Preservativity logic
An analogue of interpretability logic for constructive theories

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Abstract

In this paper we study the modal behavior of Σ-preservativity, an extension of provability which is equivalent to interpretability for classical superarithmetical theories. We explain the connection between the principles of this logic and some well-known properties of HA, like the disjunction property and its admissible rules. We show that the intuitionistic modal logic given by the preservativity principles of HA known so far, is complete with respect to a certain class of frames.

1 Introduction

An intriguing open problem in provability logic is the axiomatization of the provability logic of HA, the constructive theory of the natural numbers. Up till now no decent (r.e.) axiomatization for this logic has been found, but a proof that such an axiomatization could not exist has not been found either. Most classical theories have a finitely axiomatizable, decidable provability logic with a simple modal semantics. A remarkable thing is their *stability*: many classical theories, e.g. PA, ZF, $\mathsf{I}\Delta_0+\mathsf{exp}$, share the same provability logic, GL. Observe that the provability logic of HA could certainly not be equal to GL since GL contains classical propositional logic. However, since PA is the classical counterpart of HA one could wonder whether the provability logic of HA is, as an intuitionistic modal logic, axiomatized by the modal axioms of GL. It turns out that this is not the case, as was shown by Leivant in the 70's (a proof will be given in the next section). Given the stability of classical provability logics, this makes the study of the provability logic of HA all the more interesting.

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Some years ago Visser [10] axiomatized a fragment iPH of the so-called preservativity logic of HA. Preservativity logic is an extension of provability logic that we will introduce below. The logic iPH captured all the principles of the provability logic of HA that were discovered before that time. Furthermore, results in [4][9] implied that for well-known properties of HA that are expressible in preservativity logic, the statement that expresses this property either is not contained in the preservativity logic of HA, or it belongs to iPH (this will be discussed in more detail in the next section). This has lead us to the conjecture that iPH axiomatizes the preservativity logic of HA. The reason that we prefer to study preservativity instead of provability logic is that many principles have a more transparant formulation in this setting. At the moment of writing this paper it is not even clear whether there is a nice axiomatization of the logic consisting of all formulas in the language of provability logic that are valid in iPH.

In the remaining part of the introduction we explain what preservativity logic is and we discuss its connection with interpretability logic. The rest of the paper is devoted to the modal characterization of the principles of the given preservativity logic. The main result is the modal completeness of the preservativity logic with respect to a certain class of frames. The motivation for this research is the fact that completeness proofs for provability and interpretability logics, in the style of the landmark paper [8], partly rely on the modal completeness of the logic in question.

Preservativity logic is defined as follows. Let Σ_1 and Π_1 denote the first levels of the arithmetical hierarchy. For an arithmetical theory T and sentences φ and ψ in the language of T, φ is said to Σ_1*-preserve* ψ with respect to T, if for all Σ_1-sentences θ it holds that $T \vdash (\theta \to \varphi)$ implies $T \vdash (\theta \to \psi)$. We denote this with $\varphi \rhd_T \psi$. Since we will not consider any other forms of preservativity than Σ_1-preservativity we will, as in the title, always refer to preservativity instead.

On the modal side the notion of preservativity gives rise to a modal language $\mathcal{L}_\rhd$ with one binary modal operator, $\rhd$. Analogous to provability logic the preservativity logic of T is defined as the collection of $\mathcal{L}_\rhd$-formulas A such that $T \vdash A^*$ for any arithmetical realization *. In this context the definition of an *arithmetical realization* is extended to cover formulas in which the preservativity symbol $\rhd$ occurs: an arithmetical realization * is a mapping from $\mathcal{L}_\rhd$-formulas to arithmetical formulas which commutes with the connectives and such that $(A \rhd B)^* = \mathrm{Pres}_T(\ulcorner A^* \urcorner, \ulcorner B^* \urcorner)$, where $\mathrm{Pres}_T(x, y)$ is a formula in the language of T that is the formalized version of the statement $A \rhd_T B$. The formulas in $\mathcal{L}_\rhd$ are called modal formula

Clearly, preservativity logic is an extension of provability logic because we have

$$\Box_T \varphi \text{ iff } \top \rhd_T \varphi.$$

All the principles of the fragment of the provability logic of HA studied in [3] are derivable in the preservativity logic considered in this paper. See [5] for a discussion on this connection.

