Kripke models for subtheories of CZF

Rosalie Iemhoff*

Abstract

In this paper a method to construct Kripke models for subtheories of constructive set theory is introduced that uses constructions from classical model theory such as constructible sets and generic extensions. Under the main construction all axioms except the collection axioms can be shown to hold in the constructed Kripke model. It is shown that by carefully choosing the classical models various instances of the collection axioms, such as exponentiation, can be forced to hold as well. The paper does not contain any deep results. It consists of first observations on the subject, and is meant to introduce some notions that could serve as a foundation for further research.

Keywords: Kripke models, constructive set theory, constructible sets, generic extensions, intuitionistic logic.

1 Introduction

Constructive set theory was introduced by John Myhill in 1975. It serves as a foundation for constructive mathematics, in much the same way that Zermelo-Fraenkel set theory serves as a foundation for classical mathematics. To obtain a constructive set theory, the first naive idea would be to only restrict the logic of classical set theory to intuitionistic logic but leave the axioms unchanged. This, however, does not work since already the Axiom of Foundation implies the law of the excluded middle. Therefore the set theoretic axioms have to be chosen with care. Some classical axioms are severely restricted, such as the Separation Axiom, that in a constructive setting allows the construction of extensions based on bounded formulas only. For other axioms it suffices to replace them by a classically equivalent form. Set Induction, for example, is classically equivalent to Foundation and an accepted principle of constructive set theory.

Several systems have been proposed as a constructive set theory in the course of time, but we will concentrate in this paper on one particular brand of constructive set theory called Constructive Zermelo Fraenkel set theory CZF. CZF, which is based on Myhill's system, was introduced by Peter Aczel, who also confirmed the constructivity of the theory by providing an interpretation of it in

*Department of Philosophy, University Utrecht, Heidelberglaan 8, 3584 CS Utrecht, The Netherlands. Rosalie.Iemhoff@phil.uu.nl

Martin-Löf type theory [1, 2, 3, 12]. Later Michael Rathjen constructed a realizability interpretation of the theory, and using this showed that CZF possesses constructive properties such as the disjunction property and various existence properties.

In this paper we study the Kripke models of Constructive Zermelo Fraenkel set theory. Kripke models are a useful tool to study constructive theories and because of their simplicity have been applied with great success to non-classical logics in general, and intuitionistic logic and Heyting Arithmetic in particular. Although nowadays there exist various models of CZF, such as the two interpretations discussed above, as well as others in topos theory [10] and in the form of Heyting algebras [5], Kripke models have been less investigated. In [7] Robert Lubarsky constructed two beautiful Kripke models of CZF that refute classical principles such as the Power Set Axiom.

In this paper the aim is to construct, given a frame, a model of CZF on that frame by using constructions from classical model theory such as the constructible sets and generic extensions. The idea is to attach classical models (the so-called local models) to the nodes of the frame and see how far, by carefully choosing the local models, the axioms of CZF can be forced to hold in the Kripke model.

This paper is just a small first step in that direction. We introduce the framework and provide requirements on the local models under which the model constructed on a given frame is indeed a Kripke model. Then we proceed to show that under certain natural conditions such as transitivity, the Kripke models thus constructed satisfy the axioms of CZF minus the collection axioms, although certain instances of the latter can be shown to hold as well. These results are straightforward: they follow easily from the definition of the models. The conditions on the local models for which this subtheory of CZF is satisfied in the final model are quite general, and there are many ways to construct Kripke models that satisfy the necessary requirements.

Finally we show how to produce Kripke models for various forms of collection by using specific properties of certain generic extensions. Given a frame we attach generic extensions of a certain ground model to the leaves of the frame, and the model itself to all interior nodes. We will see that several instances and variations of the collection axioms can be recovered via requirements on the partial orders on which the generic extensions are based, such as the countable chain condition. In particular, a bounded form of Exponentiation and weak forms of Replacement can be forced to hold.

These are modest results, but we do hope that the method presented in this paper, in particular the relation between properties of Kripke models and the generic extensions on which they are based, can be pushed further, and lead to models of full CZF.

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2 CZF

Constructive Zermelo Fraenkel set theory CZF consists of the axioms and rules of intuitionistic logic with equality extended by the following axioms.

Extensionality	$a = b \leftrightarrow \forall x(x \in a \leftrightarrow x \in b)$.
Empty Set	$\exists x \forall y(y \notin x)$
Pairing	$\exists c \forall x(x \in c \leftrightarrow x = a \vee x = b)$.
Union	$\exists b \forall x\big(x \in b \leftrightarrow \exists y \in a(x \in y)\big)$.
Bounded Separation	$\exists b \forall x\big(x \in b \leftrightarrow x \in a \wedge \varphi(x)\big)$ (φ bounded).
Strong Infinity	$\exists \omega \forall x\big(x \in \omega \leftrightarrow x = \emptyset \vee \exists y \in \omega(x = y \cup \{y\})\big)$.
Set Induction	$\forall x\big(\forall y \in x \varphi(y) \rightarrow \varphi(x)\big) \rightarrow \forall x \varphi(x)$.
Strong Collection	$\forall x \in a \exists y \varphi(x,y) \rightarrow$ $\exists b\big(\forall x \in a \exists y \in b \varphi(x,y) \wedge \forall y \in b \exists x \in a \varphi(x,y)\big)$.
Subset Collection	$\exists c \forall z\big(\forall x \in a \exists y \in b \varphi(x,y,z) \rightarrow$ $\exists d \in c\big(\forall x \in a \exists y \in d \varphi(x,y,z) \wedge \forall y \in d \exists x \in a \varphi(x,y,z)\big)\big)$.

Bounded formulas are formulas in which every quantifier is bounded, thus of the form $\forall x \in y$ or $\exists x \in y$. $\mathsf{CZF}^{\mathsf{c}}$ is CZF minus the collection axioms, $\mathsf{CZF}^{\mathsf{ic}}$ is $\mathsf{CZF}^{\mathsf{c}}$ minus Set Induction. IZF is CZF in which the collection axioms are replaced by the Power Set Axiom,

$$\textit{Power Set} \quad \exists b \forall x\big(x \in b \leftrightarrow x \subseteq a\big),$$

and Replacement,

$$\textit{Replacement} \quad \forall x \in a \exists! y \varphi(x,y) \rightarrow \exists b \forall x \in a \exists y \in b \varphi(x,y).$$

Note that Set Induction is the constructive variant of Foundation, to which it is classically equivalent. It resembles the situation in Heyting Arithmetic, for which induction is one of the axioms, while the least number principle, which is classically equivalent to it, does not hold. Also note that Strong Collection is formulated in the way it is above since only full Separation would make it equivalent to the form most commonly used:

$$\forall x \in a \exists y \varphi(x,y) \rightarrow \exists b \forall x \in a \exists y \in b \varphi(x,y).$$

Bounded Strong Collection is Strong Collection in which the φ is a bounded formula. Below we will define what it means when a term is *set-bounded* in a

formula. Given these notions, *Set-bounded Subset Collection* is Subset Collection in which the $\varphi(x, y, z)$ is a bounded formula in which z is set-bounded.

The following two axioms are equivalent (over the other axioms) to respectively Bounded Separation and Subset Collection [4].

$$\textit{Binary Intersection} \qquad \exists c \forall x (x \in c \leftrightarrow x \in a \wedge x \in b)$$

$$\textit{Fullness} \qquad \exists c \forall r \in \mathrm{mv}({}^a b) \exists r' \in \mathrm{mv}({}^a b)(r' \subseteq r \wedge r' \in c)$$

In Fullness $\mathrm{mv}({}^a b)$ stands for all multi-valued relations from a to b, i.e. all $r \subseteq a \times b$ such that $\forall x \in a \exists y \in b \, \langle a, b \rangle \in r$. The property $r' \subseteq r$ expresses that r' is a refinement of r. Thus Fullness expresses that there is a set that contains at least one refinement for every $r \in \mathrm{mv}({}^a b)$. Note that if $r : a \to b$, Fullness implies that $r \in c$. Therefore Fullness and Bounded Separation imply Exponentiation, that is, that b^a is a set. This, however, does not imply that Power Set holds, because the existence of a set $\{0, 1\}^a$ does not imply that the power set of a exists: the existence of undecidable sets shows that not every subset of a corresponds to a function in $\{0, 1\}^a$.

