

The Skolemization of existential quantifiers in intuitionistic logic

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Abstract

In this paper an alternative Skolemization method is introduced that for a large class of formulas is sound and complete with respect to intuitionistic logic. This class extends the class of formulas for which standard Skolemization is sound and complete and includes all formulas in which all strong quantifiers are existential. The method makes use of an existence predicate first introduced by Dana Scott.

Keywords: Skolemization, eSkolemization, Herbrand's theorem, intuitionistic logic, Kripke models, existence predicate.

1 Introduction

Skolemization is the replacement of strong quantifiers in a sequent by fresh function symbols, where a strong quantifier is a positive occurrence of a universal quantifier or a negative occurrence of an existential quantifier. Skolemization can either be considered in the context of derivability or of satisfiability. In this paper we take the former approach; in the latter instead of the strong quantifiers the weak quantifiers are replaced.

Of course, Skolemization applies to formulas as well, but in this paper we will mainly use sequents instead of formulas and only at the end list the corresponding results for formulas. Via Skolemization one obtains a sequent that is equiderivable with the original sequent and that does not contain any strong quantifiers anymore. This idea goes back to Skolem, in his seminal paper from 1920. Together with Herbrand's theorem Skolemization provides a connection between predicate and propositional logic. For a prenex formula $A = \exists x_1 \forall y_1 \dots \exists x_n \forall y_n P(x_1, y_1, \dots, x_n, y_n)$, with P quantifier free, this correspondence takes the following form: there are terms t_{ij} such that

$$\text{LK} \vdash (\Rightarrow A) \Leftrightarrow \text{LK} \vdash \left(\Rightarrow \bigvee_{i=1}^m P(t_{i1}, f_1(t_{i1}), \dots, t_{in}, f_n(t_{i1}, \dots, t_{in})) \right),$$

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where LK is the standard Gentzen calculus for classical predicate logic, and the f_i are ‘fresh’ function symbols, not occurring in A . As in classical logic all formulas have a prenex normal form this provides a correspondence between derivability in predicate and propositional logic, that is, of derivability in predicate logic and in a decidable theory. Of course, the terms t_{ij} cannot be found recursively in A as this would make predicate logic decidable. Nevertheless, Skolemization and Herbrand’s theorem can be very useful, and are extensively used in automated theorem proving.

We let S^S denote the Skolemization of a sequent S . Note the two different uses of S : in $(\cdot)^S$ it stands for the Skolemization translation, in S it ranges over sequents. We say that Skolemization is *sound* or *complete* for L if in the following equivalence \Rightarrow respectively \Leftarrow , holds:

$$\forall S : L \vdash S \Leftrightarrow L \vdash S^S. \quad (1)$$

Thus Skolemization is sound and complete for classical logic. For prenex sequents, i.e. sequents in which all formulas in the sequent are in prenex normal form, it is known that Skolemization is also sound and complete in the setting of intuitionistic logic [16, 17]. Clearly, this does not imply soundness and completeness for all sequents, since not all sequents have a prenex form in intuitionistic logic. This, however, is not the real problem in this setting: even if one Skolemizes infix sequents on the spot (definition in Section 2) Skolemization fails to be complete, but is sound:

$$S \vdash_{LJ} S^S \quad \text{and} \quad LJ \vdash S^S \not\Leftarrow LJ \vdash S.$$

The following sequents

$$\forall x(Ax \vee B) \Rightarrow (\forall xAx \vee B) \quad \neg\neg\exists xAx \Rightarrow \exists x\neg\neg Ax. \quad (2)$$

illustrate these facts. They are not derivable in LJ, but their Skolemizations

$$\forall x(Ax \vee B) \Rightarrow Ac \vee B \quad \neg\neg Ac \Rightarrow \exists x\neg\neg Ax, \quad (3)$$

are. Many more counter examples to the completeness of Skolemization can be found in the papers [16, 17] by Mints in which a full characterization of the sequents for which (1) holds is given. This characterization and its completeness proof are complicated, which already shows that in the context of intuitionistic logic Skolemization is a non trivial affair.

In this paper we define an alternative Skolemization method, called eSkolemization and denoted by $(\cdot)^s$, that is sound and complete for a class of sequents \mathcal{S}^\exists that is a proper extension of the class of sequents for which Skolemization is sound and complete. For example, it contains the formulas in (3), and all sequents in which all strong quantifiers are existential. In the remainder of the introduction we will discuss the main idea behind this alternative Skolemization method, all definitions and precise details will follow in the next section.

1.1 The existence predicate and eSkolemization

ESkolemization makes use of a so-called *existence predicate* first introduced by Dana Scott in [21]. This predicate, E , denotes whether a term exists: Et means

t exists. The idea is that terms may denote partial objects but quantifiers range over existing objects only. Using this predicate one can formulate an extension IQCE of intuitionistic predicate logic IQC that covers the meaning of this predicate, as was done in [21]. A Gentzen calculus LJE corresponding to IQCE was first introduced in [1] and will be recalled in Section 3 below. Based on this notion of existence eSkolemization is defined as follows: in a sequent, negative occurrences $\exists xA(x, \bar{y})$ are replaced by $Ef(\bar{y}) \wedge A(f(\bar{y}), \bar{y})$ and positive occurrences $\forall xA(x, \bar{y})$ by $Ef(\bar{y}) \rightarrow A(f(\bar{y}), \bar{y})$. Thus, when considering the case that $f(\bar{y}) = c$, the existential statements are replaced by “ c exists and $A(c)$ holds” and the universal statements are replaced by “if c exists, then $A(c)$ holds”. The original sequents we consider are in a language \mathcal{L} , the eSkolemized sequents are in a language \mathcal{L}' that includes the Skolem functions and the existence predicate. $\mathcal{S}_{\mathcal{L}}$ denotes the set of sequents in \mathcal{L} .

As it turns out, eSkolemization has many of the nice properties that one would wish any reasonable alternative to Skolemization to have. For example, it satisfies:

- * The eSkolemization S^s of a sequent S is a sequent in which all quantifiers are weak.
- * There is a simple, recursive, way in which we can obtain S^s from S .

Completeness we do not have for LJ pure, as this system does not cover the existence predicate occurring in eSkolemized sequents. We therefore have to move to LJE, and then we do have (Corollary 8.14), for a set of axioms $\Sigma_{\mathcal{L}}$ and set of sequents \mathcal{S}^{\exists} to be defined below, that

$$\forall S \in \mathcal{S}^{\exists} \cap \mathcal{S}_{\mathcal{L}} : \vdash_{\text{LJ}} S \Leftrightarrow \Sigma_{\mathcal{L}} \vdash_{\text{LJE}} S^s. \quad (4)$$

The system LJE together with the axioms $\Sigma_{\mathcal{L}}$ is a very natural extension of LJ, Scott in [21] gives many reasons for extending intuitionistic logic in this way. In fact, for sequents in the language \mathcal{L} it is equal to LJ:

$$\forall S \in \mathcal{S}_{\mathcal{L}} : \vdash_{\text{LJ}} S \Leftrightarrow \Sigma_{\mathcal{L}} \vdash_{\text{LJE}} S. \quad (5)$$

Moreover, like LJ, $\Sigma_{\mathcal{L}} \vdash_{\text{LJE}}$ is decidable for quantifier free formulas (Section 4.10). For all this reasons one might say that the eSkolemization method solves the Skolemization problem for LJ with respect to existential quantifiers.

Finally, let us remark that (4) also holds for the system LJE itself (Theorem 8.13):

$$\forall S \in \mathcal{S}^{\exists} \cap \mathcal{S}_{\mathcal{L}} : \Sigma_{\mathcal{L}} \vdash_{\text{LJE}} S \Leftrightarrow \Sigma_{\mathcal{L}} \vdash_{\text{LJE}} S^s. \quad (6)$$

1.2 Alternative approaches

In the context of automated theorem proving the lack of Skolemization in intuitionistic logic is often overcome by putting certain restrictions on proofs. Here not all proofs in IQC are considered but only a proper subset \mathcal{P} of its proofs, for which it is shown that

$$\text{IQC} \vdash A \Leftrightarrow \exists P \in \mathcal{P} (P \text{ is a proof in IQC of } A^S).$$

Approaches along these lines have e.g. been introduced in [9, 10, 15, 18, 22].

In this paper we have taken a different approach. Our aim was to define an alternative Skolemization method that satisfies properties that standard Skolemization has in classical logic, properties like the $*$'s and (4) above, and for which the (partial) completeness proof does not put restrictions on proofs, like the \mathcal{P} above. Of course, our final aim is to have a method that satisfies (4) for all sequents. Although our results are still partial in this respect, we think them of interest, as in contrast to most approaches in automated theorem proving so far, they provide a canonical approach to Skolemization in intuitionistic logic which does not put restrictions on the proofs allowed and which moreover is based on a clear intuition about what terms in intuitionistic logic should mean. This intuition will be explained in more detail in the next section.

1.3 Section contents

Section 2 contains the definitions of Skolemization and eSkolemization and explains the idea behind the latter. Section 3 introduces the Gentzen calculi LJE and LJE($\Sigma_{\mathcal{L}}$). Section 4 recalls results from [1, 2] on the proof theory of these calculi. In Section 6 an alternative Kripke style semantics is introduced, which is shown to be sound and complete with respect to LJE($\Sigma_{\mathcal{L}}$) in Section 7. In Section 8 the class of sequents \mathcal{S}^{\exists} is defined and it is shown that for this class eSkolemization is sound and complete. Section 8.16 contains examples that show that eSkolemization is not complete with respect to all sequents. Section 9 contains the translation of the obtained results to IQC, i.e. to the setting of formulas.

2 Skolemization and eSkolemization

In this section we give the definitions of classical Skolemization $(\cdot)^S$, introduce the alternative Skolemization method $(\cdot)^s$, and discuss the ideas underlying the latter definition and the proofs to come.

2.1 Preliminaries

We consider languages \mathcal{L} , \mathcal{L}_s , \mathcal{L}^e and \mathcal{L}_s^e as explained in the introduction: \mathcal{L}_s is the *Skolem language* for \mathcal{L} : an extension of \mathcal{L} by function symbols that are not in \mathcal{L} , for every arity infinitely many. Constants are considered as 0-ary function symbols. \mathcal{L}^e is the extension of \mathcal{L} by the existence predicate E and by infinitely many variables. \mathcal{L}_s^e is the extension of \mathcal{L}_s by the existence predicate E and infinitely many variables. Thus $\mathcal{L} \subseteq \mathcal{L}^e$ and $\mathcal{L} \subseteq \mathcal{L}_s \subseteq \mathcal{L}_s^e$. Most of the time we denote \mathcal{L}_s^e by \mathcal{L}' to decrease the number of indices. Thus \mathcal{L}' is the biggest language that we work with. If we talk about a formula without specifying in which language it is, it is tacitly assumed to be in \mathcal{L}' . None of the languages contain equality, see the last section for the reason for this. We assume that \mathcal{L} contains at least one constant and no variables. The former requirement is needed to make the construction of the reduction trees work: see Definition 7.6, the remark at the case R \forall . The latter requirement will be clarified after the introduction of the Gentzen calculi. Given a set D , \mathcal{L}'_D denotes the language \mathcal{L}' extended by the elements of D , which are considered as constants of the language, similarly for \mathcal{L}_D .

The languages all contain \perp , and $\neg A$ is defined as $A \rightarrow \perp$. A, B, C, D, E, \dots range over formulas in \mathcal{L}_s^e , s, t, \dots over terms in \mathcal{L}_s^e . Γ, Δ, Π range over finite multisets of formulas in \mathcal{L}_s^e . A formula is *closed* when it does not contain free variables. A sequent $\Gamma \Rightarrow \Delta$ is closed if all formulas in $\Gamma \cup \Delta$ are closed. A sequent is *in* \mathcal{L}' if all its formulas are in \mathcal{L}' . For a formula A in which x does not occur as a bound variable we write $A[t/x]$ for the result of substituting t for x everywhere in A . When we write $A(x)$ then $A(t)$ denotes the result of replacing some, namely the indicated occurrences of x in A by t . Similar notions are defined for sequents. Proofs are assumed to be trees. We assume that in a proof bound variables and eigenvariables are all different. We often write Ax for $A(x)$. \bar{d} ranges over finite sequences of elements d_1, \dots, d_n . For a set D we write $\bar{d} \in D$ when all elements of \bar{d} are in D , and $\bar{d} \notin D$ when some element of \bar{d} is not in D . $\mathcal{T}_{\mathcal{L}'}$ denotes the set of terms in \mathcal{L}' , $\mathcal{F}_{\mathcal{L}'}$ denotes the set of formulas in \mathcal{L}' , $\mathcal{S}_{\mathcal{L}'}$ denotes the set of sequents in \mathcal{L}' . Similarly for the other three languages.

The notion of *positive* and *negative* occurrences of a formula are inductively defined as follows. If A is an atom A occurs positively in A . If A occurs positively (negatively) in B , then it occurs positively (negatively) in $B \wedge C$, $C \wedge B$, $B \vee C$, $C \vee B$, $C \rightarrow B$, $\forall xB$ and $\exists xB$, and negatively (positively) in $B \rightarrow C$. Similar notions are defined for sequents $\Gamma \Rightarrow \Delta$ by considering them as formulas $\bigwedge \Gamma \rightarrow \bigvee \Delta$.

2.2 Skolemization

Definition 2.3 Q denotes either \forall or \exists . A *strong quantifier* in a formula A is the occurrence of a subformula of the form $QxB(x)$ in A , where $Q = \forall$ if the occurrence is positive, and $Q = \exists$ if the occurrence is negative. The *first strong quantifier* in A is the first strong quantifier in A when reading A from left to right.

The *Skolem sequence* of a formula A is a sequence of formulas $A = A_1, \dots, A_n = A^S$ such that A_n does not contain any strong quantifiers and A_{i+1} is the result of replacing the first strong quantifier $QxB(x)$ in A_i by $B(f(y_1, \dots, y_n))$, where $f \in \mathcal{L}_s \setminus \mathcal{L}$ does not occur in A_i , and the weak quantifiers in the scope of which $QxB(x)$ occurs are exactly Qy_1, \dots, Qy_n . The formula A^S is called the *Skolemization* of A . In the proofs to come it is necessary to define a Skolemization sequence in this deterministic way because in the proof we need that the replaced formula does not occur in the scope of any other strong quantifiers anymore.