For *classical* theories T the notion of preservativity is equivalent to the notion of Π_1-conservativity: we have that φ Σ_1-preserves ψ if and only if $\neg\varphi$ is Π_1-conservative over $\neg\psi$. For many classical theories, for example PA, Π_1-conservativity is equivalent to the well-investigated notion of interpretability. Therefore, for these theories the preservativity logic is known, although the notion is not studied directly but only via the equivalence with interpretability. It is not difficult to see that all the principles of the interpretability logic of PA, ILM are inherited by HA (that is, if we reformulate ILM in terms of preservativity by replacing $A \rhd_i B$ by $\neg B \rhd \neg A$). The converse does not hold, see Section 2.0.1.

As we will see, many principles of the provability logic of HA have a elegant formulation in the setting of preservativity logic. This seems to suggest that the notion of preservativity gives the right view on questions in provability logic of constructive theories.

In the context of intuitionistic logic the notion of intuitionistic truth provability logic seems less natural, because the intuitionistic notion of truth depends heavily on the constructive thruths one is willing to accept. Thus the notion can only be meaningful once we know what our meta-theory is. We have chosen not to address this question in this paper, and therefore we will concentrate solely on provability logic. But let us note in passing that $\Box(A \vee B) \to \Box A \vee \Box B$ is an example of a principle that, for almost all reasonable meta-theories, is in the truth provability logic of HA but not in the provability logic of HA.

In the next section we introduce the preservativity logic given by the principles discovered by Visser. We discuss the meaning of the principles and explain why we conjecture the given logic to be (all of) the preservativity logic of HA. Section 3 contains preliminaries and Sections 4- 8 contain the completeness results for the individual principles. Section 9 contains the main theorem of the paper, i.e. the modal completeness of the preservativity logic. Section 10 discusses some admissible rules of the logic.

As said, our interest in the modal properties of this logic stems from the fact that completeness proofs for provability and interpretability logics partly rely on the modal completeness of the logic in question. However, the paper could also be viewed as a study in intuitionistic modal logic. The characterization of the principles requires many technical tools from modal logic. Moreover, these logics deviate considerably from the logics that are regularly studied in intuitionistic modal logic. Therefore, some proofs are quite different from the ones for modal logics than one usually encounters. Therefore, also from the modal point of view of these logics seem to be interesting.

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2 The preservativity logic of Heyting Arithmetic

To state the principles of the preservativity logic of HA we need the following notation. For formulas $A, B_1, \ldots, B_n$, the formula $(A)(B_1, \ldots, B_n)$ is inductively defined to be

$$\begin{array}{lll}
(A)(B, C_1, \ldots, C_n) & \equiv_{def} & (A)(B) \vee (A)(C_1, \ldots, C_n) \\
(A)(\bot) & \equiv_{def} & \bot \\
(A)(B \wedge B') & \equiv_{def} & (A)(B) \wedge (A)(B') \\
(A)(\Box B) & \equiv_{def} & \Box B \\
(A)(B) & \equiv_{def} & (A \to B) \\
 & & B \text{ not of the form } \bot,\, (C \wedge C') \text{ or } \Box C.
\end{array}$$

Note that we have $(A)(C_1, \ldots, C_n) = (A)(C_1) \vee \ldots \vee (A)(C_n)$, and that $(A)(\top) = (A \to \top)$, hence $(A)(\top) \leftrightarrow \top$.

The expression $(\cdot)(\cdot)$ is an abbreviation and not an operator, because applying it to equivalent formulas does not give equivalent results. For example, $\Box p$ is equivalent to $(\top \to \Box p)$, but $(A)(\top \to \Box p) = (A \to (\top \to \Box p))$ and $(A)(\Box p) = \Box p$. Hence the formulas $(A)(\top \to \Box p)$ and $(A)(\Box p)$ are in general not equivalent.

In [10] the following principles of the preservativity logic of HA known so far are given. In fact, we give here a slightly different axiomatization then the one used by Visser. In [5] it is shown that the two systems are equivalent. We denote intuitionistic propositional logic by IPC. Recall that φ is provable if and only if $\top$ preserves φ. This accounts for the definition of $\Box$ in the system.

