A Modal Analysis of some Principles of the Provability Logic of Heyting Arithmetic

R. Iemhoff*

ILLC/Dept. of Mathematics and Coputer Science

Universiteit van Amsterdam

Plantage Muidergracht 24

1018 TV Amsterdam

iemhoff@wins.uva.nl

Abstract

Besides the well-known principles $K, K4$ and L, the Leivant principle $\Box(A \vee B) \to \Box(A \vee \Box B)$ and the Formalized Markov scheme $\Box\neg\neg(\Box A \to \bigvee \Box B_i) \to \Box(\Box A \to \bigvee \Box B_i)$ are principles of the Provability Logic of (intuitionistic) Heyting Arithmetic. In this paper they are studied from a modal logical point of view. They are characterized, separate as well as in combination with the other principles, in terms of frame properties, and the finite model property is established. In order to make the paper self-contained intuitionistic modal logic is introduced and similar results for $K4$ and L, which were already known, are proved again.

1 Introduction

In this paper we study the intuitionistic modal logic given by the principles

K	$\Box(A \to B) \to (\Box A \to \Box B)$	
$K4$	$\Box A \to \Box\Box A$	
L	$\Box(\Box A \to A) \to \Box A$	(Löb's principle)
Le	$\Box(A \vee B) \to \Box(A \vee \Box B)$	(Leivant's principle)
Ma	$\Box\neg\neg(\Box A \to \bigvee \Box B_i) \to \Box(\Box A \to \bigvee \Box B_i)$	(Formalized Markov scheme)

*The author is supported by the Netherlands Organisation for Scientific Research (NWO) under grant PGS 22-262

and the rules modus ponens and necessitation. We shall characterize it in two ways in terms of frame properties, and prove that it has the finite model property.

These principles belong to the so-called Provability Logic of intuitionistic Heyting Arithmetic **HA**, which arises from an arithmetical interpretation of the $\Box$ in which $\Box\varphi$ is read as 'φ is provable in **HA**'. This will be described in detail in the next section. The full Provability Logic of **HA** is not known yet. The fact that the Leivant principle belongs to the Provability Logic of **HA** is first stated in [Leivant 75] and proved in [Visser 94]. For the Formalized Markov scheme, it is proved in [Visser 94] as well.

A motivation to characterize modal logics which are the Provability Logic of a certain theory in terms of frame properties and prove (if possible) the finite model property for them, can be to gain inside in the theory in question. Moreover, it often turned out that in showing that some modal logic is the Provability Logic of a certain theory one uses these properties heavely.

In order to make the paper self-contained intuitionistic modal logic is introduced in the third section, and completeness results for K, $K4$ and L which were already known are proved again, in sections 5-7. And although nowhere needed to understand the technicalities of the provability interpretation, we will start with a short introduction to Provability Logic, in the next section. Finally, in sections 8-10 the completeness results for the Leivant principle and the Formalized Markov scheme are given, separate as well as together with the other principles. And in the last section of the paper we show that the logics under consideration all have the disjunction property and the reflection rule (if $\vdash \Box A$ then $\vdash A$).

I would like to thank D. de Jongh and A. Visser for the many discussions we had on this subject and for their patient help, and A. Troelstra for his careful reading of the manuscript.

2 The Landscape: Provability

A *modal* language will always be, in this paper, a language for propositional logic extended with one unary modal operator, $\Box$. A formula in this language is called a modal formula.

Let $\mathbf{T}$ be an arithmetical theory which allows a formalization of 'being provable in $\mathbf{T}$' via a predicate $Proof_{\mathbf{T}}(x, y)$, which is a formalization of the statement that y is the code of a proof in $\mathbf{T}$ of the formula with Gödelnumber x. When φ is an arithmetical sentence with Gödelnumber $\ulcorner\varphi\urcorner$, $\Box_{\mathbf{T}}\varphi$ denotes the statement $\exists y Proof_{\mathbf{T}}(\ulcorner\varphi\urcorner, y)$. An arithmetical realization of the modal language into the language of $\mathbf{T}$ is a mapping * from the formulas of the modal language to sentences in the language of $\mathbf{T}$ that commutes with the propositional connectives and such that $(\Box A)^* = \Box_{\mathbf{T}}(\ulcorner A^* \urcorner)$. The provability logic of $\mathbf{T}$ is the set of modal formulas A such that $\mathbf{T}$ proves A^* for any arithmetical realization *, i.e. the set $\{A \mid \forall^* \ \mathbf{T} \vdash A^*\}$.

Note that in general the provability logic of a theory $\mathbf{T}$ may depend on T as well as on the chosen formalization of the proof predicate $Proof_{\mathbf{T}}$. We will be a bit ambiguous in this respect. When talking about 'the provability logic' of a certain theory, we will always assume that a not-to-unusual proof predicate is fixed in advance.

The famous article of Solovay [Solovay 76] may well be seen as the starting point of provability logic. In this paper Solovay proves that the the provability logic of **PA** is the logic now known as **L** or **GL**, consisting of the principles $K, K4$ and L, the tautologies of classical propositional logic and the rules of necessitation $(A/\Box A)$ and modus ponens. Moreover, the proof gives a method to construct for any formula A which is not a principle of the provability logic of **PA** an actual counterexample A^*, that is, a realization such that **PA** does not prove A^*. The way this is obtained employs a modal completeness result for **GL**. First, it is shown that **GL** is complete with respect to the class of finite, transitive, conversely well-founded Kripke models. And second, it is shown that for every such Kripke model $\mathcal{K}$ there exists a realization * such that

for all nodes k of $\mathcal{K}$ (if $\mathcal{K}, k \not\models A$, then $\neg A^*$ is consistent with **PA**).

As long as we stay on the classical side provability logics are very stable; many arithmetical theories have the same provability logic, **GL**. See, for example, De Jongh and Japardize [JJ 97] for an explication of the vague 'many' in this statement. However, there is in general no stability in going from a classical theory to its intuitionistic counterpart, as can be seen in the comparison of **HA** and **PA**. For example, the Leivant principle $\Box(A \vee B) \to \Box(A \vee \Box B)$ is not a provability principle of **PA**. This can be seen very easily. If **GL** would derive the Leivant principle it would also derive $\Box(\Box\bot \vee \Box\neg\Box\bot)$, as it clearly derives $\Box(\Box\bot \vee \neg\Box\bot)$. But then an application of L shows that it would derive $\Box\Box\bot$. Hence the provability logic of **HA** is not a part of **GL**. The converse is not true either. The principle $\Box(p \vee \neg p)$ is a theorem of the provability logic of **PA**, but not of the corresponding logic of **HA**. Note that this also shows that there is no monotonicity (converse monotonicity) in provability logics; stronger theories do not necessarily have stronger (weaker) provability logics.

In the literature on intuitionistic modal logic one often encounters logics of which it is claimed that they are the 'true' intuitionistic counterparts of some classical modal logic, for example Löb's logic. Of course, what one will accept as an intuitionistic counterpart of a given (classical) logic, will depend on the interpretation one has in mind for the box operator, hence on the properties one wants it to have. Here we always have the provability interpretation in mind. A striking difference between this interpretation and most others is that it is in *itself* of a mathematical nature. Thus verification of the validity of principles can be executed in a *formal* rigorous way.

3 Intuitionistic modal logic

3.1 The syntax

The system $\mathbf{K}_i$ is a theory in the language of propositional logic extended with one modal operator $\Box$. It is given by the axioms:

$Taut$ all intuitionistic propositional tautologies in the modal language,

K $\Box(A \to B) \to (\Box A \to \Box B)$,

together with the rules of modus ponens and necessitation.

When $\mathbf{R}$ is a set of additional axioms, we write $\mathbf{R}_i$ for the system obtained by the axioms $\mathbf{K}_i$ and $\mathbf{R}$ while keeping the rules modus ponens and necessitation. We will write $\vdash_{\mathbf{R}_i} A$ when A is derivable in $\mathbf{R}_i$. We write $B_1, \ldots, B_n \vdash_{\mathbf{R}_i} A$ for a derivation of A in $\mathbf{R}_i$ from $B_1, \ldots, B_n$ without use of necessitation, in other words, when A is derivable by modus ponens from theorems of $\mathbf{R}_i$ and $B_1, \ldots, B_n$. (When we trust it is clear from the context which derivation system we mean, we omit the subscript $\mathbf{R}_i$.)

We write $\mathbf{L}_i$ for $\mathbf{K4L}_i$, and $\mathbf{B}_i$ for $\mathbf{LLeMa}_i$.

For an extensive introduction to intuitionistic modal logic see for example [WZ 96a, 96b].

3.2 The semantics

First some notation. When R and S are two binary relations, $(R;S)$ is the relation defined via $w(R;S)u :\Leftrightarrow \exists v(wRvSu)$.

A *frame* is a triple $\mathcal{F} = (W, \preccurlyeq, R)$, where W is a nonempty set (the set of *nodes*), $\preccurlyeq$ is a partial ordering on W (the *intuitionistic relation*) and R a binary relation on W (the *modal relation*) such that $(\preccurlyeq;R) \subseteq R$. A frame is called *brilliant* if in addition $(R;\preccurlyeq) \subseteq R$.

A *model* is a pair $\mathcal{M} = (\mathcal{F}, V)$, where $\mathcal{F}$ is a frame and V a valuation relation on pairsconsisting of nodes and propositional variables. We demand that V is persistent, i.e.

$$\text{if } w \preccurlyeq v \wedge (w,p) \in V, \text{ then } (v,p) \in V.$$

We inductively define what it means for a formula A to be *valid at a node* w of a model $\mathcal{M}$ ($\mathcal{M}, w \models A$):

$$\mathcal{M}, w \models p :\Leftrightarrow wVp$$

$$\mathcal{M}, w \models A \wedge B :\Leftrightarrow \mathcal{M}, w \models A \text{ and } \mathcal{M}, w \models B$$

$$\mathcal{M}, w \models A \vee B :\Leftrightarrow \mathcal{M}, w \models A \text{ or } \mathcal{M}, w \models B$$

$$\mathcal{M}, w \models A \to B :\Leftrightarrow \forall v \geqslant w \ (\mathcal{M}, v \models A \text{ implies } \mathcal{M}, v \models B)$$

$$\mathcal{M}, w \models \Box A :\Leftrightarrow \forall v \ (\text{if } wRv \text{ then } \mathcal{M}, v \models A).$$

Remark 1 $(\preccurlyeq;R) \subseteq R$ is required for the upwards persistency of the $\Box$ (for $w \preccurlyeq w'$, $w \models \Box A$ implies $w' \models \Box A$).

When $\mathcal{M}$ is clear from the context we write $w \models A$ instead of $\mathcal{M}, w \models A$. If a sequence of models $\mathcal{M}_1, \mathcal{M}_2, \ldots$ is involved, we sometimes write $w \models_i A$ for $\mathcal{M}_i, w \models A$. When defining V we often write $w \models p :\Leftrightarrow \ldots$, instead of $wVp :\Leftrightarrow \ldots$. We say that A is *valid* in $\mathcal{M}$, notation $\mathcal{M} \models A$, if A is valid at all nodes of $\mathcal{M}$. The formula A is said to be valid in a frame $\mathcal{F}$, notation $\mathcal{F} \models A$, if A is valid in all models with underlying frame $\mathcal{F}$.

A property P on frames *corresponds* to a set R of formulas or schemes if for all frames F: $F \models R$ iff $P(F)$. Note that in this case we have

$$\vdash_{\mathbf{R}_i} A \Rightarrow A \text{ is valid on all frames with property } P.$$

Remark 2 The condition $(\preccurlyeq;R) \subseteq R$, included to guarantee persistence for formulas $\Box A$, may be weakened to

$$(\preccurlyeq;R) \subseteq (R;\preccurlyeq) \qquad (w \preccurlyeq w'Rv' \Rightarrow \exists u(wRu \preccurlyeq v')).$$

However we prefer to work with the simple condition where possible. For more discussion on this topic, see [Simpson 93].

3.3 Canonicity

To define the canonical model for a logic we have to introduce *saturated sets.* As we also need the concept of X*-saturated sets* later on, we give all these definitions here simultaneously.

A set of formulas is called *adequate* if it is closed under subformulas and contains $\top$ and $\bot$. A set of formulas Γ is called X*-saturated* if it is a consistent subset of X such that

- $\Gamma \vdash A$ implies $A \in \Gamma$, for all $A \in X$,
- $\Gamma \vdash A \vee B$ implies $A \in \Gamma$ or $B \in \Gamma$, for all $A \vee B \in X$.

If X is the set of all formulas, an X-saturated set is just called *saturated.* It is easily shown that for any adequate set X and for any A for which $\nvdash A$, there is an X-saturated set Γ such that $\Gamma \nvdash A$. Note also that any $\Delta \subseteq X$ for which $\Delta \nvdash A$, can be extended to an X-saturated Γ such that $\Gamma \nvdash A$.

For any system $\mathbf{R}_i$, the $\mathbf{R}_i$*-canonical model* is the model $(W, \preccurlyeq, R, V)$ defined as follows:

W consists of the saturated sets (with respect to $\vdash_{\mathbf{R}_i}$),

$w \preccurlyeq v :\Leftrightarrow w \subseteq v$,

$wRv :\Leftrightarrow \forall \Box A \in w (A \in v)$,

$w \models p :\Leftrightarrow p \in w$, for propositional variables p.

To see that this indeed defines a model, see the completeness proof for $\mathbf{K}_i$.

Note that the $\mathbf{R}_i$-canonical frame is always brilliant and hence that when a logic $\mathbf{R}_i$ corresponds to a frame property P, then the assumption that the $\mathbf{R}_i$-canonical model has the property P implies that

$\vdash_{\mathbf{R}_i} A$ iff A is valid on all brilliant frames which have P.

We call $\mathbf{R}_i$ *canonical* in this case, i.e. if the canonical model has the frame property to which the logic corresponds.

4 Some constructions of models

In this paragraph two methods of constructing new models from old ones are introduced, all of which will be used later on. Before starting, let us agree on some notation.

A node v in a frame is *above* w if $w \preccurlyeq v$. In this case w is called *below* v. A node t is called an *(intuitionistic) top node* if there is no element above it in the frame. $Top(F)$ is the set of all *top* nodes in a frame F. We often omit F when we trust it is clear from the context to which frame we refer. When w is a node, $T(w)$ is the set of all *top* nodes above w.

A node v in a frame is called a *successor* of w if wRv, in which case w is called a *predecessor* of v. When Z is a set of nodes, $R(Z)$ is the set of all successors of elements in Z. We write $R(w)$ for $R(\{w\})$.