We define

$$\textit{Separation}_\varphi \qquad \exists b \forall x \big(x \in b \leftrightarrow x \in a \wedge \varphi(x)\big),$$

and similarly for *Strong Collection*$_\varphi$ and *Subset Collection*$_\varphi$. Bounded Separation, Strong Collection and Subset Collection are axiom schemes, while their variants $()_\varphi$ are formulas.

An axiom that is often used in constructive set theory is the Regularity Axiom which guarantees the existence of greatest fixed points, which are used in the setting of inductive definitions. In this paper the focus is on the other axioms and the Regularity Axiom will not be discussed any further here.

3 Kripke models

In this section we introduce the method to construct Kripke models for subtheories of CZF from classical models of ZF, which is the main object of study in this paper. Intuitively, given a frame F and classical models M_i for every node i of F, the Kripke model will be the result of attaching model M_i to node i, where the forcing of atomic formulas at a node corresponds to the validity of atomic formulas in the model at that node. Of course, in order to obtain a Kripke model various requirements have to be fulfilled. For example, for upwards persistency every model M_i has to be a subset of the models at nodes above i, and every atomic formula that holds in M_i, should hold in all models at the nodes above i. The definitions below provide such restrictions. A collection of models is called *sound* for a given frame, it if satisfies all the necessary requirements.

Although in the applications to come classical models will be attached to nodes in a frame, one could also attach Kripke models to these nodes. The definitions below describe the construction on this level of generality, but it might be clarifying to keep in mind that in the theorems to come only the former restricted

version of the construction is used. Clearly, the general case covers the specific case, as classical models are instances of Kripke models, namely Kripke models consisting of one node.

Note that in the general case we deal with nodes on two levels: nodes of the underlying frame F, and nodes in the Kripke models M_i that we attach to the nodes of F. For a node m in M_i we denote forcing in M_i at this node by $M_i, m \Vdash$, and in the final Kripke model we denote forcing at node i in F and node m in M_i by $\langle i, m\rangle \Vdash$. Here follow the details.

Given a Kripke model K we denote by W^K, $\preccurlyeq^K$, D^K and I^K respectively its set of nodes, partial ordering, set of domains and its interpretations. A similar notation is used for frames. When K is clear from the context we omit the superscript. Given a node i in a model K, D_i^K denotes the domain at node i, and I_i^K the interpretation at node i.

Given a frame $F = (W, \preccurlyeq)$, we call a collection of (Kripke) models $\mathcal{M} = \{M_i \mid i \in W\}$ *sound for* F if

$$\begin{gathered}\forall i, j \in W\, \forall m \in W^{M_i} \forall n \in W^{M_j} :\ i \preccurlyeq j \ \ \Rightarrow\ \ D_m^{M_i} \subseteq D_n^{M_j},\\ \forall i, j \in W \forall m \in W^{M_i} \forall n \in W^{M_j} \forall a, b \in D_i^{M_i} :\\ i \preccurlyeq j\ \wedge\ M_i, m \Vdash P(a,b) \ \ \Rightarrow\ \ M_j, n \Vdash P(a,b) \quad (P \text{ is } = \text{ or } \in).\end{gathered}$$

Given a frame F and a collection of Kripke models $\mathcal{M} = \{M_i \mid i \in W^F\}$ that is sound for F, the Kripke model $K_F(\mathcal{M}) = (W, \preccurlyeq, D, I)$ is defined as follows:

- $W \equiv_{def} \bigcup \{\{i\} \times W^{M_i} \mid i \in W^F\}$,
- $D_{\langle i,m\rangle} \equiv_{def} D_m^{M_i}$,
- $\langle i, m\rangle \preccurlyeq \langle j, n\rangle \equiv_{def} (i = j \wedge m \preccurlyeq^{M_i} n) \vee (i \prec^F j)$,
- $I_{\langle i,m\rangle} \equiv_{def} I_m^{M_i}$.

Thus $K_F(\mathcal{M})$ is obtained from F by replacing node i in F by the Kripke model M_i. The models in $\mathcal{M}$ are called *local* models. We sometimes call $K_F(\mathcal{M})$ the *final* model.

Soundness guarantees the result to be a Kripke model:

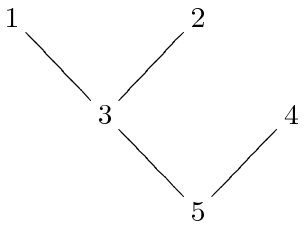
Lemma 1 If $\mathcal{M}$ is sound for F, then in $K_F(\mathcal{M})$:

$$\langle i, m\rangle \preccurlyeq \langle j, n\rangle \ \ \Rightarrow\ \ \big(\langle i, m\rangle \Vdash \varphi\ \Rightarrow\ \langle j, n\rangle \Vdash \varphi\big).$$

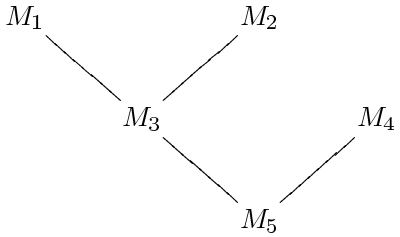
In the following, when talking about a model $K_F(\mathcal{M})$, we tacitly assume that $\mathcal{M}$ is sound for F.

Clearly, classical models are models of the form $K_F(\mathcal{M})$, where F consists of one node. We will mainly consider models of the form $K_F(\mathcal{M})$ for which $\mathcal{M}$

consists of classical models and F is finite. Let us first consider an example of this form, where F is the frame

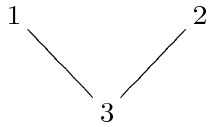


and $\mathcal{M} = \{M_i \mid i = 1, \ldots, 5\}$ consists of classical models. Then $K_F(\mathcal{M})$ is the model



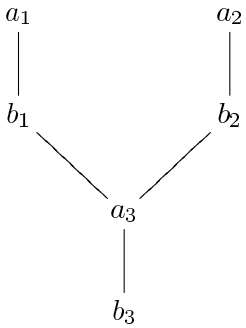
This model has 5 nodes.

An example for which the models in $\mathcal{M}$ consist of more than one node is for example given by the following frame F



and the collection of models $\mathcal{M} = \{M_i \mid i = 1,2,3\}$, where the frames of the models M_i are of the form

Thus $K_F(\mathcal{M})$ is the following 6 node model:



As mentioned above, if forcing in a model M_i is considered this will always be indicated, as in expressions $M_i, m \Vdash \varphi$. In case the forcing is relative to $K_F(\mathcal{M})$, the model will be often omitted, which means that $\langle i, m\rangle \Vdash \varphi$ is short for $K_F(\mathcal{M}), \langle i, m\rangle \Vdash \varphi$ (recall that all nodes in $K_F(\mathcal{M})$ are of the form $\langle i, m\rangle$). Similar remarks apply to $\preccurlyeq^{M_i}$ and $\preccurlyeq^{K_F(\mathcal{M})}$.

3.1 Model properties

This section discusses all the properties used to describe the requirements on F and $\mathcal{M}$ under which $K_F(\mathcal{M})$ is a Kripke model of certain axioms of CZF. The properties are listed below, but the reader might consider to not consult them before the particular point at which they are used in the theorems, since their meaning follows easily from the way in which they are applied.