Axioms:

$$\Box A \equiv_{def} \top \rhd A$$

Taut	all tautologies of IPC	
P1	$A \rhd B \wedge B \rhd C \to A \rhd C$	
P2	$C \rhd A \wedge C \rhd B \to C \rhd (A \wedge B)$	
Dp	$A \rhd C \wedge B \rhd C \to (A \vee B) \rhd C$	(Disjunctive Principle)
4p	$A \rhd \Box A$	
Lp	$(\Box A \to A) \rhd A$	(Löb's Preservativity Principle)
Mp	$A \rhd B \to (\Box C \to A) \rhd (\Box C \to B)$	(Montagna's Principle)
Vp$_n$	$(\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_{n+1} \vee A_{n+2}) \rhd (\bigwedge_{i=1}^{n}(A_i \to B_i))(A_1, \ldots, A_{n+2})$	(Visser's Principles)
Vp	$Vp_1, Vp_2, Vp_3, \ldots$	(Visser's Scheme)

Rules:

Modus Ponens	$A, (A \to B)/B$
Preservation Rule	$(A \to B)/A \rhd B$

We use the name iPH for the logic given by these principles and rules.

As mentioned in the introduction, all principles except the Disjunctive Principle hold for PA as well. It is not difficult to see that the Disjunctive Principle does not hold for PA, see below. For all of these principles, besides *Vp* and *Dp*, it is easy to verify that they are indeed principles of the preservativity logic of HA. For the arithmetical validity of *Vp* and *Dp* we refer the reader to [10].

In the remainder of this section we discuss the meaning of the given principles. We will see that these principles form a natural fragment of the preservativity logic of HA. Namely, each of them corresponds to either a principle of the provability logic of PA or to one of the following characteristic properties of HA: its propositional admissible rules, Markov's Rule and the Disjunction Property.

The definition of $\Box$ and the first two principles are easily seen to be principles of the preservativity logic of HA. The principles *4p* and *Lp* resemble the two characteristic axioms for the provability logic of PA, which are

$$4 \quad \Box A \to \Box\Box A$$

$$L \quad \Box(\Box A \to A) \to \Box A.$$

Since $A \rhd B$ implies $(\Box A \to \Box B)$ in the system (Section 3.6), the principles *4p* and *Lp* imply their provability counterparts *4* and L. The principle *4* is derivable from L, but usually it is still included in the axioms. We will see that in the same way *4p* is derivable from *Lp* (Section 6). The principle *Mp* is baptized after its classical counterpart in interpretability logic, which is discussed below. It is easy to see that it belongs to the preservativity logic of HA, using the fact that the arithmetical realization of a formula $\Box C$ is always Σ_1.

2.0.1 The Disjunctive Principle and the Disjunction Property

The Disjunctive Principle *Dp* is related to the Disjunction Property of HA, which reads

(*Disjunction Property*) if $\mathsf{HA} \vdash \varphi \vee \psi$, then $\mathsf{HA} \vdash \varphi$ or $\mathsf{HA} \vdash \psi$.

Friedman (1975) has shown that HA does not prove its disjunction property, i.e. HA does not derive the true formula $\Box(\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)$. Leivant (1975) showed that HA does prove the weaker version

$$\mathsf{HA} \vdash \Box(\varphi \vee \psi) \to \Box(\varphi \vee \Box\psi).$$

Hence the so-called Leivant Principle $\Box(A \vee B) \to \Box(A \vee \Box B)$ is part of the provability logic of HA. In the preservativity logic of HA this principle occurs as a consequence of the two principles *4p* and *Dp*. Note that the fact that *Dp* and *4p* are in the preservativity logic of HA imply the following strengthening of Leivant's Principle:

$$\mathsf{HA} \vdash (\varphi \vee \psi) \rhd (\varphi \vee \Box\psi).$$

Finally, let us show that the Leivant's Principle does not belong to GL, the provability logic of PA. This implies that de Disjunctive Principle does not belong to the preservativity logic of PA (hence the reformulation of the principle in terms of interpretability does not belong to the interpretability logic of PA). We reason as follows. If GL would derive the Leivant Principle it would also derive $\Box(\Box\bot \vee \Box\neg\Box\bot)$, as it clearly derives $\Box(\Box\bot \vee \neg\Box\bot)$. But then an application of L shows that it would derive $\Box\Box\bot$. Hence the provability logic of HA is not a part of GL. The converse is not true either. The principle $(p \vee \neg p)$ is a theorem of the provability logic of PA, but not of the corresponding logic of HA. Note that this also shows that there is no monotonicity (converse monotonicity) in provability logics; stronger theories do not necessarily have stronger (weaker) provability logics.

2.0.2 Visser's Scheme and the admissible rules

The scheme *Vp* is called after A. Visser who proved its arithmetical validity [10]. Note that it is not a principle but a collection of infinitely many principles. In [5] it is shown that no Vp_m derives Vp_n for $m < n$ over the base preservativity logic defined below. Hence the scheme consists really of infinitely many principles.