For any adequate set of formulas X we define

$$w^X_{\Box} = \{\Box B \in X \mid w \models \Box B\},$$

$$w^X_{\boxdot} = \{B, \Box B \in X \mid w \models \Box B\},$$

$$w^X_{\not\Box} = \{\Box B \in X \mid w \not\models \Box B\},$$

$$w^X_{\not\to} = \{B \to C \in X \mid w \not\models B \to C, w \not\models B\}.$$

The superscript will be sometimes omitted.

4.1 The Construction Method

The *construction method* is often used to obtain a completeness result with respect to some class of finite frames and thereby the decidability of the logic in question. The method allows us to construct a finite model $\mathcal{M}_b = (W_b, \preccurlyeq_b, R_b, V_b)$ for any node b of any canonical model $\mathcal{M} = (W, \preccurlyeq, R, V)$, for any finite adequate set X. And the domain W_b of $\mathcal{M}_b$ is (a copy of) a subset of W which intuitively is the minimal set of nodes needed to make b behave in the same manner in $\mathcal{M}$ and $\mathcal{M}_b$ with respect to the formulae in X. For instance, when $\Box B \in X$ but $\Box B \notin b$, we have to take care that there is a node w in W_b such that bR_bw and $w \not\models_b B$. Similar for implications.

Since we will apply this method only to canonical models which satisfy L, we do not give the most general form of the construction, but just a version for canonical models satisfying L. Henceforth the models considered are models which satisfy L.

The construction proceeds as follows. We choose step by step, starting with b, a finite subset of W which will be the domain W_b of our new model $\mathcal{M}_b$. Note that the elements of W are sets of formulas. Put $\alpha_{\langle\rangle} = b$. Suppose α_σ is defined. We choose elements $\alpha_{\sigma*\langle\Box B\rangle}$ and $\alpha_{\sigma*\langle C\to D\rangle}$ in W, for all elements $\Box B \in (\alpha_\sigma)_{\not\Box}$ and for all $(C \to D) \in (\alpha_\sigma)_{\not\to}$. The node $\alpha_{\sigma*\langle\Box B\rangle}$ is an element $v \in W$ such that $\alpha_\sigma R v$, $\Box B \in v$ and $B \notin v$. The node $\alpha_{\sigma*\langle C\to D\rangle}$ is an element $v \in W$ such that $\alpha_\sigma \preccurlyeq v$, $C \in v$ and $D \notin v$. Note that such elements can always be found. For the node $\alpha_{\sigma*\langle\Box B\rangle}$ this follows from the fact that the model satisfies L.

Now define $W_b = \{\sigma \mid \sigma \text{ is defined }\}$, and V_b via

$$\sigma \models_b p :\Leftrightarrow \alpha_\sigma \models p, \text{ for p} \in X.$$

We define the intuitionistic and the modal relation such that

$$\text{for all } A \in X,\ \text{for all } \sigma \in W_b:\quad \mathcal{M}, \alpha_\sigma \models A \Leftrightarrow \mathcal{M}_b, \sigma \models A.$$

As the choice of the relations will differ from case to case we do not give any specific examples here besides the obvious one;

$$\begin{array}{lll} \sigma \preccurlyeq_b \tau & :\Leftrightarrow & \alpha_\sigma \preccurlyeq \alpha_\tau \\ \sigma R_b \tau & :\Leftrightarrow & \alpha_\sigma R \alpha_\tau \end{array}$$

It is not difficult to see that this choice at least gives a model with the desired property, be it not always on a frame with the desired properties.

Remark 3 It is easy to see that W_b is finite if X is. This is in fact the reason why we chose the nodes $\alpha_{\sigma*\langle\Box B\rangle}$ such that they contain $\Box B$; in this way $(\alpha_{\sigma*\langle\Box B\rangle})_{\not\Box} \subset (\alpha_\sigma)_{\not\Box}$, or in other words, $(\alpha_\sigma)_\Box \subset (\alpha_{\sigma*\langle\Box B\rangle})_\Box$.

4.2 Neighbourhoods

The *boxdepth* of a formula is the maximum number of nested boxes of the formula. The method described here is used later to prove that some logic is conservative over another with respect to formulae of boxdepth one.

Whenever w is a node in a model $\mathcal{M}$, we define the *n-neighbourhood* $(\mathcal{M}, w)_n = (W_n, \preccurlyeq_n, R_n, \models_n)$ via two sequences of subsets $(I_i)_i$, $(M_i)_i$ of the domain of $\mathcal{M}$.

$$\begin{array}{lllllll} I_1 & = & \{v \mid w \preccurlyeq v\} & & M_1 & = & I_1 \\ I_{k+1} & = & \{v \mid \exists x \in M_k\ x \preccurlyeq v\} & & M_{k+1} & = & \{v \mid \exists x \in I_k\ xRv\} \end{array}$$

$W_n = \bigcup_{i \preccurlyeq n}(I_i \cup M_i)$. The relations of $(\mathcal{M}, w)_n$ are just the restrictions to W_n of the corresponding relations of $\mathcal{M}$. It is easy to prove with induction on the boxdepth of a formula that

$$\text{for all } A \text{ with boxdepth} \leqslant n, \forall v \in I_1 \quad (\mathcal{M}, w)_n, v \models A \text{ iff } \mathcal{M}, v \models A.$$

$(\mathcal{M}, w)$ denotes the model $(\mathcal{M}, w)_n$, where n is the length of the maximal R-chain in $\mathcal{M}$, provided it is finite.

5 The principle K

Proposition 4 $\vdash_{\mathbf{K}_i} A$ iff A is valid on all finite brilliant frames.

Proof Only from right to left. Assume $\nvdash_{\mathbf{K}_i} A$. Let X be a finite adequate set which contains A. We build a model on X as follows. This model will not force A.

Let W consist of all X-saturated sets. $\preccurlyeq, R, V$ are defined in the same way as on the $\mathbf{K}_i$-canonical model.

Note that this defines indeed a brilliant frame, as $\preccurlyeq;R = R;\preccurlyeq = R$. We show that for all B in X: $w \models B$ iff $B \in w$, and then we are done. We prove it by induction on B. For propositional variables, disjunction and conjunction it is clear.

Case $\rightarrow$. If $(C \rightarrow D) \in w$ then certainly $w \models C \rightarrow D$. For the other direction, assume $(C \rightarrow D) \notin w$. If $w \cup \{C\}$ would derive D, then also $w \vdash (C \rightarrow D)$. Thus $w \cup \{C\} \nvdash D$. This implies that $w \cup \{C\}$ is consistent. Let v be an X-saturated extension of $w \cup \{C\}$ which does not derive D, then clearly $w \preccurlyeq v$, $v \models C$ and $v \not\models D$, hence $w \not\models C \rightarrow D$.

Case $\Box$. If $\Box C \in w$, then it follows immediately that $w \models \Box C$. For the other direction, assume $\Box C \notin w$. Consider the set $\Delta = \{D \mid \Box D \in w\}$. We have $\Delta \nvdash C$, as otherwise $w \vdash \Box C$. Hence there exists an X-saturated set v containing Δ which does not derive C. As clearly wRv, this shows that $w \not\models \Box C$. **QED**

6 The principle $K4$

In the same way as in the classical case it can be shown that $\vdash_{\mathbf{L}_i} \mathbf{K4}$.

We call a frame *semi-transitive* if it satisfies $(R;R) \subseteq (R;\preccurlyeq)$.

Lemma 5 **K4** corresponds to semi-transitivity.

Proposition 6 $\vdash_{\mathbf{K4}_i} A$ iff A is valid on all finite transitive brilliant frames.

Proof Assume $\nvdash_{\mathbf{K4}_i} A$. Let X be an adequate set which contains A such that there is a X-saturated set Γ with $\Gamma \nvdash_{\mathbf{K4}_i} A$. The model $(W, \preccurlyeq, R, V)$ has the same $W, \preccurlyeq$ and V as a $\mathbf{K}_i$-canonical model. But now define

$$wRv :\Leftrightarrow \forall \Box B \in w (B, \Box B \in v).$$

It is clear that this frame is transitive. So, we only have to show that $w \models B$ iff $B \in w$, for all $B \in X$. The only step which is different from the $\mathbf{K}_i$ case is the one for $B = \Box C$, direction from left to right:
if $\Box C \notin w$, then the set $\{D, \Box D \mid \Box D \in w\}$ does not derive C. Therefore, it can be extended to an X-saturated set v which does not contain C. As wRv, this proves $w \not\models \Box C$. **QED**

Proposition 7 $\mathbf{K4}_i$ is canonical.

7 The Löb principle

We call a frame *conversely well-founded* if the modal relation on the frame is conversely well-founded. An L-frame is a frame which is both semi-transitive and conversely well-founded.

Lemma 8 **L** corresponds to semi-transitivity plus converse well-foundedness.

Proof We only treat the direction from left to right. First assume that F is not semi-transitive; choose w, v, u such that

$$wRvRu \wedge \forall v'(wRv' \to v' \not\preccurlyeq u). \tag{1}$$

Consider the model on F given by the valuation $x \models p :\Leftrightarrow (x \not\preccurlyeq v \wedge x \not\preccurlyeq u)$. We have to see that $w \models \Box(\Box p \to p)$. Therefore, take x, y with wRx, $x \preccurlyeq y$ and $y \models \Box p$. Because $u \not\models p$, $v \not\models \Box p$ and thus $y \not\preccurlyeq v$. Also $y \not\preccurlyeq u$ by (1), hence $y \models p$. But clearly $w \not\models \Box p$, since $v \not\models B$.

For the second part, assume F is a semi-transitive but non conversely well-founded frame. Let $w_0 R w_1 R w_2 R \ldots$ be a chain in F. Define a valuation on F via

$$x \models p :\Leftrightarrow x \not\preccurlyeq w_i \text{ for all } i.$$

In this model on F, if $w_0 R x$ and $x \models \Box p$ then $x \not\preccurlyeq w_i$, for all i, as $x \preccurlyeq w_i$ implies xRw_{i+1}. Hence $w \models \Box(\Box p \to p)$. But not $w \models \Box p$. **QED**

Proposition 9 $\vdash_{\mathbf{L}_i} A$ iff A is valid on all transitive conversely well-founded brilliant frames.

Proof Let A be such that $\nvdash_{\mathbf{L}_i} A$. Let X be a finite adequate set containing A, such that there is an X-saturated Γ for which $\Gamma \nvdash_{\mathbf{L}_i} A$. We build a model $(W, \preccurlyeq, R, V)$, which does not make A valid. W is the set of X-saturated sets. $\preccurlyeq, V$ are defined in the same way as for the $\mathbf{L}_i$-canonical model. But define

$$wRv :\Leftrightarrow \forall \Box B \in w(\Box B, B \in v) \wedge \exists \Box D \in v(\Box D \notin w).$$

This makes it into a brilliant, transitive, conversely well-founded frame. We show that $w \models B$ iff $B \in w$, for $B \in X$. Only the left to right direction for the case $B = \Box C$ is different:
Assume $\Box C \notin w$. Let $\Delta = \{D, \Box D \mid \Box D \in w\} \cup \{\Box C\}$. From $\Delta \vdash C$ it would follow that $w \vdash \Box(\Box C \to C)$, and hence $w \vdash \Box C$, which is false. Therefore, $\Delta \nvdash C$. Let v be a X-saturated extension of Δ which does not derive C, then wRv, and therefore $w \not\models \Box C$. **QED**

8 The Leivant principle

Since **HA** possesses the so-called *disjunction property*, i.e.

if $\mathbf{HA} \vdash A \vee B$, then $\mathbf{HA} \vdash A$ or $\mathbf{HA} \vdash B$,

it is a natural question to ask if **HA** can prove it's own disjunction property, that is, if $\Box(A \vee B) \to (\Box A \vee \Box B)$ is a principle of the provability logic of **HA**. It follows from a result of Friedman [Friedman 75] that this is not the case. Thus informally speaking, **HA** cannot prove the statement 'if there is a proof of A or B, there is a proof of A or there is a proof of B'. The Leivant principle, which *is* a principle of the provability logic of **HA**, can be seen as a second best to the formalized version of the disjunction property. For the Leivant principle clearly implies $\Box(A \vee B) \to \Box(\Box A \vee \Box B)$. And this principle expresses the statement 'if there is a proof of A or B, there is a proof that (there is a proof of A or there is a proof of B)'.

There are two noteworthy consequences of LLe which we do want to state explicitly. The first one because it is the one consequence of LLe mainly used in the completeness proof of $\mathbf{LLe}_i$. And the second one because it intuitively expresses the interesting statement 'if there is a proof that (either A holds or B is not provable), then A is provable'. Let $\boxdot A$ be short for $(A \wedge \Box A)$. The Leivant principle implies

$$\Box(A_1 \vee \ldots \vee A_n \vee B_1 \vee \ldots \vee B_m) \to \Box(A_1 \vee \ldots \vee A_n \vee \boxdot B_1 \ldots \vee \boxdot B_m).$$

We show this only for the special case of a disjunct of two formulas:

$$\begin{array}{lll} \Box(A \vee B) & \vdash_{\mathbf{Le}_i} & \Box(A \vee B) \wedge \Box(A \vee \Box B) \\ & \vdash & \Box(A \vee (A \wedge \Box B) \vee \boxdot B) \\ & \vdash & \Box(A \vee \boxdot B) \end{array}$$

The other noteworthy consequence of LLe is

$$\Box(A \vee \neg\Box B) \to \Box A.$$

This follows immediately from the observation above and the fact that $\mathbf{L}_i$ derives $(\Box\neg\Box B \to \Box B)$.

8.1 Correspondence

First, we show that Le corresponds to a certain frame property Le^∞. And then we show that on finite frames, Le corresponds to a stronger property. Since we prove completeness of $\mathbf{Le}_i$ with respect to these class of frames, this property will be called the Leivant property. .

Lemma 10 Le corresponds to the Le^∞-property, which is

Le^∞**-property** $\quad wRvRu \to \exists x(wRx \wedge x \preccurlyeq v \wedge x \preccurlyeq u)$.

Proof We only treat the direction from left to right. Assume that a frame F does not have the Le^{∞}-property. Take $wRvRu$ such that

$$\forall x \neg(wRx \wedge x \preccurlyeq v \wedge x \preccurlyeq u). \tag{2}$$

Now define a model on F via
$y \models p \leftrightarrow \exists x \preccurlyeq y(wRx \wedge x \preccurlyeq u)$,
$y \models q \leftrightarrow \exists x \preccurlyeq y(wRx \wedge x \not\preccurlyeq u)$.
With this valuation clearly $w \models \Box(p \vee q)$. It is also easy to see that $u \not\models q$, hence $v \not\models \Box q$. Moreover, $v \not\models p$. For, if $v \models p$, then there is a node $x \preccurlyeq v$ such that $wRx \wedge x \preccurlyeq u$. This contradicts (2). Hence $v \not\models p$, thus $v \not\models p \vee \Box q$. Therefore, $w \not\models \Box(p \vee \Box q)$.