The definition of Skolemization we carry over to sequents by considering a sequent $\Gamma \Rightarrow \Delta$ as the formula $\bigwedge \Gamma \rightarrow \bigvee \Delta$. Define

$$\mathcal{M} \equiv_{\text{def}} \{S \in \mathcal{S}_{\mathcal{L}} \mid S \text{ closed, LJ} \vdash S \Leftrightarrow \text{LJ} \vdash S^S\}. \quad (7)$$

As is well-known for classical predicate logic:

$$\text{LK} \vdash S \text{ if and only if } \text{LK} \vdash S^S. \quad (8)$$

This is no longer true for LJ as the examples (2) above showed.

2.4 ESkolemization

In this section we introduce the notion of eSkolemization. We will first present the definition and afterwards explain the idea behind it.

Definition 2.5 The *eSkolem sequence* of a formula A is a sequence of formulas $A = A_1, \dots, A_n = A^s$ such that A_n does not contain any strong quantifiers and A_{i+1} is the result of replacing the first strong quantifier $QxB(x)$ in A_i by

$$Ef(y_1, \dots, y_n) \rightarrow B(f(y_1, \dots, y_n)) \text{ if } Q = \forall$$

and by

$$Ef(y_1, \dots, y_n) \wedge B(f(y_1, \dots, y_n)) \text{ if } Q = \exists,$$

where $f \in \mathcal{L}_s \setminus \mathcal{L}$ does not occur in A_i , and the weak quantifiers in the scope of which $QxB(x)$ occurs are exactly Qy_1, \dots, Qy_n . Again we carry over this definition to sequents by considering a sequent $\Gamma \Rightarrow \Delta$ as the formula $\bigwedge \Gamma \rightarrow \bigvee \Delta$.

Note that if $QxB(x)$ is not in the scope of any weak quantifier, then f is a constant. Observe that in (e)Skolemization *occurrences* are replaced. For example, if $S = (\Rightarrow \forall xBx \wedge \forall xBx)$, then S^s is $(\Rightarrow Ec \wedge Bc \wedge Ed \wedge Bd)$ and not $(\Rightarrow Ec \wedge Bc \wedge Ec \wedge Bc)$. Note that the (e)Skolemization of S is unique up to renaming of the Skolem functions, therefore we speak of *the* (e)Skolemization of S .

Example 2.6

$$\begin{array}{ll} S & S^s \\ \Rightarrow \forall xP(x) & \Rightarrow Ec \rightarrow P(c) \\ \exists xP(x) \Rightarrow \exists xR(x) & Ec \wedge P(c) \Rightarrow \exists xR(x) \\ \exists xP(x) \Rightarrow \forall xR(x) & Ec \wedge P(c) \Rightarrow (Ed \rightarrow R(d)) \\ \Rightarrow \forall x\exists yP(x, y) & \Rightarrow Ec \rightarrow \exists yP(c, y) \\ \Rightarrow \exists x\forall yP(x, y) & \Rightarrow \exists x(Ef(x) \rightarrow P(x, f(x))). \end{array}$$

Once all the systems have been defined we will see that

$$S \vdash_{\text{LJE}} S^s.$$

Furthermore, we will see that for the first counterexample in (2):

$$\not\vdash_{\text{LJ}} \forall x(Ax \vee B) \Rightarrow (\forall xAx \vee B) \text{ and } \Sigma_{\mathcal{L}} \not\vdash_{\text{LJE}} \forall x(Ax \vee B) \Rightarrow ((Ec \rightarrow Ac) \vee B).$$

This shows that although the sequent is a counterexample for the completeness of Skolemization, it no longer is so for eSkolemization. The same holds for the second sequent in (2).

The guiding idea behind eSkolemization is the following. In classical logic, terms range over the same domain as quantifiers. For intuitionistic logic this is no longer the case: by the definition of Kripke models (the interpretation of) terms have to be elements of the domain at the root of a model, while quantifiers might range over objects that only exist at a later stage. The following example illustrates this.

Consider the second sequent $S = \neg\neg\exists xPx \Rightarrow \exists x\neg\neg Px$ in (2) and its Skolemization and eSkolemization

$$S^S = \neg\neg Pc \Rightarrow \exists x\neg\neg Px \quad S^s = \neg\neg(\exists c \wedge Pc) \Rightarrow \exists x\neg\neg Px.$$

A counter model K to S is given by a two node Kripke model $k \preceq l$ with respective domains $D_k = \{0\}$ and $D_l = \{0, 1\}$, and where we force $P(1)$ only at l and $P(0)$ nowhere. Then

$$k \Vdash \neg\neg\exists xPx \quad k \nVdash \exists x\neg\neg Px.$$

Since the interpretation of c has to be an element of every domain, it can only be interpreted as 0. This shows the difference between the range of the quantifier and that of the constant: c can only be 0, while \exists ranges over 0 and 1. And indeed we have $K \nVdash S$ but $K \Vdash S^S$, because $K \Vdash \neg Pc$, and therefore

$$k \Vdash \neg\neg Pc \rightarrow \exists x\neg\neg Px.$$

In the presence of the existence predicate the situation changes. As we will see, in this setting we can assume that Kripke models have constant domains. Quantifiers will range over existing objects only, and not every element will exist at the root, but every element that will ever occur in the model has already a name in the root. Therefore, terms and quantifiers range over the same domain.

Given this the (partial) completeness proof for eSkolemization will be semantical: given a counter model K to S we will construct a counter model K' to S^s . We will illustrate the construction of such a counter model via the example above. Let the set of nodes of K and K' be the same, The domains at k and l are the union of all domains, i.e. $D'_k = D'_l = \{0, 1\}$ (constant domains!). The interpretation of c is 1 and the interpretation of P is as in K . Define

$$k \nVdash E1 \quad l \Vdash E1.$$

It does not matter where we force $E0$ and where not in the model. Note that

$$k \Vdash \neg\neg(Ec \wedge Pc) \quad k \nVdash \exists x\neg\neg Px.$$

The latter follows because $k \nVdash E1 \wedge P1$ and $k \nVdash E0 \wedge P0$. Hence $K' \nVdash S^s$, as desired.

This example roughly illustrates how we will proceed in the completeness proof for eSkolemization in Theorems 8.1 and 8.13. One subtlety still has to be addressed. As the example shows, it is essential that some terms might not exist at every node in the model, i.e. that $k \nVdash Ed$ for some d in the domain of k . This should in particular be possible for terms in which Skolem functions occur. For the original sequent, however, we can assume that all its terms exist. This is where the set $\Sigma_{\mathcal{L}}$ comes into play. Our original sequents will be in the language \mathcal{L} and we will assume that all terms in \mathcal{L} exist:

$$\Sigma_{\mathcal{L}} = \{\Rightarrow Et \mid t \text{ a term in } \mathcal{L}\}.$$

Given this set of axioms, LJE and LJ are the same for formulas in \mathcal{L} (Theorem 4.8). The essential point is that we do not assume the existence of terms containing Skolem functions, i.e. of terms in \mathcal{L}' . This allows the kind of counter models as constructed in the example above.

The discussion above indicates how we want to use the existence predicate. This simple idea is what underlies the main results in the paper, which despite this simplicity turn out to have rather complicated proofs. Although not in the setting of Skolemization, logics containing an existence predicate have been studied before, and e.g. [21] provides many arguments in favor of this logic over IQC, see Section 5 for more on this.

3 The Gentzen calculus LJE

In this section we define the system LJE, an analogue of LJ for intuitionistic predicate logic extended by the existence predicate that covers the intuition that Et means t exists. Such a system was first introduced by Scott in [21], but then in a Hilbert style formulation, and called IQCE. The Gentzen calculus for this system was first introduced by the authors in [1].

Given an existence predicate, terms, including variables, typically range over existing as well as non-existing elements, while the quantifiers range over existing objects only. Proofs are assumed to be trees. We assume that in a proof bound variables and eigenvariables are all different.

The system LJE

$$\begin{array}{ll}
Ax \ \Gamma, P \Rightarrow P \quad (P \text{ atomic}) & L\perp \ \Gamma, \perp \Rightarrow C \\
L\wedge \frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} & R\wedge \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \\
L\vee \frac{\Gamma, A \Rightarrow C \quad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C} & R\vee \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1} \quad i = 0, 1 \\
L\rightarrow \frac{\Gamma, A \rightarrow B \Rightarrow A \quad \Gamma, B \Rightarrow C}{\Gamma, A \rightarrow B \Rightarrow C} & R\rightarrow \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \\
L\forall \frac{\Gamma, \forall x Ax, At \Rightarrow C \quad \Gamma, \forall x Ax \Rightarrow Et}{\Gamma, \forall x Ax \Rightarrow C} & R\forall \frac{\Gamma, Ey \Rightarrow Ay}{\Gamma \Rightarrow \forall x A[x/y]} * \\
L\exists \frac{\Gamma, Ay, Ey \Rightarrow C}{\Gamma, \exists x A[x/y] \Rightarrow C} * & R\exists \frac{\Gamma \Rightarrow At \quad \Gamma \Rightarrow Et}{\Gamma \Rightarrow \exists x Ax} \\
Cut \ \frac{\Gamma \Rightarrow A \quad \Gamma, A \Rightarrow C}{\Gamma \Rightarrow C} &
\end{array}$$

Here (*) denotes the condition that y does not occur free in Γ and C .

We write $LJE \vdash S$ if the sequent S is derivable in LJE. For a set of sequents \mathcal{S} , we say that S is derivable from \mathcal{S} in LJE, and write $\mathcal{S} \vdash_{LJE} S$, if S is derivable

in LJE extended by axioms \mathcal{S} . We define

$$\text{LJE}(\mathcal{S}) \equiv_{\text{def}} \{S \in \mathcal{S}_{\mathcal{L}'} \mid \mathcal{S} \vdash_{\text{LJE}} S\}.$$

In the system LJE no existence of any term is assumed. This implies e.g. that we cannot derive $\Rightarrow \exists xEx$, or $\forall xPx \Rightarrow Pt$, but can derive $\forall xPx, Et \Rightarrow Pt$. For $\Rightarrow \exists xEx$, note that we can derive it from $(\Rightarrow Et)$. Recall that our final aim is that terms in \mathcal{L} exist while the other terms in general do not. Therefore, we define the following sets of sequents

$$\Sigma_{\mathcal{L}} \equiv_{\text{def}} \{\Gamma \Rightarrow Et \mid t \in \mathcal{T}_{\mathcal{L}}, \Gamma \text{ a multiset in } \mathcal{L}'\}.$$

Note that for all sequents $\Gamma \Rightarrow Et$ in $\Sigma_{\mathcal{L}}$, t is a closed term, and that because of the assumptions on \mathcal{L} , $\Sigma_{\mathcal{L}}$ contains at least one sequent. We define

$$\text{LJE}(\Sigma_{\mathcal{L}}) \equiv_{\text{def}} \{S \in \mathcal{S}_{\mathcal{L}'} \mid \Sigma_{\mathcal{L}} \vdash_{\text{LJE}} S\}.$$

We sometimes write \vdash for \vdash_{LJE} when it is clear from the context, and we often write $\vdash_{\mathcal{L}}$ for $\vdash_{\text{LJE}(\Sigma_{\mathcal{L}})}$, the \mathcal{L} indicating that we assume the terms in \mathcal{L} to exist.

Example 3.1

$$\begin{aligned} \not\vdash_{\text{LJE}} \Rightarrow \exists xEx & \quad \vdash_{\text{LJE}} \Rightarrow \forall xEx. \\ \vdash_{\mathcal{L}} \Rightarrow \exists xEx \wedge \forall xEx. & \end{aligned}$$

In Proposition 4.8 the relation between LJ and LJE is explained.

It is easy to see that

Lemma 3.2 $\text{LJE} \vdash A \Rightarrow A^s$. Thus also

$$\vdash_{\mathcal{L}} S \Rightarrow \vdash_{\mathcal{L}} S^s.$$

Remark 3.3 It is now simple to see that the examples (2) are counterexamples to the completeness of Skolemization but not so for eSkolemization:

$$\begin{aligned} \not\vdash_{\text{LJ}} \neg\neg\exists xAx & \Rightarrow \exists x\neg\neg Ax, \\ \vdash_{\text{LJ}} \neg\neg Ac & \Rightarrow \exists x\neg\neg Ax \quad \not\vdash_{\mathcal{L}} \neg\neg(Ec \rightarrow Ac) \Rightarrow \exists x\neg\neg Ax. \\ \not\vdash_{\text{LJ}} \forall x(Ax \vee B) & \Rightarrow (\forall xAx \vee B), \\ \vdash_{\text{LJ}} \forall x(Ax \vee B) & \Rightarrow (Ac \vee B) \quad \not\vdash_{\mathcal{L}} \forall x(Ax \vee B) \Rightarrow ((Ec \rightarrow Ac) \vee B). \end{aligned}$$

4 Properties of LJE

4.1 Uniqueness

Observe that given another predicate E' that satisfies the same rules of LJE as E , it follows that

$$\vdash_{\mathcal{L}} Et \Rightarrow E't \quad \text{and} \quad \vdash_{\mathcal{L}} E't \Rightarrow Et.$$

We namely have that $\vdash_{\mathcal{L}} (\Rightarrow \forall xEx \wedge \forall xE'x)$, and also $\vdash_{\mathcal{L}} (\forall xEx, E't \Rightarrow Et)$ and $\vdash_{\mathcal{L}} (\forall xE'x, Et \Rightarrow E't)$. Finally, two cuts do the trick. This shows that the existence predicate E is unique up to provable equivalence.

4.2 Cut elimination

In this section we recall some results from [1] that show that LJE and LJE($\Sigma_{\mathcal{L}}$) have a restricted form of cut elimination and have weakening and contraction. Some of these results we will need later on, the others are recalled to show that the systems we consider are well-behaved. The proofs of these results are more or less straightforward, where the ECut theorem, which shows that the systems allow some partial cut-elimination, is the most involved, as usual.

Lemma 4.3 (*Substitution Lemma*)

For $\mathsf{L} \in \{\text{LJE}(\Sigma_{\mathcal{L}}), \text{LJE}\}$:

If P is a proof in L of a sequent S in \mathcal{L}' in which y occurs free, and if t is a term in \mathcal{L}' that does not contain eigenvariables or bound variables of P , then $P[t/y]$ is a proof of $S[t/y]$ in L .