The principles describe (some) admissible rules of HA. For propositional formulas A, B we say that the rule A/B is a *propositional admissible rule of* HA if $\mathsf{HA} \vdash \sigma A$ implies $\mathsf{HA} \vdash \sigma B$, for all substitutions σ which replace the propositional variables by arithmetical formulas. Note that if $(\Box A \to \Box B)$ is in the provability logic of HA, then A/B is an admissible rule of HA. Since $A \rhd B$ implies $(\Box A \to \Box B)$, it follows that if $A \rhd B$ is in the preservativity logic of HA, then A/B is an admissible rule for HA. The two most meaningful instances of *Vp* describe the propositional admissible rules and Markov's Rule for HA. We will discuss them briefly.

If one restricts Visser's Scheme to pure propositional formulas, i.e. without $\Box$ or $\rhd$, it characterizes the propositional admissible rules of HA:

for propositional formulas A, B:

A/B is a propositional admissible rule of HA iff $\mathsf{iPH} \vdash A \rhd B$.

This follows from results in [4][10][9], a proof can be found in [5].

Markov's Rule, a well-known admisible rule for HA, reads

(*Markov's Rule*) for all $\varphi \in \Pi_2$: if $\mathsf{HA} \vdash \neg\neg\varphi$, then $\mathsf{HA} \vdash \varphi$.

To see how Markov's Rule is captured by Visser's Scheme, observe that the following formula is one of the consequences of Visser's Scheme,

(1) $\neg\neg\Box A \rhd \Box A$.

Namely, $\neg\neg\Box A$ is short for $((\Box A \to \bot) \to \bot)$, and by Visser's Scheme

$$((\Box A \to \bot) \to \bot) \rhd (\Box A \to \bot)(\Box A, \bot) =$$
$$(\Box A \to \bot)(\Box A) \vee (\Box A \to \bot)(\bot) = (\Box A \vee \bot) \equiv \Box A.$$

Now (1) implies that HA proves the arithmetical realizations of the formula $(\Box\neg\neg\Box A \to \Box\Box A)$, which is a partial formalization of Markov's Rule. Thus the fact that (1) is in the preservativity logic of HA implies that HA proves Markov's Rule: $\mathsf{HA} \vdash (\Box\neg\neg\Box A \to \Box\Box A)$.

Summarizing we could say that the preservativity logic presented in [10] seems a very natural part (if not all) of the preservativity logic of HA. It contains three basic principles, *P1*, *P2* and Montagna's Principle, which arithmetical validity is trivial. It contains the (preservativity form of the) two characteristic principles of the provability logic of PA, namely *4p* and *Lp*. And it contains two axioms, the Disjunctive Principle and Visser's Scheme, which are directly related to three well-known properties of HA: the Disjunction Property, Markov's Rule and the propositional admissible rules. Thus from these properties of HA that are expressible in provability logic we do know whether they belong to the provability logic of HA or not. For another important part, namely the propositional fragment, this is known as well. Namely, it was shown in [6] that the propositional fragment of the provability logic of HA is equivalent to IPC. All these facts together has lead us to the conjecture that the preservativity logic above is (all of) the preservativity logic of HA.

3 Conventions and definitions

In this section we introduce a semantics for preservativity logic, and we define the canonical model and the construction method. These are all fairly standard definitions except for the way in which the operator $\rhd$ is interpreted in models. This semantics for $\rhd$ is an idea from Visser. We also define the 'new' notion of an extendible property. In the proofs that this or that logic is canonical we need extensions of given sets of formulas. These extensions are all special instances of a 'general' principle of extension, which gave rise to the definition of an extendible property.

3.1 Definitions

The language $\mathcal{L}_\rhd$ of preservativity logic is that of propositional logic extended with one binary modal operator, $\rhd$. We assume $\bot$ (falsum) and $\top$ (true) to be present as primitive symbols in our propositional language. Recall that $\Box A$ is defined as $\top \rhd A$. A formula of the form $A \rhd B$ is called a *preservation* and a formula of the form $\Box A$ is called a *boxed formula*. We adhere to some reading conventions and omit parentheses when possible. The negation binds stronger than $\rhd$ which binds stronger than $\wedge$ and $\vee$, which in turn bind stronger than $\to$. Further $\Gamma \rhd \Delta$ is short for $\bigwedge \Gamma \rhd \bigvee \Delta$.