QED

Remark 11 Clearly $\mathbf{Le}_i \vdash K4$. Observe that this is reflected in the corresponding frame properties; an Le^{∞}-frame is semi-transitive.

Lemma 12 A finite Le^{∞}-frame satisfies

$$wRv \to \exists x(wRx \preccurlyeq v \wedge \forall u(vRu \to x \preccurlyeq u)).$$

Proof Let F be a finite Le^{∞}-frame. Consider wRv. We show that there is a node x such that $(wRx \preccurlyeq v \wedge \forall u(vRu \to x \preccurlyeq u))$. Let n be the number of successors of v. If $n = 0$, then we can take $v = x$, and we are done. Therefore, assume $n \geqslant 1$. We show that there is an enumeration $u_1, \ldots, u_n$ of the successors of v such that there is a sequence $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n$ of nodes, such that

$$wRx_i \preccurlyeq v \wedge x_i \preccurlyeq u_i.$$

Then x_n has the desired properties, i.e. we can take $x = x_n$, since $\forall i(x_n \preccurlyeq u_i)$, and $wRx_n \preccurlyeq v$ by construction.

We show how to construct these sequences. Let u_1 be a successor of v. Since $wRvRu_1$, by the Le^{∞}-property, there is a node x_1 such that

$$wRx_1 \preccurlyeq v \wedge x_1 \preccurlyeq u_1.$$

Assume that for $j \preccurlyeq i$, x_j and u_j are defined. Let u_{i+1} be a successor of v distinct from $u_1, .., u_i$. Since $wRx_i \preccurlyeq vRu_{i+1}$, wRx_iRu_{i+1}. Hence by the Le^{∞}-property there is a node x_{i+1} such that

$$wRx_{i+1} \preccurlyeq x_i \wedge x_{i+1} \preccurlyeq u_{i+1}.$$

As clearly $x_{i+1} \preccurlyeq x_i \preccurlyeq v$, we also have $x_{i+1} \preccurlyeq v$. Thus x_{i+1} has the desired properties. **QED**

An Le-frame is a frame which has the property,

Le**-property** $\quad wRv \to \exists x(wRx \preccurlyeq v \wedge \forall u(vRu \to x \preccurlyeq u))$.

A node $x \preccurlyeq v$ for which $\forall u(vRu \to wRx \preccurlyeq u)$ holds is called a *Leivant-node for the pair* (w, v). An LLe-frame is a frame which is both an L- and an Le-frame.

Corollary 13 On finite frames, Le corresponds to the Le-property.

8.2 Minimal Leivant-nodes

We will use the following lemma only in pragraph 10, but this seems a good place to state it. It shows that on finite Le^{∞}-frames even a stronger condition than the Leivant property holds. In words, for every pair wRv in a finite Le^{∞}-frame there is Leivant-node x for (w, v) which also is a Leivant-node for itself, namely for (w, x). Such a node x for which

$$wRx \preccurlyeq v \wedge \forall u(vRu \to x \preccurlyeq u) \wedge \forall u(xRu \to x \preccurlyeq u),$$

is called a *minimal Leivant-node for* (w, v).

Lemma 14 A finite Le^{∞}-frame satisfies

$$wRv \to \exists x(wRx \preccurlyeq v \wedge \forall u(vRu \to x \preccurlyeq u) \wedge \forall u(xRu \to x \preccurlyeq u)).$$

Proof Let F be a finite Le^{∞}-frame. By lemma 12, F satisfies

$$wRv \Rightarrow \exists x(wRx \preccurlyeq v \wedge \forall u(vRu \to x \preccurlyeq u)). \tag{3}$$

Consider wRv. We show that the pair (w, v) has a minimal Leivant-node x. Define $*(y)$ via

***(y)** $\forall u(yRu \to y \preccurlyeq u)$.

We construct a sequence $x_1 \geqslant x_2 \geqslant ...$ of nodes, such that

$$wRx_i \preccurlyeq v \wedge \forall u(vRu \to x_i \preccurlyeq u) \wedge (x_{i+1} = x_i \to *(x_i)).$$

Let us first see why we are done then. The finiteness of the frame implies that $x_{i+1} = x_i$, for some i. Such a node x_i has the desired properties, i.e. we can take $x = x_i$.

We show how to construct the sequence $x_1 \geqslant x_2 \geqslant ...$ by induction. By (3) there is a node x_1 such that $wRx_1 \preccurlyeq v \wedge \forall u(vRu \to x_1 \preccurlyeq u)$. Assume x_i is already defined. Thus

$$wRx_i \preccurlyeq v \wedge \forall u(vRu \to x_i \preccurlyeq u).$$

By (3) there is node x_{i+1} such that

$$wRx_{i+1} \preccurlyeq x_i \wedge \forall u(x_iRu \to x_{i+1} \preccurlyeq u).$$

Observe that since $x_{i+1} \preccurlyeq x_i$, we also have

$$wRx_{i+1} \preccurlyeq v \wedge \forall u(vRu \to x_{i+1} \preccurlyeq u).$$

Further note that since $\forall u(x_iRu \to x_{i+1} \preccurlyeq u)$, if $*(x_i)$ does not hold, then $x_i \neq x_{i+1}$. Thus x_{i+1} has the desired properties. This completes the construction of the sequence. **QED**

8.3 Completeness

Since the completeness proofs of $\mathbf{Le}_i$ and $\mathbf{LLe}_i$ are almost the same, we just treat the last one extensively.

An adequate set X is called *Le-adequate* if it is the single closure under $\Box$ of an adequate set which is of the form $\{\bigvee Z \mid Z \subseteq X_0\}$ for some set X_0 which does not contain formulas of the form $A \vee B$. X_0 is called the *base* of X.

Proposition 15 $\vdash_{\mathbf{LLe}_i} A$ iff A valid on all finite brilliant LLe-frames.

Proof If $\nvdash_{\mathbf{LLe}_i} A$, there is a finite Le-adequate set X which contains A, and such that there is an X-saturated set Γ which does not derive A. For the base of X just take the set X_0 of all subformulas of A (and their negations) minus the disjunctive ones. We construct a frame $(W, \preccurlyeq, R, V)$ on X as follows. W is the set of all X-saturated sets, $\preccurlyeq$ and V are defined as on the $\mathbf{LLe}_i$-canonical model, and

$$\begin{array}{lll} wRv & :\Leftrightarrow & \forall Z \subseteq X_0(\Box(\bigvee Z) \in w \Rightarrow \exists Z_i \in Z(Z_i, \Box Z_i \in v)) \\ & & \text{and } \exists \Box B \notin w(\Box B \in v). \end{array}$$

First, we show that for all $B \in X$, $w \models B$ iff $B \in w$, and second that the frame in an LLe-frame.

But, first some notation. Let σ range over all functions on $\{Z \mid Z \subseteq X_0 \wedge Z \neq \emptyset\}$ for which $\sigma Z \in Z$. For any set x, x_σ denotes the set $\{\sigma Z, \Box\sigma Z \mid \Box(\bigvee Z) \in x\}$. Note that if w and v are X-saturated, wRv iff for some σ and $\Box B \notin w$, $w_\sigma \cup \{\Box B\} \subseteq v$.

For the proof of $w \models B$ iff $B \in w$, the only 'new' case is the one that $B = \Box C$, from left to right:
Assume $\Box C \notin w$. We show that if for all σ, $w_\sigma, \Box C \vdash C$, then $w \vdash \Box C$. Then we can conclude that there is a σ such that $w_\sigma, \Box C \nvdash C$, and extend this set to an X-saturated extension of w_σ which does not derive C and we are done. Let $Z_1, \ldots, Z_n$ be all the subsets Z of X_0 for which $\Box(\bigvee Z) \in w$. Assume that for all σ, $w_\sigma, \Box C \vdash C$. We show how to derive $w \vdash \Box(\Box C \to C)$, and thus $w \vdash \Box C$, a contradiction;

$$\begin{array}{rcl} \forall\sigma(w_\sigma & \vdash & \Box C \to C) \\ \forall\sigma(\boxdot\sigma Z_1, \ldots, \boxdot\sigma Z_n & \vdash & \Box C \to C) \\ \forall\sigma\forall B \in Z_1(\boxdot B, \boxdot\sigma Z_2, \ldots, \boxdot\sigma Z_n & \vdash & \Box C \to C) \\ \forall\sigma(\bigvee_{B \in Z_1} \boxdot B, \boxdot\sigma Z_2, \ldots, \boxdot\sigma Z_n & \vdash & \Box C \to C) \\ & \vdots & \\ \bigvee_{B \in Z_1} \boxdot B, \ldots, \bigvee_{B \in Z_n} \boxdot B & \vdash & \Box C \to C \\ \Box(\bigvee_{B \in Z_1} \boxdot B), \ldots, \Box(\bigvee_{B \in Z_n} \boxdot B) & \vdash & \Box(\Box C \to C). \end{array}$$

As $\vdash_{\mathbf{Le}_i} \Box(\bigvee Z) \to \Box(\bigvee_{B \in Z} \boxdot B)$, this implies that $w \vdash \Box(\Box C \to C)$.

To verify the Le-property, let wRv, $\{u_1, \ldots, u_n\} = R(v)$ and $u = v \cap u_1 \cap \ldots \cap u_n$. We have to find a node x such that wRx and $x \subseteq u$. In order to find x, we first construct w_σ, B such that

$$w_\sigma \subseteq v, \Box B \in v \cap w_{\not\Box}, w_\sigma, \Box B \nvdash E \quad \text{for } E = \bigvee\{D \in X_0 \mid u \nvdash \boxdot D\}. \quad (4)$$

Note that this implies that $w_\sigma, \Box B \subseteq u$. Next, assuming (4) to be satisfied, there is a y such that

$$w_\sigma, \Box B \subseteq y \subseteq u, \forall Z \subseteq X_0 (\text{if } y \vdash \bigvee Z \text{ then } \exists Z_i \in Z(Z_i \in y)). \tag{5}$$

Then

$$x = \{D \in X \mid y \vdash D\} \text{ is } X\text{-saturated}, x \subseteq u, wRx.$$

So it meets our conditions.

Now assuming (4) we show (5) as follows. Let $Y_1, Y_2, \ldots$ be an enumeration of all the subsets of X_0 with infinite repetition. We construct sets x_n, such that the required $y = \bigcup_n x_n$.

$$\begin{aligned} x_1 &= w_\sigma \cup \{\Box B\} \\ x_{i+1} &= \begin{cases} x_i & \text{if} \quad x_i \nvdash \bigvee Y_i \\ x_i \cup \{\bigvee Y_i, D\} & \text{if} \quad x_i \vdash \bigvee Y_i, \text{ and } D \in Y_i \text{ is such that} \\ & \phantom{\text{if}} \quad x_i \cup \{\bigvee Y_i, D\} \nvdash E \end{cases} \end{aligned}$$

Now all x_i are subsets of u. The set x_1 is a subset of u since $\Box B \in v$. Also $w_\sigma \subseteq u$; formulae in w_σ come in pairs $D, \Box D$ with $D \in X_0$. Suppose $D, \Box D \in w_\sigma$ and either of $D, \Box D \notin u$. Then $u \nvdash \boxdot D$, so $D \vdash E$, and thus $w_\sigma \vdash E$, which is not the case. So $D, \Box D \in u$. Assume we have already shown $x_i \subseteq u$. If $x_i = x_{i+1}$ it is trivial; so let $x_{i+1} = x_i \cup \{\bigvee Y_i, D\}$, and assume, argueing by contradiction, that $x_{i+1} \not\subseteq u$. Since $x_i \vdash \bigvee Y_i$, so $u \vdash \bigvee Y_i$, thus $\bigvee Y_i \in u$, this implies that $D \notin u$. Hence $u \nvdash D$; but then $D \vdash E$, and $x_{i+1} \vdash E$, a contradiction.

Now we turn to the proof of (4). We have to cases.

Case $\Box E \notin v$. If $\Box E \notin v$, then for all $\Box B \in v$, all $w_\sigma \subseteq v$ we have $w_\sigma, \Box B \nvdash E$; for otherwise we have $\Box E \in v$, since $v \vdash \Box(\bigwedge w_\sigma \wedge \Box B)$. Since wRv, there clearly is a $w_\sigma \subseteq v$ and a $\Box B \in v \cap w_{\not\Box}$, hence we are done.

Case $\Box E \in v$. We show that there is a $w_\sigma \subseteq v$ such that $w_\sigma, \Box E \nvdash E$. This would prove (4), since it is easy to see that $\Box E \notin w$. Argueing by contradiction, let us assume that for all $w_\sigma \subseteq v$ we have $w_\sigma, \Box E \nvdash E$; then we can derive the incorrect statement $w \vdash \Box E$, as follows. Again, let $Z_1, \ldots Z_n$ be all subsets $Z \subseteq X_0$ for which $\Box(\bigvee Z) \in w$. First note that

$$Z_i = \bigvee_{D \in Z_i \cap u} D \vee \bigvee_{D \in Z_i, D \notin u} D, \text{ hence } Z_i \vdash E \vee \bigvee_{D \in Z_i \cap u} D, \text{ thus}$$

$$w \vdash \Box(E \vee \bigvee_{D \in Z_i \cap u} \boxdot D).$$

Furthermore by assumption

$$\begin{array}{rcl}
\forall\sigma(\text{ if } w_\sigma \subseteq v \text{ then } w_\sigma, \Box E & \vdash & E), \text{ so} \\
\forall\sigma(\text{ if } w_\sigma \subseteq v \text{ then } \boxdot\, \sigma Z_1, \ldots, \boxdot\sigma Z_n, \Box E & \vdash & E) \\
\forall\sigma(\text{ if } w_\sigma \subseteq v \text{ then } \bigvee_{B\in Z_1\cap u} \boxdot B, \boxdot\sigma Z_2, \ldots, \boxdot\sigma Z_n, \Box E & \vdash & E) \\
& \vdots & \\
E \vee \bigvee_{D\in Z_1\cap u} \boxdot D, \ldots, E \vee \bigvee_{D\in Z_n\cap u} \boxdot D, \Box E & \vdash & E \\
\Box(E \vee \bigvee_{D\in Z_1\cap u} \boxdot D), \ldots, \Box(E \vee \bigvee_{D\in Z_n\cap u} \boxdot D) & \vdash & \Box(\Box E \to E) \\
w & \vdash & \Box(\Box E \to E) \\
w & \vdash & \Box E.
\end{array}$$

QED

Proposition 16 $\vdash_{\mathbf{Le}_i} A$ iff A is valid on all finite brilliant Le-frames.

Corollary 17 $\mathbf{Le}_i$ does not derive L or Ma.