Lemma 4.4 ([1]) (*Weakening Lemma*)

For $\mathsf{L} \in \{\text{LJE}(\Sigma_{\mathcal{L}}), \text{LJE}\}$: $\mathsf{L} \vdash \Gamma \Rightarrow C$ implies $\mathsf{L} \vdash \Gamma, A \Rightarrow C$.

Lemma 4.5 ([1]) (*Contraction Lemma*)

For $\mathsf{L} \in \{\text{LJE}(\Sigma_{\mathcal{L}}), \text{LJE}\}$: $\mathsf{L} \vdash \Gamma, A, A \Rightarrow C$ implies $\mathsf{L} \vdash \Gamma, A \Rightarrow C$.

Theorem 4.6 ([1]) (*ECut theorem*)

For $\mathsf{L} \in \{\text{LJE}(\Sigma_{\mathcal{L}}), \text{LJE}\}$: Every sequent in \mathcal{L}' provable in L has a proof in L in which the only cuts are instances of the ECut rule:

$$\text{ECut: } \frac{\Gamma \Rightarrow Et \in \Sigma_{\mathcal{L}} \quad \Gamma, Et \Rightarrow C}{\Gamma \Rightarrow C}$$

In particular, LJE has cut-elimination.

Corollary 4.7 ([1]) LJE($\Sigma_{\mathcal{L}}$) is consistent.

The cut elimination theorem allows us to proof the following correspondence between LJ and LJE($\Sigma_{\mathcal{L}}$).

Proposition 4.8 ([1]) For all closed sequents S in \mathcal{L} :

$$\vdash_{\text{LJ}} S \text{ if and only if } \vdash_{\mathcal{L}} S.$$

Corollary 4.9 For all sequents $S \in \mathcal{M}$:

$$\vdash_{\text{LJ}} S \text{ if and only if } \vdash_{\mathcal{L}} S^s.$$

(The definition of \mathcal{M} is given in Section 2.3.)

Proof The direction from left to right follows from Lemma 3.2 and the proposition above. For the direction from right to left, assume $\vdash_{\mathcal{L}} S^s$. It is not difficult to see that replacing all expression of the form Et in S^s by \top , yields a formula S' that is equivalent to S^s and derivable in LJ. Hence $\vdash_{\text{LJ}} S$. \square

4.10 Decidability and interpolation

Proposition 4.11 For quantifier free closed sequents the relations \vdash_{LJE} and $\vdash_{\mathcal{L}}$ are decidable.

Proof Show, using the theorem on ECuts above, with induction to the depth of a proof, that when t_i are terms that do not occur in a quantifier free sequent $\Gamma \Rightarrow C$ unless as an element of Γ , then

$$\vdash_{\mathcal{L}} Et_1, \dots, Et_n, \Gamma \Rightarrow C \text{ implies } \vdash_{\mathcal{L}} Et_2, \dots, Et_n, \Gamma \Rightarrow C,$$

and similarly for LJE. \square

Recall that we say that a single conclusion Gentzen calculus \mathbf{L} has *interpolation* if whenever $\mathbf{L} \vdash \Gamma_1, \Gamma_2 \Rightarrow C$, there exists an I in the common language of Γ_1 and $\Gamma_2 \cup \{C\}$ such that

$$\Gamma_1 \vdash_{\mathbf{L}} I \text{ and } I, \Gamma_2 \vdash_{\mathbf{L}} C.$$

In the context of existence logics, the *common language* of two multisets Γ_1 and Γ_2 consists of all variables, \top , \perp and E , and all predicates and non-variable terms that occur both in Γ_1 and Γ_2 .

We say that a Gentzen calculus \mathbf{L} satisfies the *Beth definability property* if whenever $A(R)$ is a formula with R an n -ary relation symbol in a language \mathcal{L} , and R', R'' are two relation symbols not in \mathcal{L} such that

$$\mathbf{L} \vdash A(R') \wedge A(R'') \Rightarrow \forall \bar{x} (R' \bar{x} \leftrightarrow R'' \bar{x}),$$

then there is a formula S in \mathcal{L} such that

$$\mathbf{L} \vdash \forall \bar{x} (S \bar{x} \leftrightarrow R \bar{x}).$$

Theorem 4.12 [2] LJE and LJE($\Sigma_{\mathcal{L}}$) have interpolation.

Theorem 4.13 [2] LJE and LJE($\Sigma_{\mathcal{L}}$) satisfy the Beth definability property.

5 IQCE and IQCE⁺

As remarked above, given an existence predicate, in our systems variables typically range over existing as well as non-existing elements, while quantifiers range over existing objects only. As to the choice of the domain for the variables, there have been different approaches. Scott in [21] introduces a system IQCE for the predicate language with the distinguished predicate E , in which variables range over all objects, like in LJE and LJE($\Sigma_{\mathcal{L}}$). On the other hand, Beeson in [5] discusses a system in which variables range over existing objects only.

The formulation of the system IQCE in [21], where logic with an existence predicate was first introduced, was in Hilbert style, where the axioms and rules

for the quantifiers are the following:

$$\begin{array}{c} \forall xAx \wedge Et \rightarrow At \\ \vdots \\ \frac{B \wedge Ey \rightarrow Ay}{B \rightarrow \forall xAx} * \\ \vdots \\ \frac{Ay \wedge Ey \rightarrow B}{\exists xAx \rightarrow B} * \end{array} \quad \begin{array}{c} \vdots \\ At \wedge Et \rightarrow \exists xAx \end{array}$$

Here the * are the usual side conditions on the eigenvariable y . The system that Scott considered also contained equality, but as our systems don't we do not discuss it here.

The following formulation of IQCE in natural deduction style first appeared in [31], it can be found also in [29]. We call the system NDE (Natural Deduction Existence). It consists of the axioms and connective rules of the standard natural deduction formulation of IQC, where the quantifier rules are replaced by the following rules:

$$\begin{array}{c} [Ey] \\ \vdots \\ \forall I \frac{Ay}{\forall xAx} * \\ \vdots \\ \exists I \frac{At \quad Et}{\exists xAx} \end{array} \quad \begin{array}{c} \vdots \quad \vdots \\ \forall E \frac{\forall xAx \quad Et}{At} \\ [Ay][Ey] \\ \vdots \quad \vdots \\ \exists E \frac{\exists xAx \quad C}{C} * \end{array}$$

Again, the * are the usual side conditions on the eigenvariable y . It is easy to see that the following holds.

Fact 5.1 $\forall A \in \mathcal{F}_{\mathcal{L}}: \vdash_{\text{IQCE}} A$ if and only if $\vdash_{\text{NDE}} A$ if and only if $\vdash_{\text{LJE}} \Rightarrow A$.

Existence logic in which terms range over all objects while quantifiers and variables only range over existing objects is denoted by IQCE^+ and has e.g. been used by Beeson in [5]. The logic is the result of leaving out Ey in the two rules for the quantifiers in IQCE given above and adding Ex as axioms for all variables x . A formulation in natural deduction style is obtained from NDE by replacing the $\forall I$ and $\exists E$ by their standard formulations for IQC and adding Ex as axioms for all variables x . We call the system NDE^+ . There are some details concerning substitutions that we skip here, and only state:

Fact 5.2 $\forall A \in \mathcal{F}_{\mathcal{L}}:$

$\vdash_{\text{IQCE}^+} A$ iff $\vdash_{\text{NDE}^+} A$ iff $\{\Gamma \Rightarrow Ex \mid x \text{ a variable, } \Gamma \text{ a multiset}\} \vdash_{\text{LJE}} \Rightarrow A$.

Unterhalt in [31] thoroughly studied the Kripke semantics of these logics and proved respectively completeness and strong completeness for the systems IQCE and IQCE^+ . In Section 6 our semantics is introduced, and Section 7 discusses his and our completeness results.

6 Semantics

Here we introduce the semantics we are going to use. It consists of Kripke models equipped with a notion of forcing, called *existence forcing* and denoted by \Vdash^e , that slightly deviates from standard forcing and for which $\text{LJE}(\Sigma_{\mathcal{L}})$ is sound and complete, as we will show in the next section. The reason that we have to use a nonstandard notion of forcing is that we have to incorporate the different interpretation of the quantifiers in LJE and $\text{LJE}(\Sigma_{\mathcal{L}})$.

In [31] Unterhalt proved strong completeness for IQCE^+ and IQCE with respect to standard forcing, as explained in [29]. These results are different from ours since we work with a nonstandard forcing method and with a different proof system.

A *classical existence structure* for \mathcal{L}'_D is a pair (D, I_k) such that D is a set and I_k is a map from \mathcal{L}'_D such that

- $I_k(E)$ is a *nonempty* unary predicate on D ,
- for every n -ary predicate P in \mathcal{L}' , $I_k(P)$ is an n -ary predicate on D ,
- for every n -ary function f in \mathcal{L}'_D , $I_k(f)$ is an n -ary function from D to D (constants are considered as 0-ary functions),
- $I_k(a) = a$ for every constant $a \in D$.

For any closed \mathcal{L}'_D -term t , $I_k(t)$ denotes the interpretation of t under I_k in D , which is defined as usual. $I_k(t_1, \dots, t_n)$ is short for $I_k(t_1), \dots, I_k(t_n)$. I_k is extended to an interpretation of all formulas in \mathcal{L}'_D as usual. For \mathcal{L}'_D -sentences A , let $(D, I_k) \models A$ denote that A holds in the structure (D, I_k) , which is defined as usual for classical structures. Note that the interpretation of any closed term in \mathcal{L}'_D is an element of and the same in all domains.

A *frame* is a pair (W, \preceq) where W is a nonempty set and \preceq is a partial order on W with a root k_0 . A *Kripke existence model* on a frame $F = (W, \preceq)$ is a triple $K = (F, D, I)$, where D is a nonempty set called the *domain*, and I is a collection $\{I_k \mid k \in W\}$, such that the (D, I_k) are classical existence structures for \mathcal{L}'_D that satisfy the persistency requirements: for all $k, l \in W$, for all predicates $P(\bar{x})$ in \mathcal{L} and for all closed \mathcal{L}'_D -terms \bar{t} ,

$$k \preceq l \Rightarrow ((D, I_k) \models P(\bar{t}) \Rightarrow (D, I_l) \models P(\bar{t})),$$

$$k \preceq l \Rightarrow I_k(\bar{t}) = I_l(\bar{t}).$$

In particular, $I_k(t) = I_{k_0}(t)$ for all k and all closed terms t in \mathcal{L}'_D . Therefore, we sometimes write $I(t)$ instead of $I_k(t)$. Note that our Kripke models have constant domains! We can do so because the notion of forcing defined below restricts the range of quantifiers to existing objects, which may be a different set from node to node. A similar approach to Kripke semantics with constant domains can be found in [14]. It is not related to the existence predicate, or to proof system, but provides a correspondence between Kripke models and Kripke models with constant domains using one distinguished predicate that more or less plays the role of the existence predicate in our setting.

Given a Kripke existence model $K = (D, \preceq, I)$, the *existence forcing relation* is defined as follows, and denoted by \Vdash^e . For our purposes it suffices to define

the forcing relation $K, k \Vdash^e A$ at node k inductively only for *sentences* in \mathcal{L}'_D . When K is clear from the context we write $k \Vdash^e A$ for $K, k \Vdash^e A$. For predicates $P(\bar{x})$ in \mathcal{L}' (including E) and closed \mathcal{L}'_D -terms t , we put

$$K, k \Vdash^e P(\bar{t}) \equiv_{def} (D, I_k) \models P(\bar{t}).$$

We extend $K, k \Vdash^e A$ to all sentences in \mathcal{L}'_D in the usual way for connectives, but differently for the quantifiers:

$$\begin{aligned} k &\not\Vdash^e \perp \\ k &\Vdash^e A \wedge B \text{ iff } k \Vdash^e A \text{ and } k \Vdash^e B \\ k &\Vdash^e A \vee B \text{ iff } k \Vdash^e A \text{ or } k \Vdash^e B \\ k &\Vdash^e A \rightarrow B \text{ iff } \forall k' \succcurlyeq k : k' \Vdash^e A \Rightarrow k' \Vdash^e B \\ k &\Vdash^e \exists x A(x) \text{ iff } \exists d \in D \ k \Vdash^e Ed \wedge A(d) \\ k &\Vdash^e \forall x A(x) \text{ iff } \forall d \in D : k \Vdash^e Ed \rightarrow A(d). \end{aligned}$$

Note that the upwards persistency requirement for sentences A is fulfilled:

$$k \preccurlyeq l \wedge k \Vdash^e A \Rightarrow l \Vdash^e A.$$

Also note that

$$k \Vdash^e \forall x A(x) \Leftrightarrow \forall l \succcurlyeq k \forall d \in D \ l \Vdash^e Ed \rightarrow Ad.$$

As all forcing and Kripke models in this paper will be existence forcing and Kripke existence models, we leave out the word existence most of the time. For sentences A in \mathcal{L}'_D we say that A is *forced in* K , $K \Vdash^e A$, if for all nodes k , $K, k \Vdash^e A$. For a formula $A(\bar{x})$, $K \Vdash^e A(\bar{x})$ if $K \Vdash^e A[\bar{a}/\bar{x}]$ for all $\bar{a} \in D$. We call K an \mathcal{L} -*model* when

$$\forall t \in \mathcal{T}_{\mathcal{L}} : K \Vdash^e Et.$$

We say that A is \mathcal{L} -*forced*, written $\Vdash^e_{\mathcal{L}} A$, when $K \Vdash^e A$ for all \mathcal{L} -models K .

6.1 The forcing of infinite sequents

In the next section we will define an analogue of LJE fitted to deal with infinite sequents, i.e. sequents $\Gamma \Rightarrow \Delta$ in which Γ and Δ may be infinite. Therefore, we define here already the notion of forcing for sequents that are possibly infinite. We write $k \Vdash^e \Gamma$ meaning that $k \Vdash^e A$ for all $A \in \Gamma$.