D-formula

Given a set D, φ is a *D-formula* if all its parameters are in D.

Set-bounded

Given a formula φ and a term t, t is *set-bounded* in φ, if, when it occurs in φ, $\varphi \to t \in s$ is derivable in intuitionistic logic, for some term s that occurs in φ. In $t \in s$, for example, t is set-bounded while s is not.

Bounded

A formula is *bounded* if all its quantified subformulas are of the form $\exists x \in a\varphi(x)$ or $\forall x \in a\varphi(x)$.

$\in$-sound

To establish that $K_F(\mathcal{M})$ satisfies certain axioms of CZF we will sometimes use that these axioms hold on the meta-level. For example, in the case of Set Induction we use that Set Induction holds in the real universe in which our proofs live. For this we need the notion of $\in$-soundness.
$\mathcal{M}$ is *$\in$-sound* or *sound for* $\in$ if

$$\forall a, b \in D_m^{M_i} : \; M_i, m \Vdash a \in b \;\Rightarrow\; a \in b.$$

Here the $a \in b$ in the conclusion of $\Rightarrow$ refers to the underlying real universe, here taken to be classical set theory, if not explicitly stated otherwise. Note that classical $\in$-models are $\in$-sound.

Transitive

To force that certain simple axioms of CZF such as pairing and union hold in $K_F(\mathcal{M})$ we require that sets do not grow when moving up in the model, a property that is captured by the notion of transitivity.
$\mathcal{M}$ is *transitive* at $\langle i, m\rangle$ if

$$\forall \langle j,n\rangle \succcurlyeq \langle i,m\rangle \forall a \in D_m^{M_i}\, \forall b \in D_n^{M_j} :\ M_j, n \Vdash (b \in a \vee b = a) \ \Rightarrow\ b \in D_m^{M_i}.$$

Note that this implies

$$\forall \langle j,n\rangle \succcurlyeq \langle i,m\rangle \forall a \in D_m^{M_i}\, \forall b \in D_n^{M_j} :\ \langle j,n\rangle \Vdash (b \in a \vee b = a) \ \Rightarrow\ b \in D_m^{M_i}.$$

$\mathcal{M}$ is transitive if it is transitive at every node. Note that if $\mathcal{M}$ consists of classical models, transitivity implies that the models are transitive in the usual sense, and vice versa.
To establish that $K_F(\mathcal{M})$ satisfies certain axioms of CZF we will often use that the local models satisfy these axioms. For example, in order to have $K_F(\mathcal{M})$ be a model of Bounded Separation, we need to understand for which formulas forcing at a node in a local model is equal to forcing at that node in $K_F(\mathcal{M})$. For in these cases $K_F(\mathcal{M})$ satisfies Bounded Separation because the local models do. The formulas for which we can establish such a property are the bounded formulas, for which we will show that they are preserved and decided in $K_F(\mathcal{M})$.

Decidable

A $D_m^{M_i}$-formula $\varphi(\bar{x})$ is *decided at* $\langle i, m\rangle$ if

$$\forall \bar{a} \in D_m^{M_i} \forall \langle j,n\rangle \succcurlyeq \langle i,m\rangle : \langle j,n\rangle \Vdash \varphi(\bar{a}) \ \Leftrightarrow\ \langle i,m\rangle \Vdash \varphi(\bar{a}).$$

This is equivalent to $\langle i,m\rangle \Vdash \varphi(\bar{a}) \vee \neg\varphi(\bar{a})$. φ is decided in $K_F(\mathcal{M})$ if it is decided at every node $\langle j,n\rangle$ for which φ is a $D_n^{M_j}$-formula. Observe that every formula is decided at the leafs of $K_F(\mathcal{M})$. $\mathcal{M}$ *decides atomic formulas* if all atomic formulas are decided in $K_F(\mathcal{M})$. Note that this means that for all $a, b \in D_m^{M_i} \cap D_n^{M_j}$:

$$M_i, m \Vdash a = b \ \Leftrightarrow\ M_j, n \Vdash a = b \qquad M_i, m \Vdash a \in b \ \Leftrightarrow\ M_j, n \Vdash a \in b.$$

Note that this notion depends on the models in $\mathcal{M}$ only.

Preserved

A $D_m^{M_i}$-formula $\varphi(\bar{x})$ is *preserved at* $\langle i, m\rangle$ if

$$\forall \bar{a} \in D_m^{M_i} : \langle i,m\rangle \Vdash \varphi(\bar{a}) \ \Leftrightarrow\ M_i, m \Vdash \varphi(\bar{a}).$$

φ is preserved in $K_F(\mathcal{M})$ if it is preserved at every node $\langle j,n\rangle$ for which φ is a $D_n^{M_j}$-formula. Observe that every formula is preserved at the leafs of $K_F(\mathcal{M})$.

As we will see, in this paper we did not succeed in showing that, under the conditions considered in this paper, $K_F(\mathcal{M})$ is a model of the collection axioms. We do, however, recover certain instances of them. The requirements under which this holds are expressibility and collapse. These properties are rather strong, and we feel that in contrast to the results on the other axioms of CZF, the results using these two properties are unnatural, and leave ample room for improvement. In the last two sections we present some possible improvements by providing particular local models for which $K_F(\mathcal{M})$ is a model of bounded Exponentiation and weak versions of Replacement.

Expressible

A $D_m^{M_i}$-formula $\varphi(x,y)$, in which x,y are the only free variables, is *expressible at* $\langle i,m\rangle$ if

$$\forall a,b \in D_m^{M_i} : \ \langle i,m\rangle \Vdash \forall x \in a \exists y \in b \varphi(x,y) \ \Leftrightarrow \ M_i, m \Vdash \forall x \in a \exists y \in b \psi_m^i(x,y).$$

The formulas ψ_m^i are called the *companions of* φ. Observe that every φ is a companion of itself at the leafs of $K_F(\mathcal{M})$. Corollary 1 below implies that every bounded $D_m^{M_i}$-formula is expressible at $\langle i,m\rangle$ by itself.

Lemma 2 If $\mathcal{M}$ is a transitive class of models of Pairing and decides atomic formulas, and $\varphi(x,y)$ is expressible by companions ψ_m^i, then

$$\forall d,e \in D_m^{M_i} : \ \langle i,m\rangle \Vdash \varphi(d,e) \ \Leftrightarrow \ M_i, m \Vdash \psi_m^i(d,e).$$

Proof Consider $d,e \in D_m^{M_i}$. Because the M_i are models of Pairing, there exists a set $\{d\} \in D_m^{M_i}$ such that

$$M_m^i \Vdash \forall x (x \in \{d\} \leftrightarrow x = d).$$

The assumption that $\mathcal{M}$ is transitive and decides atomic formulas implies that also

$$\langle i,m\rangle \Vdash \forall x (x \in \{d\} \leftrightarrow x = d).$$

This implies that

$$\langle i,m\rangle \Vdash \varphi(d,e) \ \Leftrightarrow \ \langle i,m\rangle \Vdash \forall x \in \{d\} \exists y \in \{e\} \varphi(x,y).$$

The same reasoning gives

$$M_m^i \Vdash \psi_m^i(d,e) \ \Leftrightarrow \ M_m^i \Vdash \forall x \in \{d\} \exists y \in \{e\} \psi_m^i(x,y).$$

Now we can use that φ is expressible by ψ_m^i at $\langle i,m\rangle$ to obtain the desired result. □

Collapse

Given $\langle j,n\rangle \succcurlyeq \langle i,m\rangle$, and a $D_m^{M_i}$-formula $\varphi(x,y,z)$, in which x,y,z are the only free variables, then $\varphi(x,y,z)$ *collapses from* $\langle j,n\rangle$ *to* $\langle i,m\rangle$ if

$$\forall c \in D_n^{M_j} \forall a,b \in D_m^{M_i} : \langle j,n\rangle \Vdash \forall x \in a \exists y \in b \varphi(x,y,c) \;\Rightarrow$$
$$\big(\langle j,n\rangle \Vdash \neg\exists x \in a \text{ or } \big(c \in D_m^{M_i} \text{ and } \langle i,m\rangle \Vdash \forall x \in a \exists y \in b \varphi(x,y,c)\big)\big).$$

$\varphi(x,y,z)$ *collapses* if it collapses from $\langle j,n\rangle$ to $\langle i,m\rangle$ for all $\langle j,n\rangle \succcurlyeq \langle i,m\rangle$.