Proposition 18 $\mathbf{Le}_i$ is canonical.

8.4 A stronger frame property for Le

It turns out that $\mathbf{Le}_i$ is complete with respect to frames which satisfy the stronger property of being *gathering*: $wRvRu \to v \preccurlyeq u$.
Note however that we cannot have completeness of $\mathbf{LLe}_i$ with respect to finite brilliant gathering frames, as in these frames $\Box\neg\neg\Box\bot$ is valid, whereas this is not derivable in $\mathbf{LLe}_i$. But we do have the following.

Proposition 19 $\vdash_{\mathbf{LLe}_i} A$ iff A is valid on all finite gathering conversely well-founded frames.

Proof Suppose $\nvdash_{\mathbf{LLe}_i} A$. Let $\mathcal{M} = (W, \preccurlyeq, R, V)$ be a finite brilliant LLe-model which does not validate A in some node b. We define a new relation $R' \subseteq R$ on W such that the the model $\mathcal{M}' = (W, \preccurlyeq, R', V)$ has a gathering frame and validates the same formulas as $\mathcal{M}$.

Intuitively we 'erase' those modal relationships R between elements which violate the gatheringness of the frame, i.e. between nodes w, v such that there is a vRu with $v \not\preccurlyeq u$. That is, we define

$$wR'v :\Leftrightarrow wRv \text{ and } \forall u(vRu \to v \preccurlyeq u).$$

To prove that $\mathcal{M}, w \models B$ iff $\mathcal{M}', w \models B$, is straightforward once one knows

$$wRv \to w(R';\preccurlyeq)v.$$

We will show this last fact. Let $*(x)$ be short for $\forall u \in W(xRu \to x \preccurlyeq u)$. Now, assume wRv. We show that there is a node v' with $wR'v' \preccurlyeq v$, that is, with $wRv' \preccurlyeq v$ and $*(v')$. The idea is as follows. By the Leivant property there is a node x_1 below v and all its successors (in $\mathcal{M}$), and such that wRx_1. If $x_1 = v$, $wR'v$ and we are done. If $x_1 \neq v$ we consider a node x_2 below x_1 and all its successors, and such that wRx_1, which again exists by the Leivant property. If

$x_2 = x_1$, we can take $v' = x_1$. If $x_2 \neq x_1$ we consider a node x_3 which is below x_2 and all its successors , and such that wRx_3, etc.

More formally, we construct a sequence of elements $v = x_1 \geqslant x_2 \geqslant \ldots$ in W such that for all i, wRx_i, and such that if $*(x_i)$ does not hold, then $(x_i \neq x_{i-1})$. As $\mathcal{M}$ is finite this implies we can find an element x_i with the desired properties. We show how to construct x_{n+1} from x_n. If $*(x_n)$ holds, put $x_{n+1} = x_n$. If $*(x_n)$ does not hold, x_{n+1} is a node which is below x_n and its successors, and moreover such that wRx_{n+1}. **QED**

9 The Formalized Markov scheme

The Formalized Markov scheme

$$\Box\neg\neg(\Box A \to \bigvee \Box B_i) \to \Box(\Box A \to \bigvee \Box B_i).$$

is a formalization of what is known as the Markov Rule for Heyting Arithmetic,

if $\mathbf{HA} \vdash \neg\neg A$ then $\mathbf{HA} \vdash A$, for any Π_2-sentence A.

Clearly any arithmetical realization of formulas of the form $(\Box A \to \bigvee \Box B_i)$ is Π_2. But arithmetical realizations of formulas $(\bigvee \Box A_i \to \bigvee \Box B_i)$ are Π_2 too. The only reason we did not call

$$\Box\neg\neg(\bigvee \Box A_i \to \bigvee \Box B_i) \to \Box(\bigvee \Box A_i \to \bigvee \Box B_i)$$

the Formalized Markov scheme is just because it is a consequence of Ma.

There is a nice consequence of Ma which states that for formulae of a certain form a stronger version of Löb's principle is derivable:

Lemma 20 For formulae D of the form $(\bigvee \Box A_i \to \bigvee \Box B_i)$,

$$\mathbf{LMa}_i \vdash \Box(\Box D \to \neg\neg D) \to \Box D.$$

Proof Let $D = (\bigvee \Box A_i \to \bigvee \Box B_i)$.

$$\begin{aligned} \mathbf{LMa}_i \vdash \Box(\Box D \to \neg\neg D) &\to \Box(\Box\neg\neg D \to \neg\neg D) \\ &\to \Box\neg\neg D \\ &\to \Box D. \end{aligned}$$

QED

That this stronger version of L is not for arbitrary D a principle of $\mathbf{HA}$ follows from the fact that for instance $\mathbf{HA}$ derives $\Box(\Box(\Box\bot \vee \neg\Box\bot) \to \neg\neg(\Box\bot \vee \neg\Box\bot))$, but not $\Box(\Box\bot \vee \neg\Box\bot)$.

9.1 Correspondence

In contrast to the principles encountered so far, the frame property Ma^∞ to which the Markov Scheme corresponds looks a bit unelegant. However, if we restrict ourselves to finite frames, Ma corresponds to a much nicer property, the so-called Markov property. It will turn out that the $\mathbf{Ma}_i$-canonical model has this property, and therefore that $\mathbf{Ma}_i$ is complete with respect to the class of these frames.

Lemma 21 Ma corresponds to the Ma^∞-property, which is

Ma^∞**-property** for all $w, v, u_1, .., u_n$: $\bigwedge w(R;\preccurlyeq)vRu_i \rightarrow$

$\exists t \in Top(w(R;\preccurlyeq)t \wedge \forall u(t(R;\preccurlyeq)u \rightarrow v(R;\preccurlyeq)u) \wedge \bigwedge t(R;\preccurlyeq)u_i)$.

Proof Only the direction from left to right. Let F be a frame which does not have the Ma^∞-property; there are $w, v, u_1, .., u_n$ with $w(R;\preccurlyeq)vRu_i$ and

$$\forall t \in Top[w(R;\preccurlyeq)t \wedge \forall u(t(R;\preccurlyeq)u \rightarrow v(R;\preccurlyeq)u) \rightarrow \bigvee_i \neg t(R;\preccurlyeq)u_i)].$$

Let $T_i = \{t \mid w(R;\preccurlyeq)t \wedge \forall u(t(R;\preccurlyeq)u \rightarrow v(R;\preccurlyeq)u \wedge \neg t(R;\preccurlyeq)u_i\}$. Define a valuation on F via $x \models p :\Leftrightarrow v(R;\preccurlyeq)x$, $x \models q_i :\Leftrightarrow \exists t \in T_i\ t(R;\preccurlyeq)x$.

We show that with this valuation, $w \models \Box\neg\neg(\Box p \rightarrow \bigvee \Box q_i)$ and $w \not\models (\Box p \rightarrow \bigvee \Box q_i)$. The last part is obvious, since $w(R;\preccurlyeq)v$ and $v \models \Box p$. That $v \not\models \Box q_i$ for all i, follows from the fact that $u_i \not\models q_i$, for all i. Therefore, consider any top node t above some successor z of w. It suffices to show that $t \models \Box p \rightarrow \bigvee \Box q_i$, because this would imply $z \models \neg\neg(\Box p \rightarrow \bigvee \Box q_i)$. Note that $w(R;\preccurlyeq)t$. If $t \models \Box p$, then $\forall u(t(R;\preccurlyeq)u \rightarrow v(R;\preccurlyeq)u$. Hence, by assumption about the frame, $\neg t(R;\preccurlyeq)u_i$, for some i. Therefore, $t \in T_i$, and thus $t \models \Box q_i$. **QED**

It is easy to see that for finite frames, having the Ma^∞-property is equivalent to having the Markov property, which is

Ma**-property** $w(R;\preccurlyeq)v \rightarrow \exists t \in Top(w(R;\preccurlyeq)t \wedge \forall u(v(R;\preccurlyeq)u \leftrightarrow t(R;\preccurlyeq)u)$.

On brilliant frames this requirement looks a bit neater:

$$wRv \rightarrow \exists t \in Top(wRt \wedge R(t) = R(v))$$

A top node t for which wRt and $R(t) = R(v)$ hold will be called a *Markov-node for the pair* (w, v).

Note that for any successor v of w in some brilliant frame, for every *top* node t above v, wRt. Hence if $R(v)$ is empty, every *top* node t above v has the properties wRt and $R(t) = R(v)$. Hence any frame with R chains $wRvR...$ of at most 2 nodes is automatically a Markov frame.

$\mathbf{Ma}_i$ cannot be complete with respect to finite frames which satisfy the stronger property $wRv \rightarrow \exists t \in T(v)(wRt \wedge R(t) = R(v))$. Namely, the principle $\Box(\neg\neg\Box A \rightarrow \Box A)$ is valid on these frames. As both $\mathbf{Ma}$ and $\Box(\neg\neg\Box A \rightarrow \Box A) \rightarrow \Box\Box A$ are principles of $\mathbf{HA}$, so would $\Box\Box A$ be.

9.2 Completeness

We will use the following lemma in the proof of the canonicity of $\mathbf{Ma}_i$ below.

Lemma 22 If $\Delta = \{D \mid \Gamma \vdash \Box D\}$, for some set Γ, then if the set of formulas $\Delta, \Box A_1, .., \Box A_n, \neg\Box B_1, .., \neg\Box B_m$ is inconsistent, Δ derives $(\bigwedge \Box A_i \to \bigvee \Box B_i)$.

Proof

$$\begin{array}{rcl}
\Delta, \Box A_1, .., \Box A_n, \neg\Box B_1, .., \neg\Box B_m & \vdash & \bot \\
\Delta, \Box \bigwedge A_i, \neg\Box B_1 \ldots, \neg\Box B_m & \vdash & \bot \\
\Delta & \vdash & \Box \bigwedge A_i \to \neg\neg \bigvee \Box B_i \\
\Delta & \vdash & \neg\neg(\Box \bigwedge A_i \to \bigvee \Box B_i) \\
\Gamma & \vdash & \Box\neg\neg(\Box \bigwedge A_i \to \bigvee \Box B_i) \\
\Gamma & \vdash & \Box(\Box \bigwedge A_i \to \bigvee \Box B_i) \\
\Delta & \vdash & \Box \bigwedge A_i \to \bigvee \Box B_i \\
\Delta & \vdash & \bigwedge \Box A_i \to \bigvee \Box B_i.
\end{array}$$

QED

Proposition 23 The $\mathbf{Ma}_i$-canonical frame has the Ma-property.

Proof: Let $(W, \preccurlyeq, R, V)$ be the $\mathbf{Ma}_i$-canonical model. Let w, v be two nodes such that wRv. We show that there is a *top* node t such that wRt and $R(t) = R(v)$. Let $\Delta = \{D \mid \Box D \in w\}$. It suffices to show that the set $\Delta, v_\Box, \{\neg\Box E \mid \Box E \notin v\}$ is consistent, as any maximal consistent extension of this set will have the desired properties. If it is not consistent, there are $\Box E_1, \Box E_2, \ldots, \Box E_n \notin v$ and $\Box B_1, \ldots, \Box B_m \in v$ such that $\Delta, \Box B_1, \ldots, \Box B_m, \neg\Box E_1 \ldots, \neg\Box E_n$ is inconsistent. But then lemma 22 implies that $\Delta \vdash \bigwedge \Box B_i \to \bigvee \Box E_i$. Hence $(\Box E_1 \vee \ldots \vee \Box E_n)$ is in v, and that cannot be. **QED**

Corollary 24 $\mathbf{Ma}_i$ does not derive $K4, L, Le$.

10 All principles together

Now we are going to prove the finite model property for the theory $\mathbf{B}_i$, by showing that $\mathbf{B}_i$ is complete with respect to the class of finite brilliant B-frames (frames which are both LLe- and Ma-frames). A problem arising when combining the Leivant principle and the formalized Markov scheme is that we proved the finite model property for the first one via restricting the domain to X-saturated sets, and for the latter via the construction method on the canonical model. We will show that $\mathbf{B}_i$ has the finite model property via the construction method.

10.1 Notation and Explication

We cannot avoid a lot of notation in this completeness proof for $\mathbf{B}_i$. Let us start with the introduction of this now. Many notions will be relative to an adequate set X which we will often omit in the notation, in order not to have an overdose of notation swarming around. Of course, when an adequate set X is fixed (as is done in the proof below) we will tacitly take all definitions relative to this set X.

First of all, we will use a lot of models at the same time. We will not give all the relations of these models different names. But instead we say $w \preccurlyeq v$ *(wRv) in K*, if v is above (a successor of) w in the model $\mathcal{K}$. Or we write 'we work in $\mathcal{K}$', which means that in this context $\preccurlyeq$ and R refer to the intuitionistic and modal relation of $\mathcal{K}$ respectively.

Let $\mathcal{M}$ be the $\mathbf{B}_i$-canonical model and let X be a B-adequate set. In the completenss proof for $\mathbf{B}_i$ we will construct via the construction method, a set of sequences $W = \{\sigma \mid \sigma \text{ is defined }\}$ by choosing certain nodes α_σ in the $\mathbf{B}_i$-canonical model $\mathcal{M}$.

We will choose labelled and unlabelled sequences, the elements of which are formulas $\Box B$, $B \to C$, and pairs (l, τ), (m, τ), where τ is a sequence. In the resulting model, $\sigma * \langle (l, \tau) \rangle$ and $\sigma * \langle (m, \tau) \rangle$ will be a Leivant- and a Markov-node for (σ, τ) respectively. Furthermore we will use a helpfunction h which is a partial function from sequences to sequences. The use of h will get clear in the part of the completeness proof were we prove the finiteness of the constructed frame.

10.1.1 The relations $\subseteq_\to$ and $\subseteq_{\not\to}$ on sequences

For sequences σ, τ, we write $\sigma \subseteq \tau$ when $\tau = \sigma * \sigma'$ for some possibly empty sequence σ'. We write $\sigma \subseteq_\to \tau$ if σ' consists of implications only or is empty, and $\sigma \subseteq_{\not\to} \tau$ otherwise.