We say that a sequent $S = (\Gamma \Rightarrow \Delta)$ in \mathcal{L}'_D in which all free variables are among \bar{x} , is *forced at* K and write $K \Vdash^e S$, when for all $\bar{a} \in D$ and all k : $k \Vdash^e B[\bar{a}/\bar{x}]$ for some $B \in \Delta$ or $k \not\Vdash^e A[\bar{a}/\bar{x}]$ for some $A \in \Gamma$. A sequent $S = (\Gamma \Rightarrow \Delta)$ is *forced*, written $\Vdash^e S$, when $K \Vdash^e S$ for all models K . We say that S is \mathcal{L} -*forced*, written $\Vdash^e_{\mathcal{L}} S$, when $K \Vdash^e S$ for all \mathcal{L} -models K . We say that a collection of sequents \mathcal{S} (\mathcal{L} -)forces S , $\mathcal{S} \Vdash^e S$, when for all (\mathcal{L} -)models K , if $K \Vdash^e S'$ for all $S' \in \mathcal{S}$, then $K \Vdash^e S$.

Note that for finite sequents $(\Gamma \Rightarrow \Delta)$

$$K \Vdash^e (\Gamma \Rightarrow \Delta) \text{ iff } K \Vdash^e (\bigwedge \Gamma \rightarrow \bigvee \Delta),$$

and thus

$$\Vdash^e_{\mathcal{L}} \Gamma \Rightarrow \Delta \text{ iff } \Vdash^e_{\mathcal{L}} (\bigwedge \Gamma \rightarrow \bigvee \Delta).$$

7 Soundness and completeness

In this section we show that $\text{LJE}(\Sigma_{\mathcal{L}})$ is sound and complete with respect to \Vdash^e . The proof uses a Gentzen calculus LJE^∞ that is an alternative of LJE for infinite sequents which succedent can contain more than one formula. The proof follows the pattern of the completeness proof for LJ as given in [27]. This method is similar to that of Beth tableaux. The idea will be explained below.

7.1 An alternative Gentzen calculus

LJE^∞ is similar to LJE , but has structural rules. Because of this, in $\text{L}\rightarrow$ and $\text{L}\forall$ the principal formula does not have to occur in the hypotheses. For soundness, in the rules $\text{R}\forall$ and $\text{R}\rightarrow$ the antecedent may still contain only one formula, like in LJE .

Important In this section sequents are possibly infinite.

The system LJE^∞

$$\begin{array}{c}
 Ax \quad \Gamma, P \Rightarrow P, \Delta \quad (P \text{ atomic}) \qquad L\perp \quad \Gamma, \perp \Rightarrow \Delta \\
 \\
 LW \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \qquad \qquad RW \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta} \\
 \\
 LC \quad \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \qquad \qquad RC \quad \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta} \\
 \\
 L\wedge \quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad \qquad \qquad R\wedge \quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta} \\
 \\
 L\vee \quad \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad \qquad \qquad R\vee \quad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta} \\
 \\
 L\rightarrow \quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta} \qquad \qquad \qquad R\rightarrow \quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \\
 \\
 L\forall \quad \frac{\Gamma, At \Rightarrow \Delta \quad \Gamma \Rightarrow Et, \Delta}{\Gamma, \forall x Ax \Rightarrow \Delta} \qquad \qquad \qquad R\forall \quad \frac{\Gamma, Ey \Rightarrow Ay}{\Gamma \Rightarrow \forall x A[x/y]} * \\
 \\
 L\exists \quad \frac{\Gamma, Ay, Ey \Rightarrow \Delta}{\Gamma, \exists x A[x/y] \Rightarrow \Delta} * \qquad \qquad \qquad R\exists \quad \frac{\Gamma \Rightarrow At, \Delta \quad \Gamma \Rightarrow Et, \Delta}{\Gamma \Rightarrow \exists x Ax, \Delta} \\
 \\
 Cut \quad \frac{\Gamma \Rightarrow A \quad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}
 \end{array}$$

We write $\text{LJE}^\infty(\Sigma_{\mathcal{L}})$ for the system obtained from LJE^∞ by adding the sequents $\Sigma_{\mathcal{L}}$ as axioms. We say that LJE^∞ derives $\Gamma \Rightarrow \Delta$, $\vdash_{\text{LJE}^\infty} \Gamma \Rightarrow \Delta$, when there are finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\vdash_{\text{LJE}^\infty} (\Gamma' \Rightarrow \Delta')$. We say that a set of finite sequents \mathcal{S} derives a sequent S in LJE^∞ , when there are finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\Gamma' \Rightarrow \Delta'$ is derivable in the system LJE^∞ to which the sequents in \mathcal{S} are added as axioms. We have similar notions for $\text{LJE}^\infty(\Sigma_{\mathcal{L}})$. We often write $\vdash_{\mathcal{L}}^\infty$ for $\text{LJE}^\infty(\Sigma_{\mathcal{L}}) \vdash$.

We leave it to the reader to verify that the following holds, using the fact that LJE has weakening and contraction (Lemma's 4.4 and 4.5):

Lemma 7.2 For finite Γ and Δ : $\vdash_{\mathcal{L}}^\infty \Gamma \Rightarrow \Delta$ if and only if $\vdash_{\mathcal{L}} \Gamma \Rightarrow \Delta$.

7.3 Soundness

Theorem 7.4 For all sets of finite closed sequents \mathcal{S} and all closed sequents S in \mathcal{L}' :

$$\mathcal{S} \vdash_{\mathcal{L}}^\infty S \text{ implies } \mathcal{S} \Vdash_{\mathcal{L}}^e S.$$

Proof We only consider the case that \mathcal{S} is empty and that S is a sequent with at most one formula in the succedent and leave the other cases to the reader. For a smooth induction we prove that all axioms of $\text{LJE}^\infty(\Sigma_{\mathcal{L}})$ are \mathcal{L} -forced, and that for all its rules, if the hypotheses of the rule are \mathcal{L} -forced, then so is the conclusion. The case of the axioms is simple and so are most of the rules. We treat the axiom $\Sigma_{\mathcal{L}}$ and the rules $R\forall$ and $R\exists$. Let K be an \mathcal{L} -model. Recall that we write $k \Vdash^e \Gamma$ meaning that $k \Vdash^e A$ for all $A \in \Gamma$.

Consider a sequent $\Gamma \Rightarrow Et$ in $\Sigma_{\mathcal{L}}$. Hence t is a closed term in $\mathcal{T}_{\mathcal{L}}$. Let \bar{x} be all the free variables that occur in Γ . By assumption on \mathcal{L} -models it follows that $K \Vdash^e (\Gamma \Rightarrow Et)[\bar{a}/\bar{x}]$ for all $\bar{a} \in D$.

For $R\forall$ suppose $\Vdash^e \Pi, Ey \Rightarrow Ay$ and y not free in Π . Consider k in K , suppose that the free variables in Π and Ay are among $\bar{x}y$, let $\bar{a} \in D$, and assume $k \Vdash^e \Pi[\bar{a}/\bar{x}]$. We have to show that

$$\forall d \in D : k \Vdash^e (Ed \rightarrow Ad)[\bar{a}/\bar{x}].$$

Therefore, consider $l \succ k$ and $d \in D$ such that $l \Vdash^e Ed$. We have to show that $l \Vdash^e Ad[\bar{a}/\bar{x}]$. As the side condition on $R\forall$ implies that y does not occur free in Π , we have $l \Vdash^e (\Pi \wedge Ey)[\bar{a}d/\bar{x}y]$. As $\Vdash^e \Pi, Ey \Rightarrow Ay$, this implies $l \Vdash^e Ay[\bar{a}d/\bar{x}y]$, that is, $l \Vdash^e Ad[\bar{a}/\bar{x}]$.

For $R\exists$ suppose $\Vdash^e \Pi \Rightarrow At$, $\Vdash^e \Pi \Rightarrow Et$, and let all free variables of Π and At be among \bar{x} , pick $\bar{a} \in D$ and assume $k \Vdash^e \Pi[\bar{a}/\bar{x}]$. We have to show that

$$\exists d \in D : k \Vdash^e (Ed \wedge Ad)[\bar{a}/\bar{x}].$$

Since $\Vdash^e \Pi \Rightarrow Et$ and $k \Vdash^e \Pi[\bar{a}/\bar{x}]$, this gives $k \Vdash^e Et[\bar{a}/\bar{x}]$. Similarly, $k \Vdash^e At[\bar{a}/\bar{x}]$. Let $d = t[\bar{a}/\bar{x}]$. Then we have $k \Vdash^e (Ed \wedge Ad)[\bar{a}/\bar{x}]$, as desired. \square

7.5 Completeness

As mentioned above, the completeness proof given follows the pattern of the completeness proof for LJ as given in [27]. The idea is that if a sequent is undervivable we apply the inference rules in the reversed order as long as possible, resulting in a so-called reduction tree with at least one branch along which all sequents are undervivable. This branch will be a node in the Kripke model, the model that we obtain by repeating this process, and that will refute the sequent we started with. We first have to introduce the notion of a reduction tree, a notion similar to that of a Beth tableau.

Definition 7.6 Given a (possibly infinite) sequent S , the *reduction tree* for S is inductively defined as follows. Recall that we assumed that \mathcal{L} contains at least one constant and no variables, and that \mathcal{L}' has an infinite set of variables. Furthermore, we assume that at every stage of the construction we have infinitely many fresh variables of \mathcal{L}' available, i.e. variables that do not occur in the sequents constructed so far.

The construction of the reduction tree for $S = (\Gamma \Rightarrow \Delta)$ consists of repeated application of steps 0, 1, 2, \dots , 8 which correspond to inference rules of LJE^∞ without the structural rules, $\text{R}\forall$ and $\text{R}\rightarrow$. We leave it to the reader to check that at every stage of the construction we deal with countably infinite sequents only, i.e. with sequents for which the antecedent and succedent contain countably infinite many formulas only.

Step $n = 0$: write S at the bottom of the tree.

Step $n > 0$: if every leaf is an axiom of LJE or a sequent in $\Sigma_{\mathcal{L}}$, then stop. If this is not the case, then this stage is defined according to $n \equiv 0, 1, \dots, 8 \pmod 9$. Let $\Pi \Rightarrow \Lambda$ be any leaf of the tree defined at stage $n - 1$.

$n \equiv 0$: $\text{L}\wedge$ reduction. Let α be a set such that $\{A_{i0} \wedge A_{i1} \mid i \in \alpha\}$ consists exactly of all formulas in Π with outermost logical symbol \wedge to which no reduction has yet been applied. Then above $\Pi \Rightarrow \Lambda$ write the sequent

$$\Pi, \{A_{i0}, A_{i1} \mid i \in \alpha\} \Rightarrow \Lambda.$$

$n \equiv 1$: $\text{R}\wedge$ reduction. Let α be a set such that $\{A_{i0} \wedge A_{i1} \mid i \in \alpha\}$ consists exactly of all formulas in Λ with outermost logical symbol \wedge to which no reduction has yet been applied. Then above $\Pi \Rightarrow \Lambda$ write all sequents of the form

$$\Pi \Rightarrow \{A_{if(i)} \mid i \in \alpha\}, \Lambda$$

for any map $f : \alpha \rightarrow \{0, 1\}$.

$n \equiv 2$: $\text{L}\vee$ reduction. Defined in a similar way as $\text{R}\wedge$ reduction.

$n \equiv 3$: $\text{R}\vee$ reduction. Defined in a similar way as $\text{L}\wedge$ reduction.

$n \equiv 4$: $\text{L}\rightarrow$ reduction. Let α be a set such that $\{A_i \rightarrow B_i \mid i \in \alpha\}$ consists exactly of all formulas in Π with outermost logical symbol \rightarrow to which no reduction has yet been applied. Then for all $f : \alpha \rightarrow \{0, 1\}$, write above $\Pi \Rightarrow \Lambda$ the sequent

$$\Pi, \{B_i \mid f(i) = 1\} \Rightarrow \{A_i \mid f(i) = 0\}, \Lambda.$$

$n \equiv 5$: $\text{L}\forall$ reduction. Let α be a set such that $\{\forall x_i A_i(x_i) \mid i \in \alpha\}$ consists exactly of all formulas in Π with outermost logical symbol \forall . Let \mathcal{T} consists of

all terms t for which Et occurs in Π . Above $\Pi \Rightarrow \Lambda$ write the sequent

$$\Pi, \{A_i(t) \mid i \in \alpha, t \in \mathcal{T}\} \Rightarrow \Lambda.$$

Note that if $\{Et \mid t \in \mathcal{T}_{\mathcal{L}}\} \subseteq \Pi$ we can always carry out this step, since there is at least one constant in \mathcal{L} , which implies there is at least one expression of the form Et in $\{Et \mid t \in \mathcal{T}_{\mathcal{L}}\}$, and thus in Π .

$n \equiv 6$: $L\exists$ reduction. Let α be a set such that $\{\exists x_i A_i(x_i) \mid i \in \alpha\}$ consists exactly of all formulas in Π with outermost logical symbol \exists to which no reduction has yet been applied. Introduce fresh variables $\{y_i \mid i \in \alpha\}$ of \mathcal{L}' , and above $\Pi \Rightarrow \Lambda$ write the sequent

$$\Pi, \{A_i(y_i), Ey_i \mid i \in \alpha\} \Rightarrow \Lambda.$$

$n \equiv 7$: $R\exists$ reduction. Defined in a similar way as $L\forall$ reduction. Let α be a set such that $\{\exists x_i A_i(x_i) \mid i \in \alpha\}$ consists exactly of all formulas in Λ with outermost logical symbol \exists . Let \mathcal{T} consists of all terms t for which Et occurs in Π . Above $\Pi \Rightarrow \Lambda$ write the sequent

$$\Pi \Rightarrow \{A_i(t) \mid i \in \alpha, t \in \mathcal{T}\}, \Lambda.$$

$n \equiv 8$: if $\Pi \Rightarrow \Lambda$ is an axiom of LJE^∞ or a sequent in $\Sigma_{\mathcal{L}}$, then stop. If this is not the case write the same sequent $\Pi \Rightarrow \Lambda$ above it and proceed to $n + 1$.

This completes the definition of reduction trees.

The following lemma (7.8), is non-trivial and crucial in the completeness proof. It is an analogue of a lemma in [27] for LJ, and its main ingredient is the following generalization of König's Lemma.

Proposition 7.7 (A generalized König's Lemma, Takeuti [27]) Let X be any set. Let $*(\cdot)$ be a property on partial functions $f : X \rightarrow \{0, 1\}$. If

1. $*(f)$ holds if and only if there is a finite subset $Z \subseteq X$ such that $*(f \upharpoonright Z)$ (here $f \upharpoonright Z$ is the restriction of f to Z), and
2. $*(f)$ holds for all total functions f on X ,

then there exists a finite set $X' \subseteq X$ such that $*(f)$ for any f with $X' \subseteq \text{dom}(f)$ ($\text{dom}(f)$ is the domain of f).