Lemma 3 If $\mathcal{M}$ is a transitive class of models of Set Induction that decides atomic formulas, and the $D_m^{M_i}$-formula $\varphi(x,y,z)$ is bounded, z is set-bounded in it, and x,y,z are the only free variables in it, then $\varphi(x,y,z)$ collapses.

Proof We have to show that for all $\langle j,n\rangle \succcurlyeq \langle i,m\rangle$:

$$\forall c \in D_n^{M_j} \forall a,b \in D_m^{M_i} : \langle j,n\rangle \Vdash \forall x \in a \exists y \in b \varphi(x,y,c) \;\Rightarrow$$
$$\big(\langle j,n\rangle \Vdash \neg\exists x \in a \text{ or } \big(c \in D_m^{M_i} \text{ and } \langle i,m\rangle \Vdash \forall x \in a \exists y \in b \varphi(x,y,c)\big)\big).$$

Therefore suppose $\langle j,n\rangle \Vdash \forall x \in a \exists y \in b \varphi(x,y,c)$ and $\langle j,n\rangle \nVdash \neg\exists x \in a$. Thus for some $\langle h,k\rangle \succcurlyeq \langle j,n\rangle$, $\langle h,k\rangle \Vdash \exists x \in a \wedge \forall x \in a \exists y \in b \varphi(x,y,c)$. It follows that there exist elements u,v such that $\langle h,k\rangle \Vdash u \in a \wedge v \in b \wedge \varphi(u,v,c)$. Hence $u,v \in D_m^{M_i}$ by transitivity. Since z is set-bounded in $\varphi(x,y,z)$ it follows that $\langle h,k\rangle \Vdash c \in t$, for some term in $\varphi(u,v,c)$. Because Set Induction holds in the models, $t \neq c$. Since all terms in $\varphi(u,v,c)$ except c belong to $D_m^{M_i}$, it follows that $c \in D_m^{M_i}$ by transitivity. That also $\langle i,m\rangle \Vdash \forall x \in a \exists y \in b \varphi(x,y,c)$ holds, follows from Corollary 1 below. □

Recall that an $\in$-model is a model in which the membership relation is that of the universe.

Lemma 4 If $\mathcal{M}$ consists of classical transitive $\in$-models, then $\mathcal{M}$ is transitive, $\in$-sound and decides atomic formulas.

Note that in the lemma the requirement that the models of ZF are $\in$-models is not only needed for the $\in$-soundness, but also for the decidability of atomic formulas.

3.2 Forcing in the two models

Here follow some lemmas relating forcing in the local models to forcing in the final model. As we explained in the previous section, bounded formulas will be central in some of the theorems below. In this section we show that such formulas are preserved and decided in $K_F(\mathcal{M})$.

When we consider a formula at M_i, m or $\langle i,m\rangle$ we tacitly assume that all its parameters belong to $D_m^{M_i}$, that is, that it is a $D_m^{M_i}$-formula.

Lemma 5 If $\mathcal{M}$ is transitive and decides atomic formulas, then all bounded formulas are decided in $K_F(\mathcal{M})$.

Proof We have to show that for bounded formulas φ:

$$\forall \langle j,n\rangle \succcurlyeq \langle i,m\rangle : \ \langle i,m\rangle \Vdash \varphi \ \Leftrightarrow \ \langle j,n\rangle \Vdash \varphi.$$

We use formula induction. For atomic formulas the lemma holds because $\mathcal{M}$ decides atomic formulas. That it holds for conjunction, disjunction and implication follows easily. We treat the existential quantifier, the universal quantifier being similar.

$\exists$ If $\langle j,n\rangle \Vdash \exists x \in a\psi(x)$, then for some $d \in D_n^{M_j}$, $\langle j,n\rangle \Vdash d \in a \wedge \psi(d)$. By transitivity $d \in D_m^{M_i}$, and whence $\langle i,m\rangle \Vdash d \in a \wedge \psi(d)$ by the induction hypothesis. Thus $\langle i,m\rangle \Vdash \exists x \in a\psi(x)$. □

Lemma 6 If $\mathcal{M}$ is transitive and decides atomic formulas, then all bounded formulas are preserved in $K_F(\mathcal{M})$.

Proof We have to show that for bounded $D_m^{M_i}$-formulas φ:

$$\langle i,m\rangle \Vdash \varphi \ \Leftrightarrow \ M_i, m \Vdash \varphi.$$

We use formula induction. For atomic formulas the lemma holds because atomic formulas are preserved by definition of $K_F(\mathcal{M})$. That it holds for conjunction and disjunction follows easily. For implication we use the induction hypothesis and Lemma 5 that implies that all bounded formulas are decided in $K_F(\mathcal{M})$. We treat the universal quantifier, the existential quantifier being similar.

We have to show that

$$\langle i,m\rangle \Vdash \forall x \in a\psi(x) \ \Leftrightarrow \ M_i, m \Vdash \forall x \in a\psi(x).$$

$\Rightarrow$ Suppose $n \succcurlyeq^{M_i} m$, and $d \in D_n^{M_i}$ and $M_i, n \Vdash d \in a$. Thus $d \in D_m^{M_i}$ by transitivity. Hence $\langle i,m\rangle \Vdash d \in a$ by decidability. Thus $\langle i,m\rangle \Vdash \psi(d)$. Hence $\langle i,n\rangle \Vdash \psi(d)$ by upwards persistency, and thus $M_i, n \Vdash \psi(d)$ by the induction hypothesis. This proves that $M_i, m \Vdash \forall x \in a\psi(x)$.

$\Leftarrow$ Suppose $\langle j,n\rangle \succcurlyeq \langle i,m\rangle$, and $d \in D_n^{M_j}$, and $\langle j,n\rangle \Vdash d \in a$. Hence $d \in D_m^{M_i}$ by transitivity and $M_i, m \Vdash d \in a$ by decidability. Thus $M_i, m \Vdash \psi(d)$. Hence $\langle i,m\rangle \Vdash \psi(d)$ by the induction hypothesis. Thus $\langle j,n\rangle \Vdash \psi(d)$. This proves that $\langle i,m\rangle \Vdash \forall x \in a\psi(x)$. □

Observe that the condition of atomic decidability in the previous lemma cannot be replaced by atomic preservation, since the argument for implication in the proof by induction might no longer be true.

Corollary 1 If $\mathcal{M}$ is transitive and decides atomic formulas, then all bounded formulas are decided and preserved in $K_F(\mathcal{M})$.

Corollary 2 If $\mathcal{M}$ consists of transitive $\in$-models of ZF, then all bounded formulas are decided and preserved in $K_F(\mathcal{M})$.

4 Models of CZF

4.1 Models of $\mathsf{CZF^{ic}}$

Proposition 1 If $\mathcal{M}$ is transitive and decides atomic formulas, then for all axioms φ of $\mathsf{CZF^{ic}}$ except Strong Infinity, if the models in $\mathcal{M}$ satisfy Extensionality and φ, then so does $K_F(\mathcal{M})$.

