge B_i$ is the *normal form* of S and denoted by S_{nf}.

Corollary 7 For every S in $\mathcal{L}$ there exists an $\wedge\vee$-expansion S' of S^e_{nf} such that

$$\mathsf{iMP^{dec}} \vdash S \;\Leftrightarrow\; \mathsf{iMP^{ex}} \vdash S \;\Leftrightarrow\; \mathsf{iMP^{ex}} \vdash S^e \;\Leftrightarrow\; \mathsf{iMP^{ex}} \vdash S'.$$

Smoryński uses Theorem 10 and Theorem 12 to prove that $\mathsf{iEq^{dec}}$ and $\mathsf{iMP^{dec}}$ are decidable. This does not follow directly from Corollaries 6 and 7 as one has to bind the number of expansions of a sequent to obtain it. This can be done, but because of lack of space we will not do so in this paper.

Similar theorems as the ones discussed above could be obtained for other theories. The theories treated here are just some typical examples of the possible applications of eskolemization.

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