Proof For completeness sake we repeat Takeuti's proof from [27]. Let Y be the product of $|X|$ times $\{0, 1\}$. Give $\{0, 1\}$ the discrete topology and Y the product topology. Since $\{0, 1\}$ is compact, so is Y by Tychonoff's theorem. For maps f and g call g an *extension* of f , when $\text{dom}(f) \subseteq \text{dom}(g)$ and f and g are equal on $\text{dom}(f)$. For every f with finite domain, let

$$\mathcal{N}_f \equiv_{\text{def}} \{g \mid g \text{ is total and an extension of } f\}.$$

Furthermore, let

$$\mathcal{C} \equiv_{\text{def}} \{\mathcal{N}_f \mid \text{dom}(f) \text{ is finite and } *(f)\}.$$

\mathcal{C} is an open cover of Y . Therefore, \mathcal{C} has a finite subcover, say $\mathcal{N}_{f_1}, \dots, \mathcal{N}_{f_n}$.
Let

$$X' = \text{dom}(f_1) \cup \dots \cup \text{dom}(f_n).$$

Then X' satisfies the theorem: assume $X' \subseteq \text{dom}(f)$. Let g be a total extension of f . Then $*(g)$ by 2. Also, there exists an $i \leq n$ such that $g \in \mathcal{N}_{f_i}$. Thus g is an extension of both f and f_i . Since $\text{dom}(f_i) \subseteq \text{dom}(f)$ it follows that f is an extension of f_i . Therefore, $*(f)$ by 1. \square

Lemma 7.8 If a sequent S is not provable in $\text{LJE}^\infty(\Sigma_{\mathcal{L}})$, then its reduction tree has a branch along which all sequents are underivable in $\text{LJE}^\infty(\Sigma_{\mathcal{L}})$.

Proof In this proof provable will always mean provable in $\text{LJE}^\infty(\Sigma_{\mathcal{L}})$, \vdash stands for $\text{LJE}^\infty(\Sigma_{\mathcal{L}}) \vdash$. We prove the lemma by proving the following: if in a reduction tree, $\Gamma_\beta \Rightarrow \Delta_\beta$ ($\beta = 1, 2, \dots, \alpha$) are all the immediate successors of $\Gamma \Rightarrow \Delta$, then if all these successors are provable, then so is $\Gamma \Rightarrow \Delta$. Recall that a sequent $\Pi \Rightarrow \Lambda$ is provable when there are finite $\Pi' \subseteq \Pi$ and $\Lambda' \subseteq \Lambda$ such that $\Pi' \Rightarrow \Lambda'$ is provable.

We distinguish by cases according to the rule that is applied to $\Gamma \Rightarrow \Delta$ resulting in the immediate successors $\Gamma_\beta \Rightarrow \Delta_\beta$.

$L\wedge$ reduction: then $\Gamma \Rightarrow \Delta$ has one upper sequent, which is of the form $\Gamma, \{A_{i0}, A_{i1} \mid i \in \alpha\} \Rightarrow \Delta$, where $A_{i0} \wedge A_{i1}$ are all the formulas in Γ with outermost logical symbol \wedge to which no reduction has been applied yet. By assumption there are finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ and $B_i \in \{A_{i0}, A_{i1}\}$ for $i \leq n$ such that $\vdash \Gamma', B_1, \dots, B_n \Rightarrow \Delta'$. Hence $\vdash \Gamma', \{A_{i0}, A_{i1} \mid i \leq n\} \Rightarrow \Delta'$, which again implies $\vdash \Gamma', \{A_{i0} \wedge A_{i1} \mid i \leq n\} \Rightarrow \Delta'$. Thus $\vdash \Gamma \Rightarrow \Delta$.

$R\wedge$ reduction: then $\Gamma \Rightarrow \Delta$ has immediate successors $\Gamma \Rightarrow \{A_{if(i)} \mid i \in \alpha\}, \Delta$ for any map $f : \alpha \rightarrow \{0, 1\}$, where $\{A_{i0} \wedge A_{i1} \mid i \in \alpha\}$ consists exactly of all formulas in Δ with outermost logical symbol \wedge . By assumption for all $f : \alpha \rightarrow \{0, 1\}$ there are finite $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$ and $n_f \in \omega$ such that we have $\vdash \Gamma' \Rightarrow \{A_{if(i)} \mid i \leq n_f\}, \Delta'$. Now we are going to use the generalized König's Lemma. We define a property $*(\cdot)$ on the partial functions $f : \alpha \rightarrow \{0, 1\}$ as follows ($\text{dom}(f)$ denotes the domain of f):

$$\begin{aligned} *(f) &\equiv \exists m \exists a_1 \dots a_m \in \text{dom}(f) \exists \text{finite } \Gamma' \subseteq \Gamma, \Delta' \subseteq \Delta : \\ &\quad \vdash \Gamma' \Rightarrow \{A_{if(a_i)} \mid i \leq m\}, \Delta'. \end{aligned}$$

Then conditions 1. and 2. of the generalized König's Lemma 7.7 are satisfied. Hence there is a finite subset $\beta \subseteq \alpha$ such that $*(f)$ whenever $\beta \subseteq \text{dom}(f)$. Let \mathcal{F} be the collection of f for which $\text{dom}(f) = \beta$. Thus for all $f \in \mathcal{F}$ there are finite $\Gamma^f \subseteq \Gamma$ and $\Delta^f \subseteq \Delta$ such that

$$\vdash \Gamma^f \Rightarrow \{A_{if(i)} \mid i \in \beta\}, \Delta^f.$$

Hence by weakening and repeated application of $R\wedge$, one obtains

$$\vdash \{\Gamma^f \mid f \in \mathcal{F}\} \Rightarrow \{A_{i0} \wedge A_{i1} \mid i \in \beta\}, \{\Delta^f \mid f \in \mathcal{F}\}.$$

This implies that $\vdash \Gamma \Rightarrow \Delta$.

The case $R\vee$ is similar to $L\wedge$, and $L\vee$ and $L\rightarrow$ are similar to $R\wedge$.

L \exists reduction: then $\Gamma \Rightarrow \Delta$ has immediate successor

$$\Gamma, \{A_i(y_i), Ey_i \mid i \in \alpha\} \Rightarrow \Delta,$$

where $\{\exists x_i A_i(x_i) \mid i \in \alpha\}$ consists exactly of all formulas in Δ with outermost logical symbol \exists . By assumption there are finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ and $n \in \omega$ such that

$$\vdash \Gamma', \{A_i(y_i), Ey_i \mid i \leq n\} \Rightarrow \Delta'.$$

Applications of L \exists imply that then $\Gamma \Rightarrow \Delta$ is provable too.

The cases R \exists and L \forall are similar. This proves the lemma. \square

Theorem 7.9 For all sets of finite closed sequents \mathcal{S} and all closed sequents S in \mathcal{L}' :

$$\mathcal{S} \Vdash_{\mathcal{L}}^e S \text{ implies } \mathcal{S} \vdash_{\mathcal{L}}^\infty S.$$

Proof We treat the case that \mathcal{S} is empty and leave the other case to the reader. The proof we give is similar to the elegant completeness proof for LJ in [27]. In the proof we will write \vdash for $\vdash_{\text{LJE}^\infty(\Sigma_{\mathcal{L}})}$. Let $S = (\Gamma \Rightarrow \Delta)$ be a closed sequent and assume that $\not\vdash S$. We will construct a \mathcal{L} -model K such that $K \not\models^e S$ in the following way. It will be defined in ω many steps using reduction trees, which will be the nodes of K . We assume that \mathcal{L}' contains infinitely many variables that do not occur in S . The nodes in the model will be triples of numbers, a reduction tree and a branch in the tree along which all sequents are unprovable.

Step 0: Let T_0 be the reduction tree for $\Gamma, \{Et \mid t \in \mathcal{T}_{\mathcal{L}}\} \Rightarrow \Delta$. Since $\not\vdash \Gamma \Rightarrow \Delta$, also

$$\not\vdash \Gamma, \{Et \mid t \in \mathcal{T}_{\mathcal{L}}\} \Rightarrow \Delta.$$

By Lemma 7.8 there is a branch b_0 in T_0 containing only unprovable sequents. Let $(0, T_0, b_0)$ be a node (the root) in K and proceed to the next step.

Step $i+1$. For any reduction tree T with branch b along which all sequents are unprovable constructed at step i , we consider Π and Λ , which are the respective unions of the formulas in the antecedents and succedents along b . Note that thus $\not\vdash \Pi \Rightarrow \Lambda$. Let k range over all formulas in Λ with outermost logical symbol \rightarrow or \forall . We proceed in the following way.

If k is a formula of the form $A \rightarrow B$. Then construct the reduction tree T_k for $\Pi, A \Rightarrow B$. Note that $\not\vdash \Pi, A \Rightarrow B$. Thus by Lemma 7.8 there is a branch b_k in T_k containing only unprovable sequents. We add the node $(i+1, T_k, b_k)$ to the model, let it be an immediate successor of (i, T, b) , and proceed to the next step $i+2$.

If k is a formula $\forall x A(x)$. Then construct the reduction tree T_k for $\Pi, Ey \Rightarrow A(y)$, where y is a variable in \mathcal{L} that has not yet occurred in the construction of K . Observe that if $\Pi, Ey \Rightarrow A(y)$ is derivable, then so is $\Pi \Rightarrow \forall x Ax$, since y does not occur in Π . Thus $\not\vdash \Pi, Ey \Rightarrow A(y)$, and whence by Lemma 7.8 there is a branch b_k in T_k containing only unprovable sequents. We add the node $(i+1, T_k, b_k)$ to the model, let it be an immediate successor of (i, T, b) , and proceed to the next step $i+2$.

Nodes constructed at different moments are different: if at stage i treating the A 's leads in some cases to the same reduction trees T with branch b of

unprovable sequents, then we assume that we add extra labels to (i, T, b) to account for the different occurrences of A at that step.

This process is continued ω times. Let W be the union of all triples that have been constructed, and let \preceq be the reflexive transitive closure of the immediate successor relation constructed at the stages. Define D to be the set of all terms appearing in the construction. Given a node $k = (i, T, b)$, let Π_k and Λ_k be the respective unions of the formulas in the antecedents and succedents along b . Then define an interpretation I as follows:

$$I_k(R) \equiv_{def} \{\bar{d} \in D \mid R(\bar{d}) \in \Pi_k\},$$

and I_k is the identity on function symbols: $I_k(f)(\bar{a}) = f(\bar{a}) \in D$. Since \mathcal{L} contains at least one constant c , it also implies that $I_k(E)$ is nonempty. Note that $K = ((W, \preceq), D, I)$ indeed is a Kripke existence model. The fact that we started with the sequent $\Gamma, \{Et \mid t \in \mathcal{T}_{\mathcal{L}}\} \Rightarrow \Delta$ implies that $Et \in \Pi_k$ for all k and all terms t in \mathcal{L} . Hence $K \Vdash^e Et$ for all terms $t \in \mathcal{T}_{\mathcal{L}}$, and thus K is an \mathcal{L} -model. It is not difficult to show by formula induction that we have

$$\begin{aligned} A \in \Pi_k &\Rightarrow k \Vdash^e A \\ A \in \Lambda_k &\Rightarrow k \nVdash^e A. \end{aligned}$$

We treat the case $A = B \rightarrow C$ and leave the other cases to the reader. This will complete the theorem.

First assume $B \rightarrow C \in \Pi_k$. We have to show that $k \Vdash^e B \rightarrow C$. Therefore, consider $l \succ k$ such that $l \Vdash^e B$. Thus by the induction hypothesis $B \in \Pi_l$. By the construction of the reduction tree, $C \in \Pi_l$ or $B \in \Lambda_l$. Since $B \notin \Lambda_l$, otherwise the branch would be derivable, it follows that $C \in \Pi_l$, and thus $l \Vdash^e C$.

Second, assume $B \rightarrow C \in \Lambda_k$. By the construction of the model, there is a node $l \succ k$ such that $B \in \Pi_l$ and $C \in \Lambda_l$. This implies that $l \Vdash^e B$ and $l \nVdash^e C$. Hence $k \nVdash^e B \rightarrow C$. \square

By Lemma 7.2 and the theorem it follows that :

Corollary 7.10 For all sets of finite closed sequents \mathcal{S} and all closed sequents S in \mathcal{L}' :

$$\mathcal{S} \vdash_{\mathcal{L}} S \text{ if and only if } \mathcal{S} \Vdash_{\mathcal{L}}^e S.$$

Corollary 7.11 For all sets of finite closed sequents \mathcal{S} and all closed sequents S in \mathcal{L}' : $\mathcal{S} \vdash_{\mathcal{L}} S$ if and only if $K \Vdash^e S$ for all \mathcal{L} -models K based on frames that are well-founded trees that force \mathcal{S} .

Proof Immediate from Lemma 7.2 and the proof of Theorem 7.9. \square

8 eSkolemization

In this section we prove that in $\text{LJE}(\Sigma_{\mathcal{L}})$ eSkolemization is sound and complete for all closed sequents in which all strong quantifiers are existential. Then follows the main theorem of the paper, Theorem 8.13, which is an extension

of Theorem 8.1 in that it shows that eSkolemization is sound and complete for all closed sequents in which all strong quantifiers are almost existential or that belong to \mathcal{M} (definition of \mathcal{M} in Section 2.3). The definition of almost existential quantifiers is given below.