Proof The proof of this theorem is simple, but we have included all details for completeness' sake. Essential is that bounded formulas are preserved and decided.

Extensionality We have to show that

$$\langle i, m\rangle \Vdash \forall x(x \in a \leftrightarrow x \in b) \leftrightarrow a = b.$$

Observe that the formula is bounded. Since the M_i are models of Extensionality, they satisfy this formula at all their nodes. Lemma 6 implies that whence the formula holds at all nodes in $K_F(\mathcal{M})$.

Empty Set Suppose the models in $\mathcal{M}$ are models of Empty Set. Thus for all nodes m in M_i there exists a set $\emptyset^i_m \in D^{M_i}_m$ that is the empty set at that node, i.e. such that

$$M_i, m \Vdash \forall x(x \in \emptyset^i_m \to \bot).$$

Since $\forall x(x \in \emptyset^i_m \to \bot)$ is a bounded formula, Lemma 6 implies that it is forced at $\langle i, m\rangle$ too.

Pairing Suppose the models in $\mathcal{M}$ are models of Pairing. Thus there exist sets, denoted $\{a, b\}^i_m$, in $D^{M_i}_m$ for which

$$M_i, m \Vdash \forall x(x \in \{a, b\}^i_m \leftrightarrow x = a \vee x = b).$$

Transitivity and the fact that atomic formulas are decided implies that $\forall x(x \in \{a, b\}^i_m \leftrightarrow x = a \vee x = b)$ is forced at $\langle i, m\rangle$ too.

Union Suppose the models in $\mathcal{M}$ are models of Union. The existence of sets $\cup a^i_m \in D^{M_i}_m$ such that

$$M_i, m \Vdash \forall x\big(x \in \cup a^i_m \leftrightarrow \exists y \in a(x \in y)\big),$$

implies that

$$\langle i, m\rangle \Vdash \forall x\big(x \in \cup a^i_m \leftrightarrow \exists y \in a(x \in y)\big)$$

by transitivity and the fact that atomic formulas are decided.

Bounded Separation Suppose the models in $\mathcal{M}$ are models of Bounded Separation. Let φ be a bounded formula. Thus there are sets $c^i_m \in D^{M_i}_m$ for which

$$M_i, m \Vdash \forall x\big(x \in c^i_m \leftrightarrow x \in a \wedge \varphi(x)\big).$$

Since $\forall x\big(x \in c^i_m \leftrightarrow x \in a \wedge \varphi(x)\big)$ is a bounded formula, it follows from Lemma 6 that $\langle i, m\rangle$ forces that formula too. □

4.2 Strong Infinity

Proposition 2 If $\mathcal{M}$ is a transitive class of models of $\mathsf{CZF^{ic}}$ that decides atomic formulas, then $K_F(\mathcal{M})$ is a model of Strong Infinity.

Proof Let $\emptyset^i_m$ be the sets such that

$$M_i, m \Vdash \forall x \neg (x \in \emptyset^i_m).$$

Because the models are models of Strong Infinity, there are sets ω^i_m such that

$$M_i, m \Vdash \forall x (x \in \omega^i_m \leftrightarrow x = \emptyset^i_m \vee \exists y \in \omega^i_m (x = y \cup \{y\})).$$

It suffices to show that $\langle i, m\rangle$ forces the formula. Therefore consider $\langle j, n\rangle \succcurlyeq \langle i, m\rangle$ and $x \in D^{M_j}_n$. We show that $\langle j, n\rangle$ forces

$$x \in \omega^i_m \leftrightarrow x = \emptyset^i_m \vee \exists y \in \omega^i_m (x = y \cup \{y\}).$$

We will use that for $x, y \in D^{M_i}_m$:

$$M_i, m \Vdash x = y \cup \{y\} \;\Leftrightarrow\; \langle i, m\rangle \Vdash x = y \cup \{y\}.$$

It is not difficult to see that this holds, using transitivity, the decidability of atomic formulas, and that $x = y \cup \{y\}$ is equivalent to $\forall z (z \in x \leftrightarrow z \in y \vee z = y)$.
$\rightarrow$ Suppose $\langle j, n\rangle \Vdash x \in \omega^i_m$. We show that $\langle i, m\rangle$ forces $x = \emptyset^i_m \vee \exists y \in \omega^i_m (x = y \cup \{y\})$. Transitivity and the decidability of atomic formulas implies that $x \in D^{M_i}_m$ and $\langle i, m\rangle \Vdash x \in \omega^i_m$. Thus M_i, m forces $x \in \omega^i_m$, and whence $x = \emptyset^i_m \vee \exists y \in \omega^i_m (x = y \cup \{y\})$. The observations above imply that $\langle i, m\rangle$ forces that formula too.
$\leftarrow$ First suppose $\langle j, n\rangle \Vdash x = \emptyset^i_m$. Note that $\langle i, m\rangle$ forces $\emptyset^i_m \in \omega^i_m$ since M_i, m does. Hence $\langle j, n\rangle$ forces $x \in \omega^i_m$ by the equality axioms. Second, suppose for some y, $\langle j, n\rangle$ forces $y \in \omega^i_m \wedge x = y \cup \{y\}$. Thus $y \in D^{M_i}_m$ by transitivity, and $\langle i, m\rangle \Vdash y \in \omega^i_m$ by the decidability of atoms. There is a set $x' \in D^{M_i}_m$ such that $\langle i, m\rangle$, and whence M_i, m, forces $x' = y \cup \{y\}$. Thus M_i, m, and whence $\langle i, m\rangle$, forces $x' \in \omega^i_m$. Hence $\langle j, n\rangle \Vdash x' \in \omega^i_m$. Since $\langle j, n\rangle$ forces $x = x'$ by Extensionality, it forces $x \in \omega^i_m$ by the equality axioms. □

Corollary 3 If $\mathcal{M}$ is transitive and decides atomic formulas, and all models in $\mathcal{M}$ are models of $\mathsf{CZF^{ic}}$, then $K_F(\mathcal{M})$ is a model of $\mathsf{CZF^{ic}}$.

Combining this with Lemma 4 we obtain the following corollary.

Corollary 4 If $\mathcal{M}$ consists of transitive $\in$-models of ZF, then $K_F(\mathcal{M})$ is a model of $\mathsf{CZF^{ic}}$.

4.3 Power Set

Proposition 3 If $\mathcal{M}$ is a class of transitive classical $\in$-models of Power Set, then $K_F(\mathcal{M})$ is a model of Power Set.

Proof Since the models in $\mathcal{M}$ are models of Power Set, there is a set $P(a)$ in M_i such that $M_i, m \Vdash \forall x(x \in P(a) \leftrightarrow x \subseteq a)$. The transitivity of the models in $\mathcal{M}$ implies that this then holds at all nodes in all models M_j, where $j \succcurlyeq^F i$. Note that this implies that it holds in $K_F(\mathcal{M})$. □

4.4 Models of Set Induction

As mentioned in the beginning, the background theory in this paper is taken to be ZFC, but we do think that in most cases CZF would suffice. In this section we explicitly state in which background theory we work, because in the case of Set Induction it is instructive to see how the conditions on the local models have to be changed if we only allow constructive reasoning and wish the final model to be a model of Set Induction.

Proposition 4 (CZF) If $\mathcal{M}$ is sound for $\in$, then $K_F(\mathcal{M})$ is a model of Set Induction.