10.1.2 The relations $\tilde{\preccurlyeq}$ and $\tilde{R}$ of the model $\tilde{W}$

For a given set of sequences W, the model $\tilde{W}$ is the model $(W, \tilde{\preccurlyeq}_{\restriction W}, \tilde{R}_{\restriction W}, V)$, where V is defined via

$$\tilde{W}, \sigma \models p :\Leftrightarrow \mathcal{M}, \alpha_\sigma \models p,\ p \text{ a propositional variable in } X.$$

The α_σ will be suitably chosen among the nodes of $\mathcal{M}$ as we will see later on. And $\tilde{\preccurlyeq}_{\restriction W}, \tilde{R}_{\restriction W}$ respectively are the restrictions of $\tilde{\preccurlyeq}$ and $\tilde{R}$ to W, where $\tilde{\preccurlyeq}$ and $\tilde{R}$ are defined as follows. An *lm-chain* is a sequence $(x_1, .., x_n)$ of sequences, such that for all i, one of (a)-(e) is the case:

(a) $i = n$,

(b) $x_i = y * \langle (m, x_{i+1}) \rangle$, or $x_i = y * \langle (l, x_{i+1}) \rangle$, for some y,

(c) $x_i \subseteq x_{i+1}$ and $x_i \neq x_{i+1}$,

(d) $h(x_i) = x_{i+1}$,

(e) $h(x_{i+1}) = x_{i+2}$ and $x_{i+1} = y * \langle (m, x_i) \rangle$ for some y.

$$x\tilde{R}y \quad :\Leftrightarrow \quad \text{there is an lm-chain } (x = x_1, .., x_n) \text{ and } x_n \subseteq_{\not\to} y$$
$$x\tilde{\preccurlyeq}y \quad :\Leftrightarrow \quad x \subseteq_{\to} y, \text{ or } (x = x' * \langle (l, z) \rangle \text{ and } (z \subseteq_{\to} y \text{ or } z\tilde{R}y \text{ or } x\tilde{R}y)).$$

Remark 25 Note that if $x \subseteq_{\not\to} y$ then $x\tilde{R}y$. And if $x'\tilde{R}y$ and for x there is an lm-chain $(x = x_1, .., x_n)$ such that $x_n \subseteq x'$, then $x\tilde{R}y$.

10.2 Three procedures

We shall define three ways of choosing some set of new nodes, given some set of sequences (nodes). In the completeness proof for $\mathbf{B}_i$ we will apply these three procedures one after another to construct the desired finite B-model, and proof they are correct The procedures intuitively work as follows.

Given a set W of sequences, the first procedure (**I**) extends it to a set W' in such a way that $\tilde{W'}$ has the Markov property. The second procedure (**II**) extends a set W to a set $\tilde{W'}$ in such a way that it has the Leivant property. And the third procedure (**III**) extends W in such a way to a set $\tilde{W'}$ that $\tilde{W'}$ is a model which satisfies

$$\forall B \in X, \forall \sigma \in W' (\tilde{W'}, \sigma \models B \text{ iff } \mathcal{M}, \alpha_\sigma \models B).$$

10.2.1 The models $\mathcal{M}$ and $\mathcal{N}$

Let $\mathcal{M}$ be the $\mathbf{B}_i$-canonical model. Let $\mathcal{N}$ be the model constructed in the proof for $\mathbf{LLe}_i$, proposition 15, with adequate set X. For any node w in the canonical model $\mathcal{M}$, that is, for any saturated set w, let $[w] = w \cap X$. Note that $[w]$ is a node in $\mathcal{N}$. We sometimes say $[w]R[v]$ instead of '$[w]R[v]$ in $\mathcal{N}$', and similar for $\preccurlyeq$, because in this case it is clear from to context to which model we refer.

Remark 26 Note that if for two nodes w, v in the canonical model $\mathcal{M}$ we have wRv in $\mathcal{M}$ and $w_\Box \subset v_\Box$, i.e. there is a $\Box B \in v \cap X$ such that $\Box B \notin w$, then $[w]R[v]$ in $\mathcal{N}$ by the definition of $\mathcal{N}$ in proposition 15. On the other hand, if $[w]R[v]$ in $\mathcal{N}$, then $w_\Box \subset v_\Box$. It also follows from the definition of $\mathcal{N}$ that $[w] \preccurlyeq [v]$ in $\mathcal{N}$ iff $w \cap X \subseteq v \cap X$.

10.2.2 Relevant pairs of sequences

Call a pair (x, y) *relevant-Leivant [relevant-Markov] in* $\tilde{W}$ if $y = x * \langle D \rangle$, where D is not an implication and not of the form $\langle l, z \rangle$ [$y = x * \langle D \rangle * y'$, where $x * \langle D \rangle \subseteq_{\to} y$, and where D is not an implication and not of the form $\langle m, z \rangle$], and x is not of the form $\sigma * \langle (m, \tau) \rangle$, and (x, y) does not have a Leivant-node [Markov-node] in $\tilde{W}$.

Intuitively, the relevant Leivant (Markov) pairs are the only pairs which we have to give a Leivant (Markov)-node, in order to give all pairs of nodes in W a Leivant (Markov)-node in the resulting model.

10.2.3 Procedure I

Start with a set of sequences V_0. Consider in step n all the relevant Markov pairs (σ, τ) of V_n, and choose Markov-nodes $\alpha_{\sigma * \langle (m,\tau) \rangle}$ for the corresponding pairs $(\alpha_\sigma, \alpha_\tau)$ in the $\mathbf{B}_i$-canonical canonical model $\mathcal{M}$. Let V_{n+1} be the union of V_n and these newly defined sequences, and go to the next step.

Of course, the sequences in V_0 have to be such that these Markov-nodes exist. But we shall not bother about this here. In the cases in which we this procedure is used, the set V_0 with which we start will have the desired properties, see *Case (i)* in the completeness proof, proposition 29.

10.2.4 Procedure II

Start with a set V_0. Choose for every relevant Leivant pair (σ, τ) of $\tilde{V_0}$, a node $\alpha_{\sigma * \langle (l,\tau) \rangle} = x$, such that $\alpha_\sigma R x$ in $\mathcal{M}$, and $[x]$ is a minimal Leivant-node for $([\alpha_\sigma], [\alpha_\tau])$ in $\mathcal{N}$. For the definition of a minimal Leivant-node and for the definition of $\mathcal{N}$, respectively see sections 8.2 and 10.2.1. If $\tau = \sigma * \langle (m, \tau') \rangle$, and $x_\Box = (\alpha_\tau)_\Box$, then put $h(\tau) = \sigma * \langle (l, \tau) \rangle$. Let V be the union of V_0 and these new sequences.

Remark 27 Procedure **II** will be the only procedure in which we assign an h-image to some newly chosen sequences. Note that this implies that $h(\tau)$ only exists for some τ which are of the form $\sigma * \langle (m, \tau') \rangle$ and for which (σ, τ) has been once a relevant Leivant pair. Note furthermore, that once a sequence τ has been part of a relevant Leivant pair (σ, τ), after procedure **II** is performed it will never for any sequence σ', be part of a relevant Leivant pair (σ', τ) anymore. This shows that h is indeed a (partial) function.

Remark 28 We cannot guarantee that for any Leivant-node x for (w, v) in $\mathcal{M}$, $w_\Box \subset x_\Box$. Therefore, in contrast to procedure **I**, in procedure **II** we cannot just choose $\alpha_{\sigma * \langle (l,\tau) \rangle}$ to be a Leivant-node for $(\alpha_\sigma, \alpha_\tau)$ in $\mathcal{M}$. For if we would do this, it could be that $(\alpha_\sigma)_\Box = (\alpha_{\sigma * \langle (l,\tau) \rangle})_\Box$, and we would have no guarantee that the process stops in a finite number of steps, see remark 3. Note that the way in which we choose $\alpha_{\sigma * \langle (l,\tau) \rangle}$ in procedure **II** implies that $(\alpha_\sigma)_\Box \subset (\alpha_{\sigma * \langle (l,\tau) \rangle})_\Box$, by remark 26.

10.2.5 Procedure III

Start with a set V_0. Step n consists of the following. If $\sigma \in V_n$, and there is a $\Box B \in X$, $\Box B \notin \alpha_\sigma$, for which there is no $x \in V_n$ such that $\sigma \tilde{R} x$ and $B \notin \alpha_x$, then choose a node $\alpha_{\sigma * \langle \Box B \rangle}$ which is a node w such that $\alpha_\sigma R w$ in $\mathcal{M}$, w does not contain B, and $[w]$ is a minimal Leivant-node for $([\alpha_\sigma], [w])$ in $\mathcal{N}$. For the definition of a minimal Leivant-node and for the definition of $\mathcal{N}$, respectively see sections 8.2 and 10.2.1. For $B \to C \in (\alpha_\sigma)_{\not\to}$, if there is no $x \in V_n$ such that $\sigma \tilde{\preccurlyeq} x$ and $B \in x$ and $C \notin x$, choose a node $\sigma * \langle B \to C \rangle$ as is usual in the construction method. Let V_{n+1} be the union of V_n with the newly defined sequences, and go to the next step.

10.3 Completeness

Proposition 29 $\vdash_{\mathbf{B}_i} A$ iff A is valid on all finite brilliant B-frames.

Proof Assume $\nvdash_{\mathbf{B}_i} A$. Let b be a node in the $\mathbf{B}_i$-canonical model $\mathcal{M}$ which does not force A. Let X be a finite Le-adequate set which contains A. We construct in a similar way as in the construction method, a finite model of the form $\tilde{W}$, see section 10.1.2. We construct the domain W of $\tilde{W}$ stepwise. At step n we choose nodes $\alpha_\sigma \in \mathcal{M}$. We define $W_n = \{\sigma \mid \sigma \text{ is defined in a step } \preccurlyeq n\}$. Then $W = \bigcup_n W_n$.

Step 0. Let W_0 be the result of procedure **III** (section 10.2.5), starting with $V_0 = \{b\}$.
Step $3n+1$. Let W_{3n+1} be the result of procedure **I** (section 10.2.3), starting with set $V_0 = W_{3n}$.
Step $3n+2$. Let W_{3n+2} be the result of procedure **II** (section 10.2.4), starting with set $V_0 = W_{3n+1}$.
Step $3n+3$. The set W_{3n+3} is the result of procedure **III**, starting with $V_0 = W_{3n+2}$.

Let W be the union of the W_n and consider $\tilde{W}$. From now on, unless stated otherwise, $\tilde{\preccurlyeq}$ and $\tilde{R}$ will be short for $\tilde{\preccurlyeq}_{\restriction W}$ and $\tilde{R}_{\restriction W}$. Now it remains to show the following claims.

(i) The construction is correct, i.e. the nodes we want to choose in the consecutive steps do exist.

(ii) $\tilde{W}$ is a model, i.e. $(\tilde{\preccurlyeq};\tilde{R}) \subseteq \tilde{R}$ and for all propositional variables $p \in X$, $\forall \sigma, \tau \in W$ (if $\sigma \tilde{\preccurlyeq} \tau$ and $\sigma \models p$ then $\tau \models p$).

(iii) $\tilde{W}$ has the Markov property.

(iv) $\tilde{W}$ has the Leivant property.

(v) $\forall B \in X \forall \sigma \in W (\tilde{W}, \sigma \models B \text{ iff } \mathcal{M}, \alpha_\sigma \models B)$.

(vi) The process stops in a finite number of steps, i.e. for some n, $\forall m \geqslant n$, $W_m = W_n$. Since clearly every W_m is finite, this shows that W is finite.

We will omit the tedious but straightforward proofs of *(ii)* and *(v)*, and only prove *(i)*, *(iii)*, *(iv)* and *(vi)*. Unless stated otherwise, $\sigma, \tau, x, y, z, a, b$ range over sequences, B, C, D over elements of sequences. We remind the reader that $w_\Box$ denotes $w_\Box^X$.

Claim (i). We show that the three procedures are correct. We will need the following lemmas and remark.

Lemma 30 If $\sigma \subseteq_\rightarrow \tau$ and $\sigma \neq \tau$, then $\alpha_\sigma < \alpha_\tau$ in $\mathcal{M}$.
If $\sigma \subseteq_\rightarrow \tau$ and $\sigma \neq \tau$, then $\alpha_\sigma \subset \alpha_\tau$.
If $\sigma \subseteq_\rightarrow \tau$ and $\sigma \neq \tau$, then $[\alpha_\sigma] < [\alpha_\tau]$ in $\mathcal{N}$.

Proof By examining procedure **III**, section 10.2.5, it is easy to conclude the first part. Since by definition, $\preccurlyeq = \subseteq$ on canonical models, the second part follows immediately from the first one. From the second part and remark 26 the third part follows. **QED**

Remark 31 For any two nodes w, v in the $\mathbf{B}_i$-canonical model $\mathcal{M}$ the following holds,

$$R(w) = R(v) \text{ iff } w_\Box = v_\Box.$$

Hence if t is a Markov-node for (w, v), $t_\Box = v_\Box$.

Lemma 32 For all n, for all pairs (σ, τ) in W_n which are either a relevant Leivant-pair or a relevant Markov-pair,

$$\alpha_\sigma R \alpha_\tau \text{ in } \mathcal{M} \text{ and } [\alpha_\sigma] R [\alpha_\tau] \text{ in } \mathcal{N}.$$

Proof Let (σ, τ) be any relevant pair, relevant Leivant or relevant Markov, in some W_n. By the definition of relevant pairs, section 10.2.2, $\tau = \sigma * \langle D \rangle * \sigma'$, for some D which is no implication, and σ' consists of implications only. Hence D is of the form $\Box B$, $\langle l, \tau' \rangle$ or $\langle m, \tau' \rangle$. We show that in all these cases,

$$\alpha_\sigma R \alpha_{\sigma * \langle D \rangle} \text{ in } \mathcal{M} \text{ and } [\alpha_\sigma] R [\alpha_{\sigma * \langle D \rangle}] \text{ in } \mathcal{N}. \tag{6}$$

Let us first see why we are done then. Since $\sigma * \langle D \rangle \subseteq_\rightarrow \sigma * \langle D \rangle * \sigma' = \tau$, it follows from lemma 30 that $\alpha_{\sigma * \langle D \rangle} \preccurlyeq \alpha_\tau$ in $\mathcal{M}$ and $[\alpha_{\sigma * \langle D \rangle}] \preccurlyeq [\alpha_\tau]$ in $\mathcal{N}$. As both $\mathcal{M}$ and $\mathcal{N}$ are brilliant models, this gives the desired result;

$$\alpha_\sigma R \alpha_\tau \text{ in } \mathcal{M} \text{ and } [\alpha_\sigma] R [\alpha_\tau] \text{ in } \mathcal{N}.$$

Therefore, all we have to show is that for the three possibilities of D, (6) holds.

Case $z = \sigma * \langle \Box B \rangle$. By examining procedure **III**, section 10.2.5, we see that $\alpha_\sigma R \alpha_z$ in $\mathcal{M}$. Moreover, since $[\alpha_z]$ is a minimal-Leivant node for $([\alpha_\sigma], [\alpha_z])$, it follows that $[\alpha_\sigma] R [\alpha_z]$ in $\mathcal{N}$, and we are done.