Theorem 8.1 Let $\exists xB(x, \bar{y})$ be a formula that occurs negatively in a closed sequent S in \mathcal{L} , is not in the scope of strong quantifiers, and where \bar{y} are the variables of the weak quantifiers in the scope of which B occurs. If S' is the result of replacing in S the formula $\exists xB(x, \bar{y})$ by $Ef(\bar{y}) \wedge B(f(\bar{y}), \bar{y})$, where f is a fresh function symbol not in \mathcal{L} , then

$$\vdash_{\mathcal{L}} S \text{ if and only if } \vdash_{\mathcal{L}} S'.$$

Proof The direction from left to right follows from Lemma 3.2. For the direction from right to left it is more convenient to work with sentences instead of closed sequents. Therefore, let A' be the result of replacing a negative occurrence of $\exists xB(x, \bar{y})$ in a sentence A that does not occur in the scope of strong quantifiers, by a formula $Ef(\bar{y}) \wedge B(f(\bar{y}), \bar{y})$, where f is a fresh function symbol from $\mathcal{L}' \setminus \mathcal{L}$ not occurring in A and \bar{y} are the variables of the weak quantifiers in the scope of which B occurs. Arguing by contradiction, assume $\not\vdash_{\mathcal{L}}^e A$. We show that $\not\vdash_{\mathcal{L}}^e A'$. By Corollary 7.11 we can pick a well-founded tree model K that refutes A . Let 0 be an element in D such that $E0$ is forced at the root r of K .

Some notation: in this proof, \mathcal{L} stands for the language of A . Thus it is the language of A' without f . \mathcal{V} denotes the set of variables in \mathcal{L}' . Let G be the set of all function symbols (including constants) in \mathcal{L} . So all $g \in G$ are distinct from f and G is finite. g and g_i range over elements in G . Recall that for a sequence \bar{d} and a set X , $\bar{d} \in X$ means that all elements of \bar{d} belong to X , and $\bar{d} \notin X$ means that some $d \in \bar{d}$ does not belong to X .

Case 1. We first treat the case that all functions in \mathcal{L} and f have arity 1, and after that comment on the general case. Thus a formula $\exists xB(x, y)$ is replaced by $Efy \wedge B(fy, y)$ in going from A to A' . Some notation: we will need so many brackets that we often write fd for $f(d)$ and gfd for $g(f(d))$, etc.

Let us first sketch the idea of the proof. Keep also in mind the informal discussion on constructing counter models in Section 2.4. The aim is to construct a counter model K' to A' . K' will have the same set of nodes as K but a bigger domain $D' \supseteq D$, which we construct inductively in steps i . To this end we will use a partial function

$$\epsilon : K \times (D' \cup \mathcal{V}) \rightarrow (D \cup \mathcal{V}).$$

We write $\epsilon(k, d) \downarrow$ if $\epsilon(k, d)$ is an element in $D \cup \mathcal{V}$ and $\epsilon(k, d) \uparrow$ otherwise, i.e. if $\epsilon(k, d)$ is undefined. The idea is that at K', k the element $d' \in D'$ corresponds to the element $\epsilon(k, d') \in D$ at K, k . At every step we consider the nodes in K and $\epsilon(k, d)$ for some $d \in D_i \subseteq D'$, where i is already defined. If $K, k \Vdash^e \exists xB(x, \epsilon(k, d))$ we choose a witness $\epsilon(k, fd)$ in D such that

$$K, k \Vdash^e E\epsilon(k, fd) \wedge B(\epsilon(k, fd), \epsilon(k, d)).$$

This $\epsilon(k, fd)$ will then correspond to fd in D' , i.e. we extend the domain D_i with these elements fd . Thus for these fd , $\epsilon(k, fd) \downarrow$. If $K, k \not\vdash^e \exists xB(x, \epsilon(k, d))$ we do not need an element in D to which fd corresponds, and we put $\epsilon(k, fd) \uparrow$.

Of course we also have to extend the interpretations of the function symbols in G to the new domain. That is, we have to define the interpretation $I'_k(g)$ of g in K' for the new elements fd . To this end we just add for $g = g_1 \dots g_m$, elements $g(fd)$ to D_i and let I'_k be the identity on them, i.e. $I'_k(g)(fd) = g(fd)$. We put $\epsilon(k, gd) = I_k(g)(\epsilon(k, d))$ if $\epsilon(k, d) \downarrow$ and $\epsilon(k, gd) \uparrow$ otherwise.

Our aim is for the final model K' to satisfy

$$K', k \Vdash^e E(fd) \wedge B(fd, d) \Leftrightarrow K', k \Vdash^e \exists x B(x, \epsilon(k, d)). \quad (9)$$

We will obtain this in the following way. For the \bar{d} such that $\epsilon(k, d_i) \downarrow$ for all $d_i \in \bar{d}$, for all predicates P including E we define

$$\forall \bar{d} \in D' : K', k \Vdash^e P(\bar{d}) \Leftrightarrow K', k \Vdash^e P(\epsilon(k, \bar{d})), \quad (10)$$

In particular, if $K', k \Vdash^e \exists x B(x, \epsilon(k, d))$ we put $K', k \Vdash^e E(fd)$. We cannot extend this to all d , as for some we might have $\epsilon(k, d) \uparrow$. However, we will put $K', k \not\Vdash^e Ed$ for all these d , and therefore they play no role in the evaluation of the quantifiers in K' . As we will see, this then will make K' into a counter model of A' .

Two final remarks before the formal proof starts. First, the reason for the partiality of ϵ is that if we would require that all elements in D' correspond to an element of D , then (10) would violate the upwards persistency in K' . Namely, suppose $k \preceq l$ and $k \preceq l'$, and suppose k does not force and l and l' do force $\exists x B(x, d)$ in K . Suppose $K, l \Vdash B(e, d)$ and $K, l' \Vdash B(e', d)$. Thus we put $\epsilon(l, fd) = e$ and $\epsilon(l', fd) = e'$, and we also have to associate an element of D to $\epsilon(k, d)$: it may be e , e' or another element in D . In case $e \neq e'$ this would cause problems as then $\epsilon(k, d)$ cannot be equal to both; say $\epsilon(k, d) \neq e$. Then in case we would have defined forcing in K' as in (10) we might have $K, k \Vdash P\epsilon(k, d)$ and $K, l \not\Vdash P\epsilon(l, d)$. Therefore, we would have

$$K', k \Vdash Pd \quad K', l \not\Vdash Pd,$$

which would make P not upwards persistent in K' .

Second, in the example treated in Section 2.4 we interpreted the Skolem constant c as an element in the domain D . Thus we did not extend D by c as we do in the proof sketch above. The reason is that in general we cannot always let c or fd be an element of D , as the example above illustrates.

Now we start with the formal proof. The domain D' of K' is inductively defined as follows.

$$\begin{aligned} D_0 &\equiv_{def} D \cup \{g_1 \dots g_n(d) \mid d \in D, g_j \in G\} \\ D_{i+1} &\equiv_{def} \{f(d), g_1 \dots g_n f(d) \mid d \in D_i, g_j \in G\} \\ D' &\equiv_{def} \bigcup_i D_i. \end{aligned}$$

We define the following sets and the partial function $\epsilon(k, d) : K \times (D' \cup \mathcal{V}) \rightarrow (D' \cup \mathcal{V})$ simultaneously in an inductive way. We start by stipulating

$$\forall d \in D' \cup \mathcal{V} : \epsilon(k, d) = d \wedge \epsilon(k, g_1 \dots g_m(d)) = I_k(g_1) \dots I_k(g_m)(d).$$

(Recall that \mathcal{V} is the set of variables in \mathcal{L}' .) We write $\epsilon(k, d) \uparrow$ if $\epsilon(k, d)$ is not defined, and $\epsilon(k, d) \downarrow$ otherwise. Next we give for a given D_i such that $\epsilon(k, d)$ is

defined for all $d \in D_i$, the definitions of $\epsilon(k, d)$ for all $d \in D_{i+1}$, i.e. for $\epsilon(k, fd)$ and $\epsilon(k, g_1 \dots g_n fd)$. Define for $d \in D_i$

$$X^d = \{k \in K \mid \epsilon(k, d) \downarrow, K, k \Vdash^e \exists x B(x, \epsilon(k, d)), \\ \forall l \prec k (K, l \not\Vdash^e \exists x B(x, \epsilon(k, d)))\},$$

$$Z^d = \{k \mid \epsilon(k, d) \downarrow, K, k \Vdash^e \exists x B(x, \epsilon(k, d))\}.$$

Note that for all k such that $\epsilon(k, d) \downarrow$, either $k \in Z^d$, or $K, k \Vdash^e \neg \exists x B(x, \epsilon(k, d))$, or there exists at least one $l \succ k$ such that $l \in X^d$. For $k \in Z^d$ we define

$$k_d = \text{smallest } l \prec k \text{ s.t. } l \in Z^d$$

Note that k_d is well-defined because K is a well-founded tree. Observe that $l \succ k \in Z^d$ implies $K, l \Vdash^e \exists x B(x, \epsilon(k, d))$. That is not the same as $l \in Z^d$, i.e. $K, l \Vdash^e \exists x B(x, \epsilon(l, d))$. However, we will see that the latter follows from the former in the next claim.

Now for all nodes $k \in X^d$ choose an element $\epsilon(k, fd) \in D$ such that

$$K, k \Vdash^e E\epsilon(k, fd) \wedge B(\epsilon(k, fd), \epsilon(k, d)).$$

We extend this definition to all nodes k and all elements in D_{i+1} via

$$\epsilon(k, fd) = \begin{cases} \epsilon(k, fd) & \text{if } k \in X^d \\ \epsilon(k_d, fd) & \text{if } k \in Z^d \\ \uparrow & \text{if } k \notin Z^d \end{cases}$$

$$\epsilon(k, g_1 \dots g_m fd) = \begin{cases} I_k(g_1) \dots I_k(g_m)(\epsilon(k, fd)) & \text{if } \epsilon(k, fd) \downarrow \\ \uparrow & \text{if } \epsilon(k, fd) \uparrow \end{cases}$$

Here the g_i range over elements in G . Define

$$U_k \equiv_{def} D_0 \cup \{fd, g_1 \dots g_n f(d) \mid k \in Z^d, g_i \in G\} = \\ = D_0 \cup \{fd, g_1 \dots g_n f(d) \mid g_i \in G, \epsilon(k, d) \downarrow, K, k \Vdash \exists x B(x, \epsilon(k, d))\}.$$

Claim 8.2

$$k \in Z^d \wedge l \succ k \Rightarrow \quad (11)$$

$$\epsilon(k, d) \downarrow \wedge \epsilon(l, d) \downarrow \wedge \epsilon(k, d) = \epsilon(l, d) \wedge l \in Z^d \wedge k_d = l_d.$$

$$\forall k \forall d \in U_k \forall l \succ k : U_k \subseteq U_l \wedge \epsilon(k, d) \downarrow \wedge \epsilon(l, d) \downarrow \wedge \epsilon(k, d) = \epsilon(l, d). \quad (12)$$

Proof of Claim We first show that (11) implies (12). For an expression $e = g_1 \dots g_n fd$ or $e = fd$, where $g_i \in G$, assume $l \succ k$, $e \in U_k$. Hence $k \in Z^d$. Thus $\epsilon(k, d) \downarrow$, $\epsilon(l, d) \downarrow$, $\epsilon(k, d) = \epsilon(l, d)$ and $l \in Z^d$ by (11). Therefore, $e \in U_l$, which shows that $U_k \subseteq U_l$. The definition of ϵ implies that $\epsilon(k, fd) \downarrow$, $\epsilon(l, fd) \downarrow$, and $\epsilon(k, fd) = \epsilon(l, fd)$. Therefore, also $\epsilon(k, e) \downarrow$, $\epsilon(l, e) \downarrow$ and $\epsilon(k, e) = \epsilon(l, e)$. This proves (12).

To show (11), assume $l \succ k \in Z^d$. It is easy to see that $\epsilon(l, d) \downarrow$ and $\epsilon(k, d) = \epsilon(l, d)$ imply $l \in Z^d$ and $k_d = l_d$. Therefore, we only have to show that $\epsilon(l, d) \downarrow$ and $\epsilon(k, d) = \epsilon(l, d)$. We use induction on the i such that $d \in D_i$. For D_0 the claim clearly holds. For the induction step, first consider fd such that

$k \in Z^{fd}$. Hence $\epsilon(k, fd) \downarrow$. Observe that whence $k \in Z^d$. By the induction hypothesis $\epsilon(l, d) \downarrow$, $\epsilon(k, d) = \epsilon(l, d)$, $l \in Z^d$ and $k_d = l_d$. Thus $\epsilon(k, fd) = \epsilon(k_d, fd) = \epsilon(l_d, fd) = \epsilon(l, fd)$ by the definition of ϵ . Hence also $\epsilon(l, fd) \downarrow$.

Second, consider $e = g_1 \dots g_n fd$ and assume $k \in Z^e$. Whence $\epsilon(k, e) \downarrow$, thus $\epsilon(k, fd) \downarrow$, and thus $k \in Z^d$. Reasoning as in the case above, we conclude that $\epsilon(k, fd) = \epsilon(l, fd)$. Thus also $\epsilon(l, e) \downarrow$ and $\epsilon(k, e) = \epsilon(l, e)$ by the definition of ϵ . This proves the claim.

Note that the claim implies that

$$k \in Z^d \Rightarrow K, k \Vdash^e E\epsilon(k, fd) \wedge B(\epsilon(k, fd), \epsilon(k, d)). \quad (13)$$

Namely, read k for l and k_d for k : since $k \succcurlyeq k_d \in Z^d$ it follows that $\epsilon(k, d) = \epsilon(k_d, d)$. The definition of ϵ and $k \in Z^d$ imply that $\epsilon(k, fd) = \epsilon(k_d, fd)$. Since $K, k_d \Vdash^e E\epsilon(k_d, fd) \wedge B(\epsilon(k_d, fd), \epsilon(k_d, d))$, this proves that (13) holds.

To define K' we start by defining the interpretations I'_k , which are defined as the identity:

$$\forall d \in D' : I'_k(f)(d) \equiv_{def} f(d) \quad \forall d \in D' : I'_k(g)(d) \equiv_{def} g(d).$$

Note that the upwards persistency requirement on terms for the collection of I'_k is satisfied.

We now define the interpretations I'_k on predicates P including E as follows. For $e = e_1, \dots, e_n$ we let $\epsilon(k, \bar{e}) = \epsilon(k, e_1), \dots, \epsilon(k, e_n)$.

$$I'_k(P) \equiv_{def} \{\bar{d} \in D' \mid \bar{d} \in U_k, \epsilon(k, \bar{d}) \in I_k(P)\},$$

in other words, we put

$$\begin{aligned} \forall k \forall \bar{d} \in U_k : K', k \Vdash^e P(\bar{d}) &\equiv_{def} K, k \Vdash^e P(\epsilon(k, \bar{d})), \\ \forall k \forall \bar{d} \notin U_k : K', k \not\Vdash^e P(\bar{d}). \end{aligned}$$

What is important from the definition of the I'_k 's and what we will use often is that

$$\forall d \in D' : K', k \Vdash^e Ed \Leftrightarrow d \in U_k \wedge K, k \Vdash^e E\epsilon(k, d). \quad (14)$$

Note that this implies that the interpretation of E in the root of K' is nonempty, as it is so in K .