Proof *Set Induction* We show that $\forall x(\forall y \in x\varphi(y) \to \varphi(x)) \to \forall x\varphi(x)$ holds in $K_F(\mathcal{M})$. Note that it suffices to show that

$$\langle i, m\rangle \Vdash \forall x(\forall y \in x\varphi(y) \to \varphi(x)) \quad \Rightarrow \quad \langle i, m\rangle \Vdash \forall x\varphi(x).$$

Therefore suppose $\langle i, m\rangle \Vdash \forall x(\forall y \in x\varphi(y) \to \varphi(x))$. We have to show that

$$\forall \langle j, n\rangle \succcurlyeq \langle i, m\rangle \, \forall x \in D_n^{M_j} \ \langle j, n\rangle \Vdash \varphi(x).$$

Let

$$\psi(x) \equiv_{def} \forall \langle j, n\rangle \succcurlyeq \langle i, m\rangle \big(x \in D_n^{M_j} \to \langle j, n\rangle \Vdash \varphi(x)\big).$$

It suffices to show that $\forall x\big(\forall y \in x\psi(y) \to \psi(x)\big)$, since an application of Set Induction to this formula on the meta level then gives $\forall x\psi(x)$, i.e. $\langle i, m\rangle \Vdash \forall x\varphi(x)$. Thus assume $\forall y \in x\psi(y)$. To show $\psi(x)$, we prove that for $\langle j, n\rangle \succcurlyeq \langle i, m\rangle$ and $x \in D_n^{M_j}$, we have $\langle j, n\rangle \Vdash \forall y \in x\varphi(y)$. Since $\langle i, m\rangle$ forces $\forall x(\forall y \in x\varphi(y) \to \varphi(x))$, so does $\langle j, n\rangle$, and hence $\langle j, n\rangle \Vdash \varphi(x)$. This will show that $\psi(x)$, and completes the argument.
Therefore consider $\langle h, k\rangle \succcurlyeq \langle j, n\rangle$, $y \in D_k^{M_h}$ and $\langle h, k\rangle \Vdash y \in x$. Then $y \in x$ by $\in$-soundness. Because we have $\forall y \in x\psi(y)$, this implies $\langle h, k\rangle \Vdash \varphi(y)$. Thus indeed $\langle j, n\rangle \Vdash \forall y \in x\varphi(y)$. □

By Lemma 4, Corollary 4, and the previous theorem we obtain the following corollary.

Corollary 5 If $\mathcal{M}$ consists of transitive $\in$-models of ZF, then $K_F(\mathcal{M})$ is a model of $\mathsf{CZF^c}$.

Proposition 5 (ZF) If $\mathcal{M}$ consists of models of Foundation, and F is a frame without infinite ascending branches, then $K_F(\mathcal{M})$ is a model of Set Induction.

Proof We have to show that

$$i \Vdash \forall x(\forall y \in x \varphi(y) \to \varphi(x)) \quad \Rightarrow \quad i \Vdash \forall x \varphi(x).$$

Since all M_i consist of one node, $\langle i, m\rangle$ becomes i. We write M_j for D^{M_j}. Therefore suppose $i \Vdash \forall x(\forall y \in x \varphi(y) \to \varphi(x))$. Suppose $i \nVdash \forall x \varphi(x)$. Thus $i_1 \nVdash \varphi(a_1)$ for some $a_1 \in M_{i_1}$ and $i_1 \succcurlyeq i$. This implies that $i_1 \nVdash \forall x \in a_1 \varphi(x)$, say $i_2 \nVdash \varphi(a_2)$ for some $a_2 \in M_{i_2}$ such that $i_2 \succcurlyeq i_1$ and $i_2 \Vdash a_2 \in a_1$. Hence $i_3 \nVdash \varphi(a_3)$ for some $a_3 \in M_{i_3}$ such that $i_3 \succcurlyeq i_2$ and $i \Vdash a_3 \in a_2$, etcetera. Since every branch of F is finite there exists an i_j such that for all $h \geq j$ with $h \in \omega$, $a_h \in M_{i_j}$ and $M_{i_j} \Vdash a_{h+1} \in a_h$. But this contradicts the fact that the models in $\mathcal{M}$ are models of Foundation. $\Box$

4.5 Models of Strong Collection

The theorem below implies that under the standard conditions we have used so far, Bounded Collection holds in models of the form $K_F(\mathcal{M})$. Although many applications of collection require only Bounded Collection, we feel that the result in this section is rather weak. We chose to include it because the treatment of collection here is similar to the treatment of the other axioms we encountered so far. In Section 7 several variations of this theorem will be proved showing that by a more subtle application of the method to construct $K_F(\mathcal{M})$, other forms of collection can be forced to hold.

Proposition 6 If $\mathcal{M}$ is transitive and decides atomic formulas, then for every φ that is expressible by companions ψ^i_m, if the models in $\mathcal{M}$ are models of Strong Collection$_{\psi^i_m}$ and Pairing, then $K_F(\mathcal{M})$ is a model of Strong Collection$_\varphi$.

Proof Let ψ^i_m be the companions of φ. We show that for $a \in D^{M_i}_m$

$$\langle i, m\rangle \Vdash \forall x \in a \exists y \varphi(x, y) \to \exists b\big(\forall x \in a \exists y \in b \varphi(x, y) \wedge \forall y \in b \exists x \in a \varphi(x, y)\big).$$

Therefore suppose $\langle i, m\rangle \Vdash \forall x \in a \exists y \varphi(x, y)$. We first show that

$$M_i, m \Vdash \forall x \in a \exists y \psi^i_m(x, y).$$

Namely, if $n \succcurlyeq m$ and $b \in D^{M_i}_n$ such that $M_i, n \Vdash b \in a$, then $b \in D^{M_i}_m$ by transitivity, and $M_i, m \Vdash b \in a$ by decidability. Thus $\langle i, m\rangle \Vdash b \in a$ and whence $\langle i, m\rangle \Vdash \varphi(b, c)$ for some $c \in D^{M_i}_m$. Thus $M^i_m \Vdash \psi^i_m(b, c)$ by Lemma 2, and therefore $\exists y \psi^i_m(x, y)$, is forced at M_i, m, whence also at M_i, n.

Since the M_i are models of Strong Collection$_{\psi^i_m}$, this implies that there exist sets $b^i_m \in D^{M_i}_m$ such that

$$M_i, m \Vdash \forall x \in a \exists y \in b^i_m \psi^i_m(x,y) \wedge \forall y \in b^i_m \exists x \in a \psi^i_m(x,y).$$

By the expressibility of φ this implies

$$\langle i, m\rangle \Vdash \forall x \in a \exists y \in b^i_m \varphi(x,y) \wedge \forall y \in b^i_m \exists x \in a \varphi(x,y).$$

□

Corollary 6 If $\mathcal{M}$ is transitive and decides atomic formulas, then if the models in $\mathcal{M}$ are models of Bounded Strong Collection, so is $K_F(\mathcal{M})$.

Proof If φ is bounded, Corollary 1 implies that φ is expressible by φ. Apply the previous theorem. □

4.6 Models of Subset Collection

The theorem below implies that under the standard conditions we have used so far, Set-bounded Subset Collection holds in models of the form $K_F(\mathcal{M})$. The same remark as in the case of Strong Collection applies here: the result is not strong since the expressive power of the formulas in the axiom is limited. In Section 8 we will encounter several variations of this theorem showing that under reasonable conditions a restricted version of Exponentiation holds in $K_F(\mathcal{M})$.

Proposition 7 If $\mathcal{M}$ is transitive and decides atomic formulas, then for every φ that collapses and is expressible by formulas ψ^i_m, if the models in $\mathcal{M}$ are models of Subset Collection$_{\psi^i_m}$, then $K_F(\mathcal{M})$ is a model of Subset Collection$_\varphi$.