Case $z = \sigma * \langle (l, \tau') \rangle$. By examining procedure **II**, section 10.2.4, we see that $\alpha_\sigma R \alpha_z$ in $\mathcal{M}$. Moreover, since $[\alpha_z]$ is a minimal-Leivant node for $([\alpha_\sigma], [\alpha_{\tau'}])$, it follows that $[\alpha_\sigma] R [\alpha_z]$ in $\mathcal{N}$.

Case $z = \sigma * \langle (m, \tau') \rangle$. By examining procedure **I**, section 10.2.3, we see that α_z is a Markov-node for $(\alpha_\sigma, \alpha_{\tau'})$ in $\mathcal{M}$. Hence by the definition of Markov-nodes, section 9.1, $\alpha_\sigma R \alpha_z$ in $\mathcal{M}$. Thus we only have to show $[\alpha_\sigma] R [\alpha_z]$. Since $\alpha_\sigma R \alpha_z$, by remark 26 it suffices to show that

$$(\alpha_\sigma)_\Box \subset (\alpha_z)_\Box. \tag{7}$$

Note that the existence of $\alpha_{\sigma * \langle (m, \tau') \rangle}$ implies that (σ, τ') must have been a relevant Markov-pair in some W_m. Thus, by the definition in section 10.2.2 of a relevant Markov-pair, $\tau' = \sigma * \langle D' \rangle * \sigma''$, where D' is no implication and not of the form $\langle m, \tau'' \rangle$, and σ'' consists of implications only. Hence D' is of the form

$\Box B$ or $\langle l, \tau'' \rangle$. But for these two cases we just proved that $[\alpha_\sigma]R[\alpha_{\sigma * \langle D' \rangle}]$. Now it follows from remark 26 that

$$(\alpha_\sigma)_\Box \subset (\alpha_{\sigma * \langle D' \rangle})_\Box$$

Again, by lemma 30, it follows that $\alpha_{\sigma * \langle D' \rangle} \subseteq \alpha_{\sigma * \langle D' \rangle * \sigma''} = \alpha_{\tau'}$ in $\mathcal{M}$. Thus

$$(\alpha_\sigma)_\Box \subset (\alpha_{\tau'})_\Box. \tag{8}$$

Again, since α_z is a Markov-node for $(\alpha_\sigma, \alpha_{\tau'})$ in $\mathcal{M}$, we certainly have $R(\alpha_z) = R(\alpha_{\tau'})$ in $\mathcal{M}$. By remark 31 this implies that $(\alpha_z)_\Box = (\alpha_{\tau'})_\Box$. From this and (8), (7) follows. **QED**

Now we are ready to show that procedure **I** is correct. We have to show that for all n, for any relevant Markov-pair (σ, τ) in W_n, there is a node t in the canonical model $\mathcal{M}$ which is a Markov-node for $(\alpha_\sigma, \alpha_\tau)$ in $\mathcal{M}$. Since from lemma 32 we know that for any relevant Markov-pair $\alpha_\sigma R \alpha_\tau$ holds in $\mathcal{M}$, it suffices to show that $\mathcal{M}$ has the Markov property, i.e.

in $\mathcal{M}$: if wRv then $\exists t \in Top(wRt \wedge R(t) = R(v))$.

The proof that $\mathcal{M}$ has the Markov-property is completely analogous to the proof that the $\mathbf{Ma}_i$-canonical model has the Markov property, proposition 23.

To see that procedure **II** is correct, we have to show that for all n, for any relevant Leivant-pair (σ, τ) in W_n, there is a node $x \in \mathcal{M}$ such that $\alpha_\sigma R x$ in $\mathcal{M}$ and such that $[x]$ is a minimal Leivant-node for $([\alpha_\sigma], [\alpha_\tau])$ in $\mathcal{N}$. From lemma 32 we know that for any relevant Leivant-pair $\alpha_\sigma R \alpha_\tau$ holds in $\mathcal{M}$, and $[\alpha_\sigma]R[\alpha_\tau]$ holds in $\mathcal{N}$. Therefore, it suffices to show the following lemma.

Lemma 33 For any nodes w, v, for which wRv in the canonical model, and also $[w]R[v]$ in $\mathcal{N}$, there exists a node x such that wRx in $\mathcal{M}$, and $[x]$ is a minimal Leivant-node for $([w], [v])$ in $\mathcal{N}$. Minimal Leivant nodes and $\mathcal{N}$ are introduced in sections 8.2 and 10.2.1 respectively.

The proof that there is a node x such that wRx and $[x]$ is a Leivant-node for $([w], [v])$, for w, v as in the lemma, is almost the same as the part of the completeness proof for $\mathbf{LLe}_i$, proposition 15, in which it is shown that the frame has the Leivant property. Instead of sets w_σ consider sets $\{D \mid \Box D \in w\} \cup w_\sigma$. To conclude from this that x can be chosen in such a way that $[x]$ is in fact a minimal Leivant-node for $([w], [v])$, is similar to the proof of proposition 14. **QED**

To see that procedure **III** is correct, consider any W_n. Let $\sigma \in W_n$ for which $\Box B \notin \alpha_\sigma$. We show that there is node w in the canonical model $\mathcal{M}$ such that $\Box B \in w$, $B \notin w$ and $\alpha_\sigma R w$ in $\mathcal{M}$, and moreover such that $[w]$ is a minimal Leivant-node for $([\alpha_\sigma], [w])$. This will prove that procedure **III** is correct. In the canonical model there is a node v such that $\alpha_\sigma R v$, $\Box B \in v$, and $B \notin v$.

Hence $[\alpha_\sigma]R[v]$ by remark 26. If $[v]$ is a minimal Leivant-node for $([\alpha_\sigma],[v])$, let w be this node v. If not, by lemma 33 there is a node u such that $\alpha_\sigma Ru$ and $[u]$ is a minimal Leivant-node for $([\alpha_\sigma],[v])$. Since $[u] \preccurlyeq [v]$ in $\mathcal{N}$, it follows that $u \cap X \subseteq v \cap X$ by remark 26. Since $B \in X$ and $B \notin v$, $B \notin u$. So in this case we can choose u for w.

Claim (iii). We show that $\tilde{W}$ has the Markov property, i.e.

$$x\tilde{R}y \to \exists z \in Top(x\tilde{R}z \wedge \tilde{R}(y) = \tilde{R}(z)).$$

First we need some lemmas and a remark.

Remark 34 By examining the steps in which W is constructed it is easy to see that if $x * \langle (m,y)\rangle \in W$, then (x,y) must have been a relevant Markov-pair in some W_m. By the definition of a relevant Markov-pair, section 10.2.2, this implies that x and y are not of the form $\sigma * \langle (m,\tau)\rangle$. In a similar way one can see that if $x * \langle (l,y)\rangle \in W$, then x is not of the form $\sigma * \langle (m,\tau)\rangle$ and y is not of the form $\sigma * \langle (l,\tau)\rangle$.

Lemma 35 The relation $\tilde{R}$ is transitive and $x\tilde{R}y$ implies $x \neq y$. Moreover $(\tilde{\preccurlyeq};\tilde{R};\tilde{\preccurlyeq}) = \tilde{R}$. The relation $\tilde{\preccurlyeq}$ is a partial order.

Lemma 36 For all n, $x * \langle (m,y)\rangle \in \tilde{W}_n$ is a Markov-node for (x,y) in $\tilde{W}_n$. If $z = x * \langle (m,z')\rangle$ and $h(z) = y$, then z is a Markov-node for (x,y) in $\tilde{W}_n$.

Proof We work in $\tilde{W}_n$. It is convenient to treat the last part first. Therefore, consider $z \in W_n$ such that $h(z) = y$ and $z = x * \langle (m,\tau)\rangle$ for some τ. We have to show that

$$z \text{ is a top node in } \tilde{W}_n \text{ and } x\tilde{R}z \text{ and } \tilde{R}(y) = \tilde{R}(z).$$

First note that the fact that $h(z) = y$, gives $y = x * \langle (l,z)\rangle \in \tilde{W}_n$, see remark 27.

To see that z is a top node in $\tilde{W}_n$, observe that since α_z is a Markov-node in $\mathcal{M}$, it is a top node in $\mathcal{M}$. Therefore, it follows from procedure **III** that sequences of the form $z * \langle B \to C\rangle$ will never be defined. Now from the definition of $\tilde{\preccurlyeq}$, section 10.1.2, it follows that z is a top node in $\tilde{W}$.

From the definition of $\tilde{R}$, it follows immediately that $x\tilde{R}z$.

Thus it remains to show that $\tilde{R}(z) = \tilde{R}(y)$. Since by definition 10.1.2 of $\tilde{\preccurlyeq}$, we have $y\tilde{\preccurlyeq}z$, it follows that $\tilde{R}(z) \subseteq \tilde{R}(y)$ by lemma 35. To see that $\tilde{R}(y) \subseteq \tilde{R}(z)$, assume $y\tilde{R}a$. Hence there is an lm-chain $(y = y_1, .., y_m)$ such that $y_m \subseteq_{\not\to} a$. As clearly $(z, y_1, .., y_m)$ is an lm-chain too, $z\tilde{R}a$.

Now consider $z = x * \langle (m,y)\rangle \in \tilde{W}_n$. Again, we have to show that

$$z \text{ is a top node in } \tilde{W} \text{ and } x\tilde{R}z \text{ and } \tilde{R}(y) = \tilde{R}(z).$$

To see that z is a top node in $\tilde{W}$ and that $x\tilde{R}z$ is analogous to the case above. Therefore, it remains to show that $\tilde{R}(y) = \tilde{R}(z)$.

$\tilde{R}(y) \subseteq \tilde{R}(z)$: Assume $y\tilde{R}a$. Hence there is an lm-chain $(y = y_1, .., y_m)$ such that $y_m \subseteq_{\not\to} a$. As clearly $(z, y_1, .., y_m)$ is an lm-chain too, $z\tilde{R}a$.
$\tilde{R}(z) \subseteq \tilde{R}(y)$: Assume $z\tilde{R}a$. Let $(z = z_1, .., z_m)$ be an lm-chain such that $z_m \subseteq_{\not\to} a$. We have to show that $y\tilde{R}a$. By the definition of an lm-chain, section 10.1.2, we have for $z = z_1$ either

(a) $z = z_m$, hence $z \subseteq_{\not\to} a$,

(b) $z = z' * \langle(m, z_2)\rangle$, or $z = z' * \langle(l, z_2)\rangle$, for some z',

(c) $z \subseteq z_2$ and $z \neq z_2$,

(d) $h(z) = z_2$,

(e) $h(z_2) = z_3$ and $z_2 = z' * \langle(m, z)\rangle$.

We show that the only cases that can occur are (b) or (d). Now observe that in case (b) $z_2 = y$, thus $(y = z_2, .., z_n)$ is an lm-chain too. And in case (d) $(y, z_1, .., z_m)$ is an lm-chain. Hence in both the cases (b) and (d) $y\tilde{R}a$ follows. Therefore, we are done if we can show that only the cases (b) or (d) can occur.

Case (e) cannot occur by remark 34. So, case (a) and (c) remain. In both these case there is a D such that $z * \langle D\rangle \in W$. But this contradicts the following lemma, lemma 37. This completes the proof. **QED**

Lemma 37 For any $z = x * \langle(m, y)\rangle \in W$, for any D, there is no element $z * \langle D\rangle \in W$.

Proof Consider a sequence $z = x * \langle(m, y)\rangle \in W$. Argueing by contradiction, assume $z * \langle D\rangle \in W$, for some D. There are of course four possibilities for D: (a) $D = B \to C$, (b) $D = \Box B$, (c) $D = (l, \tau)$, (d) $D = (m, \tau)$. We show that none of these cases can occur. First note, by examining procedure **I**, section 10.2.3, that α_z is a Markov-node for (α_x, α_y) in $\mathcal{M}$. Thus

$$R(\alpha_z) = R(\alpha_y) \text{ and } \alpha_z \text{ is a top node in } \mathcal{M}.$$

Hence by remark 31

$$(\alpha_z)_\Box = (\alpha_y)_\Box \text{ and } \alpha_z \text{ is a top node in } \mathcal{M}. \tag{9}$$

For (a), observe that in this case, $z \subseteq_{\to} z * \langle D\rangle$. Hence by lemma 30, $\alpha_z < \alpha_{z*\langle D\rangle}$ in $\mathcal{M}$, which contradicts the fact that α_z is a top node in $\mathcal{M}$.

Case (c) and (d) cannot occur by remark 34.

Thus case (b) remains. By examinig procedure **III** we see that $\Box B \notin (\alpha_z)_\Box$. Hence by (9) also $\Box B \notin (\alpha_y)_\Box$. We show that this implies that $z * \langle\Box B\rangle$ cannot be defined. Assume y is defined in step m and z in step n. We show that

$$\exists y' \in W_{n-1}(z\tilde{R}y' \wedge B \notin y'). \tag{10}$$

By examining procedure **III** one easily conclude that this implies that a sequence $z * \langle \Box B \rangle$ will never be defined. Hence we have established that case (b) cannot occur either.

First, observe that by lemma 36, $\tilde{R}(y) \subseteq \tilde{R}(z)$ (note that for the part of the proof of lemma 36 where $\tilde{R}(y) \subseteq \tilde{R}(z)$ is established, we do not need this lemma, so there is no circle argument here). Therefore, to prove (10) it suffices to show that

$$\exists y' \in W_{n-1}(y\tilde{R}y' \wedge B \notin y'). \tag{11}$$

Clearly $m \leqslant n$, because $z = x * \langle (m, y) \rangle$. Note furthermore that $n = 3k+1$, for some k. Observe that by remark 34, y is not of the form $\sigma * \langle (m, \tau) \rangle$. Thus $m = 0$ or $m = 3k+2$ or $m = 3k+3$, for some k. Hence if $m = 0$ or $m = 3k+3$, $m < n$. And if $m = 3k+2$, $m+1 < n$. Therefore, we can prove (11), by showing

$$(m = 0 \vee m = 3k+3) \to \exists y' \in W_m(y\tilde{R}y' \wedge B \notin y'), \tag{12}$$

$$m = 3k+2 \to \exists y' \in W_{m+1}(y\tilde{R}y' \wedge B \notin y'). \tag{13}$$

We only show (13). One can prove (12) in a similar way. Assume $m = 3k+2$ for some k. Note that since $y \in W_m = W_{3k+2}$, W_{m+1} is the result of procedure **III** starting with set $V_0 = W_m$. By the definition of procedure **III**, section 10.2.5, (13) follows immediately. **QED**

Now we are ready to show that every pair $x\tilde{R}y$ in $\tilde{W}$ has a Markov-node. Consider such a pair $x\tilde{R}y$. Let $(x_1, .., x_n)$ be an lm-chain, $x = x_n \subseteq_{\not\to} y$. There are x', y', D such that

$$x_n \subseteq x' \subseteq_{\not\to} x' * \langle D \rangle = y' \subseteq_{\to} y.$$

The pair (x', y) cannot be a relevant Markov pair in $\tilde{W}$, otherwise W would not be the union of all W_n. By the definition of a relevant Markov-pair this implies that either (a) D is an implication, (b) $D = (m, \tau)$, for some τ, (c) $x' = \sigma * \langle (m, \tau) \rangle$, for some σ, τ, (d) the pair (x', y) has a Markov-node in $\tilde{W}$. Since $x' \subseteq_{\not\to} x' * \langle D \rangle$, D cannot be an implication. Thus case (a) cannot occur. Since $x' * \langle D \rangle \in W$, by lemma 37, case (c) cannot occur either. Therefore, case (b) and (d) remain. First, we show that in both these cases the pair (x', y) has a Markov-node, i.e. that

$$\exists z \in W(x'\tilde{R}z \text{ and } \tilde{R}(z) = \tilde{R}(y) \text{ and } z \text{ is a top node in } \tilde{W}). \tag{14}$$

And then we show that

$$\forall z \in W(\text{if } z \text{ is a Markov-node for } (x', y) \text{ it is one for } (x, y) \text{ too}). \tag{15}$$

This will complete the proof.