Claim 8.3 The upwards persistency requirement is fulfilled:

$$\forall k \forall l \succcurlyeq k : K', k \Vdash^e P(\bar{d}) \Rightarrow K', l \Vdash^e P(\bar{d}).$$

Proof of Claim If $\bar{d} \notin U_k$ then the implication is immediately clear. Suppose $\bar{d} \in U_k$ and $k \preccurlyeq l$. Hence $\bar{d} \in U_l$ by (12). Thus it suffices to show that

$$\forall k \forall l \succcurlyeq k : K, k \Vdash^e P(\epsilon(k, \bar{d})) \Rightarrow K, l \Vdash^e P(\epsilon(l, \bar{d})),$$

which follows from the persistency of K and the fact that $\epsilon(k, \bar{d}) = \epsilon(l, \bar{d})$, which follows from (12). This proves the claim.

Observe that here upwards persistency would not hold if we would have defined

$$\forall k \forall \bar{d} \in D' : K', k \Vdash^e P(\bar{d}) \equiv_{def} K, k \Vdash^e P(\epsilon(k, \bar{d})).$$

Namely, we then would not have that $l \preceq k$ implies $\epsilon(k, d) = \epsilon(l, d)$.

To finish the proof of the theorem we have to consider instances $C(\bar{d})$ of subformulas $C(\bar{x})$ of A . Since we assume that all bound variables in a proof are different, when we speak about an occurrence of the formula $\exists xB(x, y)$ in A , it can only be that one occurrence that is replaced in going from A to A' . Thus also when $C(\bar{z}, y)$ is a subformula of A the y can only be the y that occurs in $\exists xB(x, y)$. For subformulas $C(\bar{z}, y)$ of A and for all $\bar{e}c \in D' \cup \mathcal{V}$, we let $C(\bar{e}, c)^f$ stand for the replacement in $C(\bar{e}, c)$ of $\exists xB(x, c)$ by $E(fc) \wedge B(fc, c)$. If $C(\bar{e}, c)$ does not contain $\exists xB(x, c)$, then $C(\bar{e}, c)^f = C(\bar{e}, c)$. Thus $A^f = A'$. For convenience we also write $C(\bar{z}, y)$ in case y does not occur or occurs bounded in C . A few examples:

$$\begin{aligned} (\exists xB(x, y) \rightarrow Py)^f &= (Efy \wedge B(fy, y) \rightarrow Py) \\ (\exists uB(u, v) \wedge \exists xB(x, c))^f &= \exists uB(u, v) \wedge Efc \wedge B(fc, c) \\ \exists y(\neg \exists xB(x, y))^f &= \exists y(\neg(Efy \wedge B(fy, y))) \\ B(x, y)^f &= B(x, y). \end{aligned}$$

Recall that for $e = e_1, \dots, e_n$ we defined $\epsilon(k, \bar{e}) = \epsilon(k, e_1), \dots, \epsilon(k, e_n)$.

Claim 8.4 For every subformula $C(\bar{z}, y)$ of A :

$$\forall k \forall c \bar{e} \in U_k : K', k \Vdash^e C(\bar{e}, c)^f \Leftrightarrow K, k \Vdash^e C(\epsilon(k, \bar{e}), \epsilon(k, c)).$$

Proof of Claim With formula induction on C . Let $c\bar{e} \in U_k$. Observe that every subformula $C(\bar{x}, y)$ of A is a formula in \mathcal{L} . Therefore, if C is a predicate, then it does not contain $\exists xB(x, y)$, whence $C(\bar{e}, c)^f = C(\bar{e}, c) = P(\bar{e}, c)$ for some predicate P . We have to show that

$$K', k \Vdash^e P(\bar{e}, c) \Leftrightarrow K, k \Vdash^e P(\epsilon(k, \bar{e}), \epsilon(k, c)),$$

which follows directly from the definition of \Vdash^e in K' . The cases for the connectives follow by induction since $(\cdot)^f$ commutes with the connectives. In case of implication, use $U_k \subseteq U_l$ for $k \preceq l$ (12). We treat the quantifiers.

\forall : Suppose $C(\bar{z}, y) = \forall uD(u, \bar{z}, y)$, where y might possibly be u . Hence $C^f(\bar{e}, y) = \forall u(D(u, \bar{e}, y)^f)$. Thus we have to show:

$$\forall a \in D' : K', k \Vdash^e Ea \rightarrow D(a, \bar{e}, c)^f \Leftrightarrow \forall a \in D : K, k \Vdash^e Ea \rightarrow D(a, \epsilon(k, \bar{e}c)).$$

\Rightarrow : Suppose the left side of \Leftrightarrow holds, and consider an element $a \in D$ such that $K, k \Vdash^e Ea$. By (14) and the fact that $D \subseteq U_k$, $K', k \Vdash^e Ea$. Thus $K', k \Vdash^e D(a, \bar{e}, c)^f$. Whence by the induction hypothesis, $K, k \Vdash^e D(\epsilon(k, a\bar{e}c))$. Since $a \in D$, $\epsilon(k, a) = a$. Therefore, $K, k \Vdash^e D(a, \epsilon(k, \bar{e}c))$, and we are done.

\Leftarrow : Suppose the right side of \Leftrightarrow holds and pick an $a \in D'$ such that $K', k \Vdash^e Ea$. By (14), it follows that $a \in U_k$ and $K, k \Vdash^e E\epsilon(k, a)$. Since $\epsilon(k, a) \in D$ and the right side of \Leftrightarrow holds, $K, k \Vdash^e D(\epsilon(k, a), \epsilon(k, \bar{e}c))$. Therefore, $K', k \Vdash^e D(a, \bar{e}, c)^f$ by the induction hypothesis, and we are done.

\exists : Suppose $C(\bar{z}, y) = \exists xD(u, \bar{z}, y)$. There are three possibilities: $u = y$, $u = x$ or $u \neq x$ and $u \neq y$. The case $u = y$, and $u \neq x$ and $u \neq y$, are the same.

We start with this case. Hence $C(\bar{e}, y)^f = \exists u(D(u, \bar{e}, y)^f)$, where u is possibly equal to y . Thus we have to show that for all k

$$\exists a \in D' : K', k \Vdash^e Ea \wedge D(a, \bar{e}, c)^f \Leftrightarrow \exists a \in D : K, k \Vdash^e Ea \wedge D(a, \epsilon(k, \bar{e}c)).$$

\Rightarrow : Suppose the left side of \Leftrightarrow holds, and consider an element $a \in D'$ such that $K', k' \Vdash^e Ea \wedge D(a, \bar{e}, c)^f$. By (14), it follows that $a \in U_k$ and $K, k \Vdash^e E\epsilon(k, a)$. By the induction hypothesis we then also have $K, k \Vdash^e D(\epsilon(k, a), \epsilon(k, \bar{e}c))$, and we are done.

\Leftarrow : Suppose the right side of \Leftrightarrow holds and pick an $a \in D$ such that $K, k \Vdash^e Ea \wedge D(a, \epsilon(k, \bar{e}c))$. By (14) and the fact that $D \subseteq U_k$, it follows that $K', k \Vdash^e Ea$. And since $a \in D$, $\epsilon(k, a) = a$. Hence $K', k \Vdash^e D(a, \bar{e}, c)^f$ by the induction hypothesis, and we are done.

The final case $u = x$: $C(\bar{z}, y) = \exists xB(x, y)$ and $C(c)^f = E(fc) \wedge B(fc, c)$. We have to show that

$$K', k \Vdash^e E(fc) \wedge B(fc, c) \Leftrightarrow K, k \Vdash^e \exists xB(x, \epsilon(k, c)).$$

\Rightarrow : Suppose $K', k \Vdash^e E(fc) \wedge B(fc, c)$. By (14) also $K, k \Vdash^e E\epsilon(k, fc)$ and $f(c) \in U_k$. Hence $K, k \Vdash^e B(\epsilon(k, fc), \epsilon(k, c))$ by the induction hypothesis, as $B^f(fc, c) = B(fc, c)$. Since $\epsilon(k, fc) \in D$ this implies $K, k \Vdash^e \exists xB(x, \epsilon(k, c))$.

\Leftarrow : Only here the trick in the construction of f is used. Suppose $K, k \Vdash^e \exists xB(x, \epsilon(k, c))$. Observe that $c \in U_k$ implies $\epsilon(k, c) \downarrow$ by (12). Hence $k \in Z^c$. Thus by (13),

$$K, k \Vdash^e E\epsilon(k, fc) \wedge B(\epsilon(k, fc), \epsilon(k, c)).$$

Also, since $k \in Z^c$, $f(c) \in U_k$. Observe that $(Efc \wedge B(fc, c))^f = Efc \wedge B(fc, c)$. Therefore, $K', k \Vdash^e E(fc) \wedge B(fc, c)$ by the induction hypothesis. This proves the claim.

It follows from the last claim that

$$K', k \Vdash^e A' \Leftrightarrow K, k \Vdash^e A.$$

Therefore, $K' \not\Vdash^e A'$, as desired. This completes Case 1. of the proof.

Case 2. We will not treat this part of the proof in all detail, but just mention the main differences. In this case f and the functions in G may have any arity ≥ 0 . Thus a formula $\exists xB(x, \bar{y})$ is replaced by $B(f\bar{y}, \bar{y})$ in going from A to A' , where \bar{y} are the variables of all the weak quantifiers in the scope of which B occurs. Note that B contains no other free variables as it is not in the scope of another strong quantifier by assumption. Assume f has arity s and $g_i \in G$ has arity s_i . Thus \bar{y} has length s . For any set X , we write X^s for the set of sequences of elements in X of length s . Again, we will often write $f\bar{d}$ for $f(\bar{d})$ and $g(f\bar{d})$ for $g(f(\bar{d}))$, etc. When we write $g_i(\bar{d})$ we implicitly assume that the length of \bar{d} is s_i , and similarly for f . $\epsilon(k, \bar{d}) = \epsilon(k, d_1), \dots, \epsilon(k, d_n)$. We write $\epsilon(k, \bar{d}) \downarrow$ if $\epsilon(k, d) \downarrow$ for all $d \in \bar{d}$, and $\epsilon(k, \bar{d}) \uparrow$ otherwise. $d \in \bar{e}$ means that d occurs in some element of \bar{e} .

To define the D_i we need an auxiliary operation. Given a set X , $X^G = \bigcup X_i$ is the closure of X under the functions in G :

$$X^0 \equiv_{\text{def}} X \cup \{g(\bar{d}) \mid \bar{d} \in X, g \in G\}$$

$$\bar{X}^{i+1} \equiv_{def} X_i \cup \{g(\bar{d}) \mid \bar{d} \in X_i, g \in G\}.$$

Then we define

$$\begin{aligned} D_0 &\equiv_{def} D^G \\ D_{i+1} &\equiv_{def} (\bigcup_{j \leq i} D_j \cup \{f(\bar{d}) \mid \bar{d} \in \bigcup_{j \leq i} D_j\})^G \\ D' &\equiv_{def} \bigcup_i D_i. \end{aligned}$$

Again, we define the following sets and partial function $\epsilon : K \times (D' \cup \mathcal{V}) \rightarrow (D \cup \mathcal{V})$ simultaneously in an inductive way. They are only slight variations of the definitions above, fit to deal with arities other than 1. We start by stipulating

$$\forall d \in D \cup \mathcal{V} : \epsilon(k, d) = d \text{ and } \forall g(\bar{d}) \in D^G : \epsilon(k, g(\bar{d})) = I_k(g)(\epsilon(k, \bar{d})).$$

Given the definition of ϵ for D_i , we show how to extend it to D_{i+1} . Consider a $\bar{d} \in D_i^s$ such that at least one element of \bar{d} is not in $\bigcup_{j < i} D_j$.

$$\begin{aligned} X^{\bar{d}} = \{k \mid k \in K, \epsilon(k, \bar{d}) \downarrow, K, k \Vdash^e \exists x B(x, \epsilon(k, \bar{d})), \\ \forall l \prec k (K, l \not\Vdash^e \exists x B(x, \epsilon(k, \bar{d})))\} \end{aligned}$$

$$Z^{\bar{d}} = \{k \mid k \in K, \epsilon(k, \bar{d}) \downarrow, K, k \Vdash^e \exists x B(x, \epsilon(k, \bar{d}))\}$$

Note that the $Z^{\bar{d}}$ are only defined for sequences \bar{d} of length s . For $k \in Z^{\bar{d}}$ we define

$$k_{\bar{d}} = \text{smallest } l \preceq k \text{ s.t. } l \in Z^{\bar{d}}$$

Now for all nodes $k \in X^{\bar{d}}$ we choose a node $\epsilon(k, f\bar{d}) \in D$ such that

$$K, k \Vdash^e E\epsilon(k, f\bar{d}) \wedge B(\epsilon(k, f\bar{d}), \epsilon(k, \bar{d})).$$

Then we extend this definition to all k via

$$\epsilon(k, f\bar{d}) = \begin{cases} \epsilon(k, f\bar{d}) & \text{if } k \in X^{\bar{d}} \\ \epsilon(k_{\bar{d}}, f\bar{d}) & \text{if } k \in Z^{\bar{d}} \\ \uparrow & \text{if } k \notin Z^{\bar{d}} \end{cases}$$

Then we define inductively for $\bar{d} \in (\bigcup_{j \leq i} D_j \cup \{f(\bar{d}) \mid \bar{d} \in \bigcup_{j \leq i} D_j\})^G$:

$$\epsilon(k, g\bar{d}) = \begin{cases} I_k(g)(\epsilon(k, \bar{d})) & \text{if } \epsilon(k, \bar{d}) \downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Define

$$\begin{aligned} U_k &\equiv_{def} (D_0 \cup \{f(\bar{d}) \mid k \in Z^{\bar{d}}\})^G = \\ &= (D_0 \cup \{f(\bar{d}) \mid \epsilon(k, \bar{d}) \downarrow, K, k \Vdash \exists x B(x, \epsilon(k, \bar{d}))\})^G. \end{aligned}$$

Claim 8.5

$$k \in Z^{\bar{d}} \wedge l \succ k \Rightarrow \quad (15)$$

$$\epsilon(k, \bar{d}) \downarrow \wedge \epsilon(l, \bar{d}) \downarrow \wedge \epsilon(k, \bar{d}) = \epsilon(l, \bar{d}) \wedge l \in Z^{\bar{d}} \wedge k_{\bar{d}} = l_{\bar{d}}.$$

$$\forall k \forall d \in U_k \forall l \succ k : U_k \subseteq U_l \wedge \epsilon(k, d) \downarrow \wedge \epsilon(l, d) \downarrow \wedge \epsilon(k, d) = \epsilon(l, d). \quad (16)$$

Proof of Claim The proof of (15) is analogues to the proof of (11) above; use induction on i such that $\bar{d} \in D_i$, with a subinduction on the number of symbols in \bar{d} . We only show that (15) implies (16). Assume (15) and $d \in U_k$ and $k \preceq l$. Let $f(\bar{e}_1), \dots, f(\bar{e}_n)$ be all the terms of the form $f(\bar{e})$ in d that are not in the scope of another term of the form $f(\bar{a})$. Observe that $k \in Z^{\bar{e}_i}$. Hence $\epsilon(k, \bar{e}_i) \downarrow$, $\epsilon(l, \bar{e}_i) \downarrow$, $\epsilon(k, \bar{e}_i) = \epsilon(l, \bar{e}_i)$, $k_{\bar{e}_i} = l_{\bar{e}_i}$, and $l \in Z^{\bar{e}_i}$ by (15). The definition of ϵ implies $\epsilon(k, f\bar{e}_i) \downarrow$, $\epsilon(l, f\bar{e}_i) \downarrow$, and $\epsilon(k, f\bar{e}_i) = \epsilon(l, f\bar{e}_i)$. Whence $\epsilon(k, d) \downarrow$, $\epsilon(l, d) \downarrow$, and $\epsilon(k, d) = \epsilon(l, d)$ by the definition of ϵ on elements of the form $g(\bar{e})$. This proves (16), and thereby the claim.