Proof We show that for $a, b \in D^{M_i}_m$,

$$\begin{gathered}\langle i, m\rangle \Vdash \exists c \forall z [\forall x \in a \exists y \in b\, \varphi(x,y,z) \rightarrow \\ \exists d \in c (\forall x \in a \exists y \in d\, \varphi(x,y,z) \wedge \forall y \in d \exists x \in a\, \varphi(x,y,z))].\end{gathered} \tag{1}$$

Let ψ^i_m be the companions of φ. Since the M_i are models of Subset Collection$_{\psi^i_m}$, there are $c^i_m \in D^{M_i}_m$ for which

$$\begin{gathered}M_i, m \Vdash \forall z [\forall x \in a \exists y \in b\, \psi^i_m(x,y,z) \rightarrow \\ \exists d \in c^i_m (\forall x \in a \exists y \in d\, \psi^i_m(x,y,z) \wedge \forall y \in d \exists x \in a\, \psi^i_m(x,y,z))].\end{gathered} \tag{2}$$

It suffices to show that for all $\langle j, n\rangle \succcurlyeq \langle i, m\rangle$ and all $z \in D^{M_j}_n$

$$\begin{gathered}\langle j, n\rangle \Vdash \forall x \in a \exists y \in b\, \varphi(x,y,z) \\ \Rightarrow \\ \langle j, n\rangle \Vdash \exists d \in c^i_m (\forall x \in a \exists y \in d\, \varphi(x,y,z) \wedge \forall y \in d \exists x \in a\, \varphi(x,y,z)).\end{gathered}$$

Therefore assume $\langle j,n\rangle \succcurlyeq \langle i,m\rangle$, $z \in D_n^{M_j}$, and $\langle j,n\rangle \Vdash \forall x \in a \exists y \in b \varphi(x,y,z)$. Thus $\langle j,n\rangle \Vdash \neg\exists x \in a$ or $z \in D_m^{M_i}$ and $\langle i,m\rangle \Vdash \forall x \in a \exists y \in b \varphi(x,y,z)$. In the latter case, by expressibility $M_i, m \Vdash \forall x \in a \exists y \in b \psi_m^i(x,y,z)$. And thus for some $d \in D_m^{M_i}$, M_i, m forces

$$d \in c_m^i \wedge \forall x \in a \exists y \in d\, \psi_m^i(x,y,z) \wedge \forall y \in d \exists x \in a\, \psi_m^i(x,y,z). \tag{3}$$

By expressibility $\langle i,m\rangle$ forces

$$d \in c_m^i \wedge \forall x \in a \exists y \in d\, \varphi(x,y,z) \wedge \forall y \in d \exists x \in a\, \varphi(x,y,z). \tag{4}$$

Thus so does $\langle j,n\rangle$, which is what we had to show. In the former case, $\langle j,n\rangle$ forces $\neg\exists x \in a$, it follows that $\langle i,m\rangle$, and whence M_i, m, forces $\Vdash \neg\exists x \in a$ too, by transitivity and decidability. Hence $M_i, m \Vdash \forall x \in a \exists y \in b \psi_m^i(x,y,z)$. Consider the d for which (**??**). It follows that M_i, m, and whence $\langle i,m\rangle$, forces $\neg\exists x \in d$. Thus so does $\langle j,n\rangle$, which implies that $\langle j,n\rangle$ forces (**??**), which is what we had to show. □

Corollary 7 If $\mathcal{M}$ is transitive and decides atomic formulas, then if the models in $\mathcal{M}$ are models of Set Induction and Set-bounded Subset Collection, $K_F(\mathcal{M})$ is a model of Set-bounded Subset Collection.

Proof Let $\varphi(x,y,z)$ be a bounded formula in which z is set-bounded. By Corollary 1, φ is expressible by itself, and Lemma 3 implies that φ collapses. Thus we can apply the previous theorem. □

If we combine all results above we obtain the following corollaries.

Corollary 8 If $\mathcal{M}$ is sound for $\in$, consists of transitive models of CZF, and decides atomic formulas, then $K_F(\mathcal{M})$ is a model of CZF^c and Bounded Strong Collection and Set-bounded Subset Collection.

Corollary 9 If $\mathcal{M}$ decides atomic formulas and consists of transitive models of ZF, such that these are $\in$-models or F has no infinite branches, then $K_F(\mathcal{M})$ is a model of CZF^c plus Bounded Strong Collection and Set-bounded Subset Collection. If the models are $\in$-models, $K_F(\mathcal{M})$ is a model of Power Set.

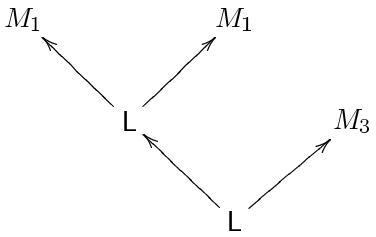
5 Examples of models

Here follow some examples of classes of models satisfying Corollary 9.

5.1 Constructible set models

A collection $\mathcal{M}$ of models sound for a frame F is called an L-*extension* if every model that does not correspond to a leaf of F is L, that is, if all M_i, where i

is not a leaf of F, equal L. In this case we denote $K_F(\mathcal{M})$ by $K_F^{\mathsf{L}}(\mathcal{M})$. The following is an example of such a model:



Recall that for every classical transitive proper class model M of ZF, $\mathsf{L}^M = \mathsf{L}$ (Theorem 3.5 in [6]), and thus L is contained in M. Therefore a class of models $\mathcal{M}$ is an L-extension on a frame F if the models in $\mathcal{M}$ corresponding to the leaves of F are classical transitive proper class models of ZF, and all other models are L.

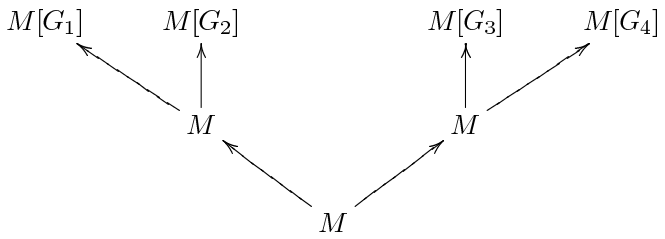
Proposition 8 If $\mathcal{M}$ is an L-extension of transitive $\in$-models of ZF, then $K_F^{\mathsf{L}}(\mathcal{M})$ is a model of $\mathsf{CZF^c}$, Bounded Strong Collection, and Set-bounded Subset Collection.

5.2 Generic models

A collection $\mathcal{M}$ of models that is sound for a frame F is called an *M-extension* if M_i is a generic extension $M[G_i]$ of M if i is a leaf of F, and it is the model M otherwise. In this case we denote $K_F(\mathcal{M})$ by $K_F^M(\mathcal{M})$. Note that such models decide atomic formulas.

We follow the notation of the book by Kenneth Kunen on set theory [6]. We let P_i be the partial order in M with respect to which G_i is generic. Since for every generic set G every m corresponds to a name $\check{m}$, such that $(\check{m})_G = m$, it follows that M is a subset of every $M[G_i]$.

The following is an example of a model $K_F^M(\mathcal{M})$:



Proposition 9 If $\mathcal{M}$ is an M-extension of transitive models of ZF sound for a frame F with no infinite branches, then $K_F^M(\mathcal{M})$ is a model of $\mathsf{CZF^c}$, Bounded Strong Collection, and Set-bounded Subset Collection.

6 Replacement

The use of constructible sets and generic extensions in the results above does not seem to rely on constructibility or genericity in an essential way. The theorems merely serve as natural applications of the more abstract lemma's that were treated before. The results in this section, however, show that by carefully choosing generic models for the leaves of a frame, the final model can be forced to satisfy stronger forms of the collection axioms than Bounded Strong Collection. We will see that under well-known conditions on generic sets certain instances of Replacement can be recovered.