In showing (14), we may restrict attention to case (b), as it follows trivially for case (d). It suffices to show that y is a Markov-node for (x', y), i.e. that

$$x'\tilde{R}y \text{ and } \tilde{R}(y) = \tilde{R}(y) \text{ and } y \text{ is a top node in } \tilde{W}. \tag{16}$$

Since $x' \subseteq_{\not\to} y$, we have $x'\tilde{R}y$ by remark 25. By lemma 36, y' is a Markov-node for (x', τ) in $\tilde{W}$, and thus, by the definition of a Markov-node, a top node in $\tilde{W}$. Since, by lemma 37,we have $y = y'$, we have shown (16) and hence (14).

To show (15), consider a Markov-node z for (x', y), i.e.

$$z \text{ is a top node in } \tilde{W},\ x'\tilde{R}z \text{ and } \tilde{R}(z) = \tilde{R}(y).$$

We have to show that z is a Markov-node for (x, y), i.e.

$$z \text{ is a top node in } \tilde{W},\ x\tilde{R}z \text{ and } \tilde{R}(z) = \tilde{R}(y).$$

Thus we only have to infer $x\tilde{R}z$. But this follows immediately from remark 25 and the fact that $(x = x_1, .., x_n)$ is an lm-chain and that $x_n \subseteq x'$.

Claim (iv). One can show that any pair xRy in $\tilde{W}$ has a Leivant-node in a completely similar way. Instead of (x', y), consider the pair (x', y'). And instead of lemma 36 for Markov-nodes, use the following corresponding lemma for Leivant-nodes.

Lemma 38 For all n, $x * \langle (l, y) \rangle \in \tilde{W}_n$ is a Leivant-node for (x, y) in $\tilde{W}_n$.

Proof Immediate from definition 10.1.2 of $\tilde{W}$. **QED**

Claim (vi). We show that $\exists n \forall m \geqslant n(W_n = W_m)$. First, some lemmas. Let $l(x)$ be the length of the sequence x.

Lemma 39 $\forall x \in W \exists n \forall (x * \langle D \rangle) \in W$ (if $x * \langle D \rangle \notin W_n$, then D is either of the form $\langle (l, \tau) \rangle$ or $\langle (m, \tau) \rangle$).

Proof By examining procedure **III** one sees that if x is defined in step n, no sequence of the form $x * \langle B \to C \rangle$ or $x * \langle \Box B \rangle$ is defined after step $n+3$ anymore. **QED**

Lemma 40 $\forall x \in W \exists n \forall x * y \in W$ (if $l(y) \leqslant 1$, then $x * y \in W_n$).

Proof Argueing by contradiction, assume there is a sequence $x \in W$ such that

$$\forall n \exists y (x * y \in W \wedge l(y) \leqslant 1 \wedge x * y \notin W_n).$$

This implies that

$$\forall n \exists m > n \exists y (l(y) \leqslant 1 \wedge x * y \notin W_n \wedge x * y \in W_m). \tag{17}$$

Let us start with three observations.

First, by (17) and lemma 39 there is an infinite sequence $y_1, y_2, ..$ in W, where y_i is defined in a step before the one in which y_{i+1} is defined, and y_i is either of the form $\sigma * \langle(l,\tau)\rangle$ or $\sigma * \langle(m,\tau)\rangle$.

Second, by examing the way in which W is constructed in consecutive steps, it is not difficult to see that if $x * \langle(l,z)\rangle$ or $x * \langle(m,z)\rangle$ is defined in step k, z must be defined in step k', for some $k-2 \leqslant k' \leqslant k$.

Third, if $x * \langle(l,z)\rangle \in W$, (x,z) must have been relevant-Leivant in some $\tilde{W}_k$. Hence by definition 10.2.2 of relevant pairs, z is not of the form $\sigma * \langle(l,\tau)\rangle$. Similar for $x * \langle(m,z)\rangle$.

These observations imply that w.l.o.g. we can assume $y_{2i+1} = x * \langle(l, y_{2i})\rangle$ and $y_{2i+2} = x * \langle(m, y_{2i+1})\rangle$. Hence, by procedure **I**, $\alpha_{y_{2i+2}}$ is a Markov-node for $(\alpha_x, \alpha_{y_{2i+1}})$ in $\mathcal{M}$. Hence $R(\alpha_{y_{2i+2}}) = R(\alpha_{y_{2i+1}})$. Thus by remark 31

$$(\alpha_{y_{2i+2}})_\Box = (\alpha_{y_{2i+1}})_\Box. \tag{18}$$

From procedure **II** we conclude that $[\alpha_{y_{2i+1}}]$ is a Leivant-node for $([\alpha_x],[\alpha_{y_{2i}}])$ in $\mathcal{N}$. Hence $[\alpha_{y_{2i+1}}] \preccurlyeq [\alpha_{y_{2i}}]$ in $\mathcal{N}$. Thus by remark 26,

$$(\alpha_{y_{2i+1}})_\Box \subseteq (\alpha_{y_{2i}})_\Box. \tag{19}$$

Combining (18) and (19) we get

$$.. (\alpha_{y_{2i+2}})_\Box = (\alpha_{y_{2i+1}})_\Box \subseteq (\alpha_{y_{2i}})_\Box = \ ..$$

The finiteness of the sets $(\alpha_{y_j})_\Box$ implies that for some i, $(\alpha_{y_{2i+1}})_\Box = (\alpha_{y_{2i}})_\Box$. But then we should have $h(y_{2i}) = y_{2i+1}$. Now lemma 36 shows that y_{2i} is a Markov-node for (x, y_{2i+1}). Therefore, (x, y_{2i+1}) can never have been relevant-Markov in any W_k, contradicting the existence of y_{2i+2}. **QED**

Lemma 41 $\forall x \in W \forall m \exists n \forall x * y \in W$ (if $l(y) \leqslant m$, then $x * y \in W_n$).

Proof By lemma 40 and the fact that every W_k is clearly finite, for every n there is a number, denoted with $f(n)$, such that

$$\forall x \in W_n \forall x * y \in W \ (\text{if } l(y) \leqslant 1, \text{ then } x * y \in W_{f(n)}).$$

In other words,

$$\forall x \in W_n \forall x * y \in W \ (\text{if } x * y \notin W_{f(n)}, \text{ then } l(y) \geqslant 2).$$

We will show that

$$\forall x \in W_n \forall m \forall x * y \in W \ (\text{if } x * y \notin W_{f^m(n)}, \text{ then } l(y) \geqslant m+1).$$

This will prove the lemma.

Therefore, consider some $x \in W_n$ and $x * y \in W$ such that $x * y \notin W_{f^m(n)}$. Let $y = \langle D_1, .., D_k\rangle$ and assume $k \leqslant m$. We derive a contradiction. Observe that for any $z * \langle D \rangle \in W$, if $z * \langle D \rangle \notin W^{m+1}_{f(n)}$, then $z \notin W^{m}_{f(n)}$. Hence $x * \langle D_1, .., D_{k-1} \rangle \notin W^{m-1}_{f(n)}$. And again, $x * \langle D_1, .., D_{k-2} \rangle \notin W^{m-2}_{f(n)}$, etc.etc. Whence $x \notin W^{m-k}_{f(n)} \supseteq W_n$, a contradiction. Thus $k > m$, and therefore $l(y) \geqslant m + 1$. This proves the lemma. **QED**

Now we are ready to show that

$$\exists n \forall m \geqslant n (W_m = W_n).$$

First, observe by examining procedure **III** that if $x \subseteq_{\rightarrow} y$ and $x \neq y$, then $(\alpha_y)_{\nrightarrow} \subset (\alpha_x)_{\nrightarrow}$. And that if $x \subseteq_{\nrightarrow} y$, then $(\alpha_y)_{\not\Box} \subset (\alpha_x)_{\not\Box}$. Let n_0 be the number of implications in X, and let n_1 be the number of formulas in $b_{\not\Box}$. From the observation above it follows that no $x \in W$ can contain more than n_0 consecutive implications, or more that n_1 elements which are no implication. Hence $l(x) \leqslant (n_0+1).n_1+n_0$, for all $x \in W$. Now apply lemma 41 to $x = \langle\rangle \in W_0$ and $m = (n_0+1).n_1+n_0$; thus there exists a number n such that for all $y \in W$, if $l(y) \leqslant m$ then $y \in W_n$. Hence $W = W_n$. Thus the finiteness of W is established. **QED**.

Corollary 42 $\mathbf{B}_i$ is complete with respect to the class of finite B-frames in which every node is either a top node or above a minimal Leivant-node.

Proof By examining the way the finite model in the completeness proof for $\mathbf{B}_i$ above, proposition 29, is constructed. **QED**

We now show that also for $\mathbf{B}_i$ we have a completeness proof with respect to gathering frames; it is complete with respect to the class of finite gathering B-frames. (We know already that we lose the brilliancy of the frames in this case.)

Proposition 43 $\vdash_{\mathbf{B}_i} A$ iff A is valid on all finite gathering B-frames.

Proof Similar to the proof of the completeness of $\mathbf{LLe}_i$ with respect to gathering frames, proposition 19. In the notation of this proof; the property

$$wRv \Rightarrow w(R';\preccurlyeq)v$$

is sufficient to preserve the Ma-property of the frame. The model $\mathcal{M}$ has the property

$$wRv \Rightarrow \exists t \in Top(wRt \wedge R(t) = R(v)),$$

And the model $\mathcal{M}'$ has the property

$$wRv \Rightarrow \exists t \in Top(w(R';\preccurlyeq)t \wedge \forall u(t(R';\preccurlyeq)u \leftrightarrow v(R';\preccurlyeq)u)),$$

thus certainly

$$wR'v \Rightarrow \exists t \in Top(w(R';\preccurlyeq)t \wedge \forall u(t(R';\preccurlyeq)u \leftrightarrow v(R';\preccurlyeq)u)).$$

QED

Proposition 43 is not a strengthening of proposition 29, since we loose the brilliancy when we restrict ourselves to gathering frames. This was already pointed out in section 8.4.

But we do have a real strengthening of the completeness result in proposition 29. This is an immediate corollary of lemma 14.

Corollary 44 $\mathbf{B}_i$ is complete with respect to the class of finite brilliant L-frames which have the Markov-property and satisfy

$$wRv \rightarrow \exists x(wRx \preccurlyeq v \wedge \forall u(vRu \rightarrow x \preccurlyeq u) \wedge \forall u(xRu \rightarrow x \preccurlyeq u)).$$

11 Some remarks

Looking at the form of the Leivant principle and the Formalized Markov scheme one expects that they only influence formulas with a boxdepth greater than 1. This is indeed the case, as the following proposition shows.

Proposition 45 $\mathbf{B}_i$ is conservative over $\mathbf{L}_i$ with respect to formulae of box-depth 1.

Proof: If $\mathbf{L}_i$ does not derive A, there is a model $\mathcal{M}$ which does not validate A in some node w. This implies that A is not valid in the 1-neighbourhood $(M, w)_1$, which clearly is a B-model. Hence $\mathbf{B}_i$ does not derive A. **QED**.

One may wonder if the Formalized Markov scheme is derivable from a principle like

Ma^- $\quad \Box(\neg\neg \bigvee \Box A_i) \rightarrow \Box(\bigvee \Box A_i)$,

or from one of these

$Ma(n)$ $\quad \Box(\neg\neg(\Box A \rightarrow \bigvee_{i=1}^n \Box B_i)) \rightarrow \Box(\Box A \rightarrow \bigvee_{i=1}^n \Box B_i)$.

The following propositions and corollaries show that this is not the case. First a lemma.

Lemma 46 For formula $D = \bigvee \Box A_i$,

$$\mathbf{LMa}_i^- \vdash \Box(\Box B \rightarrow \neg\neg D) \wedge \Box(\Box B \rightarrow (D \rightarrow B)) \rightarrow \Box D.$$

Proof Let $C_1 = \Box(\Box B \rightarrow \neg\neg D)$ and $C_2 = \Box(\Box B \rightarrow (D \rightarrow B))$.

$$\begin{array}{lrcl}
\vdash & C_2 & \to & \Box(\Box\Box B \to (\Box D \to \Box B)) \\
\vdash & C_1 & \to & \Box(\Box\Box B \to \Box D) \\
\vdash & C_1 \wedge C_2 & \to & \Box(\Box\Box B \to \Box B) \\
\vdash & C_1 \wedge C_2 & \to & \Box\Box B \\
\vdash & C_1 & \to & \Box\Box B \to \Box D \\
\vdash & C_1 \wedge C_2 & \to & \Box D.
\end{array}$$

QED

Proposition 47 $\vdash_{\mathbf{LMa}_i^-} A$ iff A is valid on all finite brilliant L-frames which satisfy: $wRt \Rightarrow \exists t \in Top(wRt$ and $R(v) \subseteq R(t))$.

Proof Only from right to left. Assume $\nvdash_{\mathbf{LMa}_i^-} A$. Let $\mathcal{M} = (W, \preccurlyeq, R, \models)$ be the $\mathbf{LMa}_i^-$-canonical model and let b be a node which does not force A. Let X be an adequate set which contains A, and which is the single closure under negation of a finite LLe-adequate set with base X_0. With the construction method we create a finite model M_b with the desired properties which does not validate A.

We start with $\alpha_{<>} = b$. Assume σ is defined. For $(B \to C) \in (\alpha_\sigma)^X_{\not\to}$, nodes $\alpha_{\sigma * \langle B \to C\rangle}$ are chosen as is usual in the construction method. For $\Box B \in (\alpha_\sigma)^X_{\not\Box}$ we chose two nodes, $\alpha_{\sigma*\langle \Box B\rangle}$, $\alpha_{\sigma*\langle(\Box B)\rangle}$, the last one of which will be a top node in the model to be constructed. We first say how they will be chosen and then show that they can be chosen in such a way.