Note that the claim implies

$$k \in Z^{\bar{d}} \Rightarrow K, k \Vdash^e E(k, f\bar{d}) \wedge B(\epsilon(k, f\bar{d}), \epsilon(k, \bar{d})). \quad (17)$$

To define K' we define the interpretations I'_k for K' for terms and predicates similarly as in Case 1:

$$\forall \bar{d} \in D' : I'_k(f)(\bar{d}) \equiv_{def} f(\bar{d}) \quad \forall \bar{d} \in D' : I'_k(g)(\bar{d}) \equiv_{def} g(\bar{d}).$$

$$I'_k(P) \equiv_{def} \{\bar{d} \in D' \mid \bar{d} \in U_k, \epsilon(k, \bar{d}) \in I_k(P)\},$$

in other words, we put

$$\forall k \forall \bar{d} \in U_k : K', k \Vdash^e P(\bar{d}) \equiv_{def} K, k \Vdash^e P(\epsilon(k, \bar{d})),$$

$$\forall k \forall \bar{d} \notin U_k : K', k \not\Vdash^e P(\bar{d}).$$

It is not difficult to see that

$$\forall d \in D' : K', k \Vdash^e Ed \Leftrightarrow d \in U_k \wedge K, k \Vdash^e E\epsilon(k, d). \quad (18)$$

Claim 8.6 The upwards persistency requirement is fulfilled:

$$\forall k \forall l \succ k : K', k \Vdash^e P(\bar{d}) \Rightarrow K', l \Vdash^e P(\bar{d}).$$

Proof of Claim Analogues to the proof of Claim 8.3 above.

As in Case 1, for subformulas $C(\bar{z}, \bar{y})$ of A and for all sequences $\bar{e}\bar{c} \in D' \cup \mathcal{V}$, we let $C(\bar{e}, \bar{c})^f$ stand for the replacement in $C(\bar{e}, \bar{c})$ of $\exists x B(x, \bar{c})$ by $E(f\bar{c}) \wedge B(f\bar{c}, \bar{c})$. Thus $A^f = A'$.

Claim 8.7 For every subformula $C(\bar{z}, \bar{y})$ of A :

$$\forall k \forall \bar{c}\bar{e} \in U_k : K', k \Vdash^e C^f(\bar{e}, \bar{c}) \Leftrightarrow K, k \Vdash^e C(\epsilon(k, \bar{e}), \epsilon(k, \bar{c})).$$

Proof of Claim Analogues to the proof of Claim 8.4 above.

Given this definitions and claims, the proof proceeds in exactly the same way as in Case 1. This completes the second case, and thereby the proof of the theorem. \square

Corollary 8.8 For each closed sequent S in \mathcal{L}' in which all strong quantifiers are existential:

$$\vdash_{\mathcal{L}} S \text{ if and only if } \vdash_{\mathcal{L}} S^s.$$

Proof Let $S = S_1, \dots, S_n = S^s$ be the Skolem sequence of S . Then Theorem 8.1 implies that $\vdash_{\mathcal{L}} S_i$ if and only if $\vdash_{\mathcal{L}} S_{i+1}$, and we are done. \square

Corollary 8.9 For all closed sequents S in \mathcal{L} in which all strong quantifiers are existential:

$$\vdash_{\perp} S \text{ if and only if } \vdash_{\mathcal{L}} S^s.$$

8.10 Extension of the main result

In this section we present an extension of the above result by extending the class of sequents for which Theorem 8.1 applies. This class of formulas is not syntactically defined, and therefore less useful. However, it contains a syntactically decidable subclass that strictly extends the closed sequents in which all strong quantifiers are existential: the class of closed sequents in which all strong universal quantifiers are of the form $\forall x \neg \neg Ax$.

Definition 8.11 For a formula A that occurs in a sequent S , $S[B/A]^p$ (p for positive) denotes the result of replacing every positive occurrence of A in S by B . Note that we do not put restrictions on the possible occurrences of free variables in A or S . We say that *all strong quantifiers in S are almost existential* if for every subformula $\forall x Ax$ of S , it holds that

$$S[\neg \exists x \neg Ax / \forall x Ax]^p \vdash_{\mathcal{L}} S.$$

Note that we always have

$$S \vdash_{\mathcal{L}} S[\neg \exists x \neg Ax / \forall x Ax]^p.$$

Thus almost existential sequents are sequents that, as a formula, are equivalent to a formula in which all strong quantifiers are existential.

Define

$$\mathcal{S}^{\exists} \equiv_{def} \{S \mid S \text{ a closed sequent in } \mathcal{L}' \text{ in which all strong quantifiers are almost existential}\} \cup \mathcal{M}.$$

Remark 8.12 \mathcal{S}^{\exists} contains all sequents in which all strong universal quantifiers are of the form $\forall x \neg \neg Ax$. But other universal quantifiers might be allowed: $\perp \Rightarrow \forall x Ax$ does not belong to the mentioned classes but every quantifier in this formula is almost existential.

Theorem 8.13

$$\forall S \in \mathcal{S}^{\exists} : \vdash_{\mathcal{L}} S \text{ if and only if } \vdash_{\mathcal{L}} S^s.$$

Proof The direction from left to right is clear. If S belong to \mathcal{M} , the equivalence follows from Corollary 4.9. Therefore, for the other direction we assume that all strong quantifiers in S are almost existential and that $\vdash_{\mathcal{L}} S^s$. Let S' be the result of replacing every strong quantifier $\forall x Bx$ in S by $\neg \exists x \neg Bx$. Then all strong quantifiers in S' are existential. Note that because

$$\vdash_{\mathcal{L}} (Et \rightarrow Bt) \Rightarrow \neg(Et \wedge \neg Bt)$$

we have $S^s \vdash_{\mathcal{L}} (S')^s$. Since by Corollary 8.8:

$$\vdash_{\mathcal{L}} S' \text{ if and only if } \vdash_{\mathcal{L}} (S')^s,$$

$\vdash_{\mathcal{L}} S$ follows. □

Corollary 8.14

$$\forall S \in \mathcal{S}^{\exists} \cap \mathcal{S}_{\mathcal{L}} : \vdash_{\text{LJ}} S \text{ if and only if } \vdash_{\mathcal{L}} S^s.$$

Using the result in Section 4.10 we can conclude the following.

Corollary 8.15 For the fragment of sequents in \mathcal{S}^{\exists} without weak quantifiers, derivability in $\vdash_{\mathcal{L}}$ is decidable.

8.16 No total Skolemization

In this section we give examples that show that eSkolemization cannot be sound and complete for all sequents. Furthermore, we discuss some conditions that, when satisfied by an alternative Skolemization method, imply that the method cannot be complete.

Counterexamples showing that eSkolemization is not complete are e.g.

$$\Rightarrow \forall x \neg \neg (Ax \vee \neg Ax) \quad \Rightarrow \neg \neg \forall x (Ax \vee \neg Ax).$$

The first formula is derivable, the second one is not. Their eSkolemizations are the derivable, and equivalent, sequents

$$\Rightarrow (Ec \rightarrow \neg \neg (Ac \vee \neg Ac)) \quad \Rightarrow \neg \neg (Ec \rightarrow Ac \vee \neg Ac).$$

Another example of the incompleteness of eSkolemization is given by the *double negation shift DNS*

$$\forall x \neg \neg Ax \Rightarrow \forall x \neg \neg Ax. \tag{19}$$

The eSkolemization DNS^s of DNS is

$$\forall x \neg \neg Ax \Rightarrow \neg \neg (Ec \rightarrow Ac).$$

Now $(\Rightarrow DNS^s)$ is derivable in $\text{LJE}(\Sigma_{\mathcal{L}})$ while $(\Rightarrow DNS)$ is not. This is no coincidence, as the following theorem shows.

Proposition 8.17 If $(\cdot)^*$ is a transformation on sequents that inductively replaces strong quantifiers $\forall x A(x)$ by $Ef(\bar{y}) \rightarrow A^*(f(\bar{y}))$ and that commutes with the connectives, then it cannot hold that for all S

$$\vdash_{\mathcal{L}} S \Leftrightarrow \vdash_{\mathcal{L}} S^*. \tag{20}$$

Proof Let c denote $f(\bar{y})$. Let $(\cdot)^*$ be as in the theorem. Then

$$(\forall x \neg \neg A(x))^* = (Ec \rightarrow \neg \neg A^*(c)) \equiv \neg \neg (Ec \rightarrow A^*(c)) = (\neg \neg \forall x A(x))^*.$$

Since

$$\not\vdash_{\mathcal{L}} \forall x \neg\neg A(x) \Rightarrow \neg\neg \forall x A(x),$$

(20) implies that

$$\not\vdash_{\mathcal{L}} \forall x \neg\neg A(x) \Rightarrow (Ec \rightarrow \neg\neg A^*(c)).$$

Since

$$\vdash_{\mathcal{L}} \forall x \neg\neg A(x) \Rightarrow \forall x \neg\neg A(x),$$

(20) implies also

$$\vdash_{\mathcal{L}} \forall x \neg\neg A(x) \Rightarrow (Ec \rightarrow \neg\neg A^*(c)).$$

A contradiction. \square

The following proposition is the analogue of the proposition above for LJ.

Proposition 8.18 If $(\cdot)^*$ is a transformation on sequents that inductively replaces strong quantifiers $QxA(x)$ by $A^*(f(\bar{y}))$ and that commutes with the connectives, then it cannot hold that for all A

$$\text{LJ} \vdash S \Leftrightarrow \text{LJ} \vdash S^*. \quad (21)$$

Clearly, the above theorems imply that at least the two “natural” ways of Skolemization, the standard one $(\cdot)^S$ and the alternative one $(\cdot)^s$, cannot work.

The following is a slight strengthening of the above observations.

Proposition 8.19 Let $(\cdot)^*$ be a transformation on sequents such that in LJ $(\neg\exists x \neg Ax)^* \Rightarrow (\neg\neg\forall x Ax)^*$ is derivable. Then it cannot hold that for all S

$$\text{LJ} \vdash S \Leftrightarrow \text{LJ} \vdash S^*.$$

And similarly for $\text{LJE}(\Sigma_{\mathcal{L}})$ instead of LJ.

Proof The following observation suffices

$$\text{LJ} \vdash \neg\exists x \neg Ax \Rightarrow \neg\exists x \neg Ax$$

$$\text{LJ} \not\vdash \neg\exists x \neg Ax \Rightarrow \neg\neg\forall x Ax.$$

\square

9 Corollaries for IQCE

By the equivalence between LJE and IQCE as discussed in Section 5, we obtain the following corollaries for IQC . We let $\Phi_{\mathcal{L}}$ be the equivalent of $\Sigma_{\mathcal{L}}$ fro formulas:

$$\Phi_{\mathcal{L}} \equiv_{\text{def}} \{Et \mid t \in \mathcal{T}_{\mathcal{L}}\}.$$

Define

$$\mathcal{F}^{\exists} \equiv_{\text{def}} \{A \mid A \text{ a sentence in } \mathcal{L}', \text{ all strong quantifiers in } A \text{ are almost existential}\} \cup \{A \mid \Rightarrow A \text{ belongs to } \mathcal{M}\}.$$

Corollary 9.1

$\forall A \in \mathcal{F}^\exists : \Phi_{\mathcal{L}} \vdash_{\text{IQCE}} A$ if and only if $\Phi_{\mathcal{L}} \vdash_{\text{IQCE}} A^s$.

Corollary 9.2

$\forall A \in \mathcal{F}^\exists \cap \mathcal{F}_{\mathcal{L}} : \vdash_{\text{IQC}} A$ if and only if $\Phi_{\mathcal{L}} \vdash_{\text{IQCE}} A^s$.

Corollary 9.3 For the fragment of sentences S in \mathcal{F}^\exists without weak quantifiers, $\Phi_{\mathcal{L}} \vdash_{\text{IQCE}}$ is decidable.

10 Questions

There are too many topics for further research to list them all, but among the most important ones are the following.

- A syntactic proof of the (partial) completeness of eSkolemization.
- A full description of the class of formulas for which eSkolemization is sound and complete.
- An alternative Skolemization method that is sound and complete in IQC for all formulas.
- Extension of the results to intuitionistic logic plus equality. In [25, 19] it is shown that in this case standard Skolemization is not even complete for prenex formulas.
- Generalization of the results to other logics, e.g. minimal logic.

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