For cardinals κ and natural numbers n we define

$$\neg\neg\mathit{Replacement}^n_\kappa$$
$$\text{for all } a \in M \text{ with } |a| < \kappa\text{: } \neg\neg\forall x \in a \exists! y \in b \varphi(x,y) \rightarrow$$
$$\exists f_1, \ldots, f_n \in b^a \forall x \in a \forall y \in b \neg\neg \textstyle\bigvee_{i=1}^n (\varphi(x,y) \leftrightarrow f_i(x) = y).$$

The "for all a with $|a| < \kappa$" is meant externally, in the real universe. The $\neg\neg$ refers to the fact the antecedent of the implication is a double negated formula as well as to the fact that the f_i are only not not equivalent to φ. The property states that the existence of certain functions at the leaves imply the existence, at the interior nodes, of a finite number of functions that at the leaves behave like the original functions.

Theorem 1 If κ is a cardinal, $\mathcal{M}$ is an M-extension of transitive models of ZF, and F is a finite frame with n leaves, and all P_i associated with the $M[G_i]$ are κ-closed, then $K^M_F(\mathcal{M})$ is a model of $\neg\neg\text{Replacement}^n_\kappa$.

Proof Since the models in $\mathcal{M}$ are classical, we write i instead of $\langle i, m\rangle$, and M_i instead of D^{M_i}. Suppose $i \Vdash \neg\neg\forall x \in a \exists! y \in b \varphi(x,y)$, where $a, b \in M_i$ and $|a| < \kappa$. Hence $j \Vdash \forall x \in a \exists! y \in b \varphi(x)$ for all leafs $j \succcurlyeq i$. Since all formulas are preserved at leaves, it follows that φ represents a function $f_j : a \rightarrow b$ that belongs to $M_j = M[G_j]$. If $i = j$ we are done. So suppose i is not a leaf.

In Theorem 6.14 in [6] it is shown that if P is κ-closed, then for all G that are P-generic over M, for all $a, b \in M$ with $|a| < \kappa$, if $g : a \rightarrow b$ is in $M[G]$, then it is in M. Therefore $f_j \in M$. Hence $f_j \in b^a$ in M. That $\neg\neg \bigvee_{j=1}^n (\varphi(x,y) \leftrightarrow f_j(x) = y)$ follows easily. □

$$\mathit{Weak}\ \neg\neg\mathit{Replacement}^n$$
$$\neg\neg\forall x \in a \exists! y \in b \varphi(x,y) \rightarrow \exists f_1, \ldots, f_n \in \mathcal{P}(b)^a$$
$$\forall x \in a \forall y \in b \big(\textstyle\bigwedge_{i=1}^n |f_i(x)| \leq \omega \wedge \neg\neg \bigvee_{i=1}^n (\varphi(x,y) \rightarrow y \in f_i(x)) \big).$$

The Weakness refers to the fact that the f_i belong to $\mathcal{P}(b)^a$ instead of b^a, but in contrast to $\neg\neg\text{Replacement}^n_\kappa$, there is no restriction on the cardinality of a.

Note that in the requirement $|f_i(x)| \leq \omega$ lies the non triviality of the property, as otherwise we could take $f_i(x) = b$, and the property would trivially hold.

Theorem 2 If $\mathcal{M}$ is an M-extension of transitive models of ZF sound for a finite frame F with n leaves, and $(P_i \text{ has c.c.c.})^M$, then $K_F^M(\mathcal{M})$ is a model of Weak Replacement^n.

Proof Since the models in $\mathcal{M}$ are classical, we write i instead of $\langle i, m\rangle$, and M_i instead of D^{M_i}. Suppose $i \Vdash \neg\neg\forall x \in a \exists! y \in b \varphi(x, y)$, where $a, b \in M_i$. Hence $j \Vdash \forall x \in a \exists! y \in b \varphi(x)$ for all leafs $j \succcurlyeq i$. Since all formulas are preserved at leaves, it follows that φ represents a function $g_j : a \to b$ that belongs to $M_j = M[G_j]$. If $i = j$ we can take $f_j(x) = \{g_j(x)\}$ and are done. So suppose i is not a leaf.

In Lemma 5.5 in [6] it is shown that if $(P \text{ has c.c.c.})^M$ and $g : a \to b$ is in $M[G]$ where $a, b \in M$, there is a map $f : a \to \mathcal{P}(b)$ in M such that for all $x \in a$, $g(x) \in f(x)$ and $(|f(x)| \leq \omega)^M$.

Thus there exist $f_j \in M$ such that $f_j : a \to \mathcal{P}(b)$ and

$$\forall x \in a (g_j(x) \in f_j(x)) \wedge (|f_j(x)| \leq \omega)^M.$$

We have to show that the following formula is forced at i:

$$\forall x \in a \forall y \in b \Big(\bigwedge_{j=1}^{n} |f_j(x)| \leq \omega \wedge \neg\neg \bigvee_{j=1}^{n} \big(\varphi(x, y) \to y \in f_j(x)\big)\Big).$$

The second part clearly holds as for leaves j, $j \Vdash \varphi(x, y) \leftrightarrow g_j(x) = y$. For the first part, note that since ω is absolute for transitive models, it belongs to all models in $\mathcal{M}$. Since $\bigwedge_{j=1}^{n} |f_j(x)| \leq \omega$ holds in M, there exist injections $h_j^x : f_j(x) \to \omega$ in M. It is easy to see that by Corollary 1, i forces that the h_j^x are injections too, since injectivity can be expressed by a bounded formula. □

7 Exponentiation

In this section we show that under certain conditions on the generic sets instances of exponentiation can be recovered. We write $f : a \to b$ as an abbreviation of "f is a function from a to b". Exponentiation is the axiom stating that the set b^a of functions from a to b exists for all a and b:

$$\textit{Exponentiation} \quad \forall a \forall b \exists c \forall f (f \in c \leftrightarrow f : a \to b).$$

For cardinals κ we define a bounded version of Exponentiation:

$$\textit{Exponentiation}_\kappa$$
$$\text{for all } a \in M \text{ with } |a| < \kappa\text{: } \forall b \exists c \forall f (f \in c \leftrightarrow f : a \to b)$$

The "for all a with $|a| < \kappa$" is meant externally, in the real universe.

Theorem 3 If κ is a cardinal, $\mathcal{M}$ is an M-extension of transitive models of ZF sound for a finite frame F with n leaves, and all P_i associated with the $M[G_i]$ are κ-closed, then $K_F^M(\mathcal{M})$ is a model of Exponentiation$_\kappa$.

Proof Since Exponentiation holds in the leaves of $K_F^M(\mathcal{M})$, it suffices to show that for all i that are not leaves, for all $a, b \in M$ with $|a| < \kappa$, there is a set $c_i \in M$ such that i forces $\forall f(f \in c_i \leftrightarrow f : a \to b)$. Take for all c_i the set b^a in M consisting of all functions in M from a to b. This set exists because M is a model of ZF, and thus of Exponentiation. We show that this c has the right property, that is, that for any node i:

$$i \Vdash \forall f(f \in c \leftrightarrow f : a \to b).$$

If i is a leaf and $f \in c$, this means that $f \in M$ by the transitivity of M, and thus $f : a \to b$ holds in $M_i = M[G_i]$ too. If $f : a \to b$ in $M[G_i]$, then we use Theorem 6.14 in [6] stating that if P is κ-closed, then for all G that are P-generic over M, for all $a, b \in M$ with $|a| < \kappa$, if $g : a \to b$ is in $M[G]$, then it is in M. Thus $f \in M$, and whence $f \in c$.

If i is not a leaf, we only have to consider the case that $f \in M$, as $f \in M[G_i]$ has been treated above. But in this case $f \in c \leftrightarrow f : a \to b$ is clearly forced at i. □

The results in the last two sections show how the choice of the generic sets can force the final Kripke model to be a model of certain instances of collection. The obvious open problem is whether this method can be pushed further to obtain models of larger subtheories of CZF than the theories treated in this paper.

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