Let $w = \alpha_\sigma$ and $E = \bigwedge\{\bigvee \Box D_i \mid w \vdash \Box(\Box B \to \neg\neg(\bigvee \Box D_i)), D_i \in X_0\}$. $\alpha_{\sigma*\langle \Box B\rangle}$ is a node v in $\mathcal{M}$ such that wRv, $\Box B \in v$, $B \notin v$ and $E \in v$. $\alpha_{\sigma*\langle(\Box B)\rangle}$ is a node t in $\mathcal{M}$ for which wRt and $\Box B \in t$, and $\neg\Box C \in t$ for all $C \in X_0$ for which $\Box C \notin v$ and such that t is a top node in the canonical model. An explication of the choice of t: $\Box B \in t$ makes the set $t_{\not\Box}$ smaller than $w_{\not\Box}$, the third requirement allows us to put tR_bu for all vR_bu, in the new model, and the last one guarantees that also in the model to be constructed, t is a top node.

We can always find $\alpha_{\sigma*\langle\Box B\rangle}$. For the case that $\Box E \in w$ this is immediately clear. If $\Box E \notin w$, we cannot have $w_{\boxdot}, \Box B, E \vdash B$. For if this would be the case, we would have $w \vdash \Box(\Box B \wedge E \to B)$. As we also have $w \vdash \Box(\Box B \to \neg\neg E)$ by the definition of E, lemma 46 implies that $w \vdash \Box E$.

We show that we can find $\alpha_{\sigma*\langle(\Box B)\rangle}$ as stated. Let $\{C_1, \ldots, C_n\} = \{D \in X_0 \mid \Box D \notin \alpha_{\sigma*\langle\Box B\rangle}\}$. It suffices to show that $w_{\boxdot}, \Box B, \neg\Box C_1, \ldots, \neg\Box C_n$ is consistent, then we can take for $\alpha_{\sigma*\langle(\Box B)\rangle}$ any maximal consistent extension of this set. Assume it is not. Then $w_{\boxdot}, \Box B \vdash \neg\neg(\bigvee \Box C_i)$. Hence $w \vdash \Box(\Box B \to \neg\neg(\bigvee \Box C_i))$, and thus $\vdash E \to (\bigvee \Box C_i)$. But then, by the construction of $\alpha_{\sigma*\langle\Box B\rangle}$ it follows that $\bigvee \Box C_i$ is in $\alpha_{\sigma*\langle\Box B\rangle}$, which cannot be.

Now we define the model. We use the notation of paragraph 10.

$$\begin{array}{rcl}
W_b & = & \{\sigma \mid \sigma \text{ is defined}\}, \\
\sigma \preccurlyeq_b \tau & :\Leftrightarrow & \sigma \subseteq_{\to} \tau, \\
\sigma R_b \tau & :\Leftrightarrow & \sigma \subseteq_{\not\to} \tau, \text{ or} \\
 & & \sigma = \sigma' * \langle(\Box B)\rangle, \sigma' * \langle \Box B\rangle \subseteq_{\not\to} \tau, \\
\sigma \models_b p & :\Leftrightarrow & p \in \alpha_\sigma.
\end{array}$$

This makes $(W_b, \preccurlyeq_b, R_b, V_b)$ into a transitive, brilliant model. To see that it is finite, conversely well-founded, that it satisfies the property stated in the proposition, and that

$$\text{for all } B \in X, \text{ for all } \sigma \in W_b : \sigma \models_b B \Leftrightarrow \alpha_\sigma \models B,$$

is straightforward. **QED**

Corollary 48 $\mathbf{LMa}_i^- \nvdash Ma$.

Proposition 49 $\vdash_{\mathbf{LMa(n)}_i} A$ iff A is valid on all finite brilliant L-frames which satisfy: $wRvRu_i$, for $i = 1, \ldots, n \Rightarrow \exists t \in Top(wRtRu_i \wedge R(t) \subseteq R(v))$.

Proof Only the direction from right to left. For $\nvdash_{\mathbf{LMa(n)}_i} A$, let b a node in the $\mathbf{LMa(n)}_i$-canonical model $\mathcal{M} = (W, \preccurlyeq, R, V)$ at which A is not valid. Let X be a finite adequate set which contains A, and such that it is the single closure under negation of an LLe-adequate set. With the construction method we construct a model $\mathcal{M}_b = (W_b, \preccurlyeq_b, R_b, V_b)$ which does not validate A and has a finite brilliant L-frame and in addition satisfies

$$wR_bvR_bu_i, \text{ for } i = 1, \ldots, n \;\Rightarrow \exists t \in Top(wR_btR_bu_i \wedge R_b(t) \subseteq R_b(v)).$$

It is easy to see that the $\mathbf{LMa(n)}_i$-canonical model has the frame property stated in the proposition; for $wRvRu_i$, for $i = 1, \ldots, n$, the *top* node t with $R(t) \subseteq R(v)$ and $wRtRu_i$ is just a maximal consistent extension of the set $w_\Box, v_\Box, \neg\Box E_1, .., \neg\Box E_n$, where $E_i = \bigvee\{D \mid D \notin u_i\}$, which consistency follows from the $Ma(n)$ principle.

Start, as is usual, with $\alpha_{<>} = b$. Assume σ is defined. Nodes $\alpha_{\sigma * \langle B \to C\rangle}$ and $\alpha_{\sigma * \langle \Box B\rangle}$ are chosen as usual, for all $(B \to C) \in (\alpha_\sigma)^X_{\not\to}$ and $\Box B \in (\alpha_\sigma)^X_{\not\Box}$. Assume $\sigma = \sigma' * \langle \Box B\rangle * \sigma''$ is defined, and its last element is not of the form $\tau * \langle (v, u_1, .., u_k)\rangle$, $k \leqslant n$. Assume furthermore that $u_i = \sigma * \sigma_i * \langle D_i\rangle$ are defined, for $1 \preccurlyeq i \leqslant k \leqslant n$, where $\sigma \subseteq_\to \sigma_i$, and D_i is no implication. Then choose a node α_τ, where $\tau = \sigma' * \langle (\sigma, u_1, .., u_k)\rangle$, which is a top node t in the canonical model such that $\alpha_{\sigma'} RtR\alpha_{u_i}$ and $R(t) \subseteq R(\alpha_\sigma)$.

Assume this process stops after a finite number of steps. Define

$$\begin{array}{rcl} W_b & = & \{\sigma \mid \sigma \text{ is defined }\}, \\ \sigma \preccurlyeq_b \tau & :\Leftrightarrow & \sigma \subseteq_\to \tau, \\ \sigma R_b \tau & :\Leftrightarrow & \sigma \subseteq_{\not\to} \tau, \text{ or} \\ & & \sigma = \sigma' * \langle (\sigma'', u_1, .., u_k)\rangle, k \leqslant n, u_i \subseteq \tau, \text{ for some } i, \text{ or} \\ & & \exists \tau'(\tau' = \tau'' * \langle (\sigma, u_1, .., u_k)\rangle, k \leqslant n, \tau' \subseteq_{\not\to} \tau), \\ \sigma \models_b p & :\Leftrightarrow & p \in \alpha_\sigma. \end{array}$$

It is not difficult to see that this defines a finite brilliant L-model, which moreover has the property stated in the proposition, and satisfies

$$\text{for all } B \in X, \text{ for all } \sigma \in W_b : \sigma \models_b B \Leftrightarrow \alpha_\sigma \models B.$$

Therefore, what remains to show is that the process stops in a finite number of steps, hence that W_b is finite. This follows from the following observation, which we will prove below. Every σ which is defined has only finitely many successors and only finitely many elements above it. As every σ which is defined is above or a successor of $\langle\rangle$, this shows that the process is finite.

We only prove the successor part of the observation. Let $n(\sigma)$ be the number of formulas in $b_{\not\Box}$ minus the number of elements in σ which are no implication. We prove the statement with induction to $n(\sigma)$.

The case $n(\sigma) = 0$ is clear, as in this case $(\alpha_\sigma)_{\not\Box} = \emptyset$ and $\sigma R \sigma'$ implies $(\alpha_{\sigma'})_{\not\Box} \subset (\alpha_\sigma)_{\not\Box}$.

Consider a σ with $n(\sigma) = m + 1$. It suffices to show that σ has finitely many successors $\tau_1, .., \tau_k$ of the form $\sigma * \sigma' * \langle D\rangle * \sigma''$, where $\sigma \subseteq_{\rightarrow} \sigma * \sigma' \subseteq_{\not\rightarrow} \sigma * \sigma' * \langle D\rangle \subseteq_{\rightarrow} \sigma * \sigma' * \langle D\rangle * \sigma''$. As $n(\tau_i) = m$, the induction hypothesis guarantees that τ_i has only finitely many successors. Because every successor of σ is one of $\tau_1, .., \tau_k$ or a successor of one of them, this implies that σ has only finitely many successors.

It is clear that σ has only finitely many successors of the form $\sigma * \sigma' * \langle D\rangle * \sigma''$, where $\sigma \subseteq_{\rightarrow} \sigma * \sigma' \subseteq_{\not\rightarrow} \sigma * \sigma' * \langle D\rangle \subseteq_{\rightarrow} \sigma * \sigma' * \langle D\rangle * \sigma''$, and $D = \Box B$ for some B. Let $x_1, .., x_l$ be a lsit of these successors. Consider the case in which $D = \langle(v, u_1, .., u_k)\rangle$, for some $k \leqslant n$. By examining the construction one sees that either $v = x_i$ for some i, or $n(v) < m$. In the last case, v is a successor of some v', with $n(v') = m$. Hence the induction hypothesis implies that there are only finitely many of these v. There are only finitely many of these u_i, since $n(v) \leqslant m$ and u_i is a successor of v. This proves that σ has only finitely many successors. **QED**

Corollary 50 If $m < n$, $\mathbf{LMa(m)}_i \nvdash Ma(n)$.

12 Admissible Rules

The notion of an *admissible rule* is one which emerges once one leaves classical propositional logic for its intuitionistic counterpart. If $B(p_1, \ldots, p_n)$ is a formula in the variables $p_1, \ldots, p_n$ and $\vec{A}$ is a sequence of formulas $A_1, \ldots, A_n$, we write $B(\vec{A})$ for the formula which is the result of substituting A_i for p_i in B. If for a theory T in a language $\mathcal{L}$ and for two formulas B, C of a language $\mathcal{L}'$ in the variables $p_1, \ldots, p_n$, it holds that $\vdash_T B(\vec{A})$ implies $\vdash_T C(\vec{A})$, for all sequences $\vec{A} = A_1, \ldots, A_n$ of formulas in $\mathcal{L}$, then we say that B *admissibly derives* C *in* T, or that B/C is an *($\mathcal{L}'$-)admissible rule for* T. Of course, if $B \vdash C$ then certainly B/C, and there are systems, like for instance classical propositional logic, in which these are the only admissible rules, so the notion is superfluous there. However, there are systems in which this is not the case. We already encountered one such; $\mathbf{K}_i$ with for example its admissible rule $\Box A/A$.

Admissible rules are interesting from the viewpoint of provability logic, as we conjecture that for any propositional admissible rule B/C of $\mathbf{HA}$, $\Box B \rightarrow \Box C$

is a principle of its provability logic. For the modal admissible rules this is certainly not the case, as the admissible rule $\Box A/A$ shows.

It is known that the admissible rules for intuitionistic propositional calculus **IPC** are the same as the propositional ones for **HA** [Visser 98], and also that it is decidable whether something is an admissible rule for **IPC** or not [Rybakov 97]. But we do not yet have a transparent way of formulating them. We discuss two rules; one is an admissible rule for **IPC** and the other is formulated in the modal language. Both are admissible rules for **HA**.

12.1 The Independence of Premise Rule

$IPR \quad \vdash \neg A \to B \vee C \Rightarrow \vdash (\neg A \to B) \vee (\neg A \to C).$

Proposition 51 All logics encountered so far have the independence of premise rule.

Proof We show that $\mathbf{B}_i$ has the independence of premiss rule, the other cases are similar. Assume that $(\neg A \to B) \vee (\neg A \to C)$ is not derivable in $\mathbf{B}_i$. It is easy to see that we then have two B-models $\mathcal{M}_i = (W_i, \preccurlyeq_i, R_i, V_i)$ for $i = 1, 2$, with roots respectively w_1, w_2, such that $\neg A$ is valid in both the roots, but B is not valid in w_1 and C is not valid in w_2. We show that $(\neg A \to B \vee C)$ is not derivable in $\mathbf{B}_i$ by constructing a countermodel $\mathcal{N} = (W, \preccurlyeq, R, V)$. This model is the result of putting one new node b below all nodes in both $\mathcal{M}_1$ and $\mathcal{M}_2$;

$W = \{b\} \cup W_1 \cup W_2,$

$w \preccurlyeq v :\Leftrightarrow w = b \text{ or } w \preccurlyeq_i v \text{ for some } i,$

$wRv :\Leftrightarrow wR_iv \text{ for some } i, \text{ or } w = b \text{ and } uR_iv \text{ for some } u,$

$w \models p :\Leftrightarrow w \in W_i \text{ and } \mathcal{M}_i, w \models p, \text{ for some } i.$

It is easy to see that this defines a B-model such that if $\mathcal{N}, b \models D$, then for all $v \in W_i$, $\mathcal{M}_i, v \models D$ for $i = 1, 2$. Therefore, $\neg A$ is valid at b and $B \vee C$ is not. Whence $(\neg A \to B \vee C)$ is not valid in b. **QED.**

12.2 The Reflection Rule

$R \quad \vdash \Box A \Rightarrow \vdash A.$

Proposition 52 All logics encountered so far have the reflection rule.

Proof We only treat the case $\mathbf{LLe}_i$. Let A be a formula which is violated in some finite LLe-model $\mathcal{M} = (W', \preccurlyeq', R', V')$. We construct a countermodel $\mathcal{N} = (W, \preccurlyeq, R, V)$. The idea is to make all the nodes of $\mathcal{M}$ successors of a new node b. But to guarantee that the new model has the iLeivant property we have to add another node, a, which will be below all nodes in W and which will function as a Leivant-node for every pair (b, w).

$W = \{b, a\} \cup W',$

$w \preccurlyeq v :\Leftrightarrow w = a,\ \text{or}\ w \preccurlyeq' v,$
$wRv :\Leftrightarrow wR'v,\ \text{or}\ w = b$ and $v \neq b$, or $w = a$ and $v \in W'$,
$w \models p :\Leftrightarrow w \models' p.$

QED

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