A *logic* is a theory closed under substitution. We call the logic which has as axioms all tautologies of IPC and the principles *P1* and *P2* (and Dp) and as rules Modus Ponens and the Preservation Rule, the *arithmetical (semantical) base preservativity logic* and denote it with iP^- (iP). For any principles A and B, $\mathsf{iP}(A \oplus B)$ is the preservativity logic consisting of the axioms of iP plus A and B, and the rules Modus Ponens and Necessitation. When T denotes the infinite set of principles $A_1, A_2, \ldots$, we also write $\mathsf{iP}(T)$ for $\mathsf{iP}(A_1 \oplus A_2 \oplus \ldots)$. When Ap is one of the principles given above we write iPA for $\mathsf{iP}(Ap)$. We write $\vdash_{\mathsf{iT}} A$ when A is derivable in iT. We write $\Gamma \vdash_{\mathsf{iT}} A$ when there is a derivation of A in iT from Γ without use of Necessitation, in other words, when A is derivable by Modus Ponens from theorems of iT and formulae in Γ.

The name 'semantical base preservativity logic' for iP comes from from the fact that it is sound and complete with respect to a certain kind of frame semantics defined in Section 3.2. Thus, semantically seen, it is a base preservativity logic. On the other hand, the only axioms of iP which *trivially* hold for all arithmetical interpretations in HA are only *P1* and *P2*, which accounts for the name 'arithmetical base preservativity logic' for iP^-.

It is not very difficult to see that the logic consisting of the rules *P1* and *P2* plus

$$\Box(A \to B) \to A \rhd B$$

and the rules Modus Ponens and Necessitation is equivalent to iP^-.

3.2 A semantics

A possible semantics for preservativity logic can be produced via frames: we just add one extra clause for the interpretation of $\rhd$. The frames we use occur already in the literature on intuitionistic modal logic, e.g. in [7][1][11].

First some notation. When R and S are two binary relations, $(R;S)$ is the relation defined via $w(R;S)u \equiv \exists v(wRvSu)$.

A *frame* is a triple $\mathcal{F} = (W, \preccurlyeq, R)$, where W is a nonempty set (the set of *nodes*), $\preccurlyeq$ is a partial ordering on W (the *intuitionistic relation*) and R a binary relation on W (the *modal relation*) such that $(\preccurlyeq;R) \subseteq R$.

A *model* is a quadruple $\mathcal{M} = (W, \preccurlyeq, R, V)$, where $(W, \preccurlyeq, R)$ is a frame and V a *valuation relation* on pairs consisting of nodes and propositional variables. We demand that V is persistent, i.e.

(*persistence*) if $w \preccurlyeq v$ and wVp, then vVp.

We inductively define what it means for a formula A to be *forced (or valid) at* *a node* w of a model $\mathcal{M}$ ($\mathcal{M}, w \Vdash A$):

$$\begin{array}{lll}
\mathcal{M}, w \Vdash p & \equiv_{def} & wVp \\
\mathcal{M}, w \Vdash A \wedge B & \equiv_{def} & \mathcal{M}, w \Vdash A \text{ and } \mathcal{M}, w \Vdash B \\
\mathcal{M}, w \Vdash A \vee B & \equiv_{def} & \mathcal{M}, w \Vdash A \text{ or } \mathcal{M}, w \Vdash B \\
\mathcal{M}, w \Vdash A \to B & \equiv_{def} & \forall v \succcurlyeq w \ (\mathcal{M}, v \Vdash A \text{ implies } \mathcal{M}, v \Vdash B) \\
\mathcal{M}, w \Vdash A \rhd B & \equiv_{def} & \forall v \ (\text{if } wRv \text{ and } \mathcal{M}, v \Vdash A \text{ then } \mathcal{M}, v \Vdash B) \\
\mathcal{M}, w \Vdash \Box A & \equiv_{def} & \forall v \ \ (\text{if } wRv \text{ then } \mathcal{M}, v \Vdash A).
\end{array}$$

Note that the definition of forcing for $\Box A$ agrees with the fact that $\Box A$ is defined as $\top \rhd A$, and that $\Box A$ gets the standard interpretation on frames. When $\mathcal{M}$ is clear from the context we write $w \Vdash A$ instead of $\mathcal{M}, w \Vdash A$. The formula A is *valid* in $\mathcal{M}$, notation $\mathcal{M} \models A$, if A is forced in all nodes in $\mathcal{M}$. The formula A is *valid* in a frame $\mathcal{F}$, notation $\mathcal{F} \models A$, if A is valid in all models with underlying frame $\mathcal{F}$.

A node v in a frame is called a *successor* of w if wRv, in which case w is called a *predecessor* of v. We use an abbreviation for the relation $(R;\preccurlyeq)$:

$$\tilde{R} \equiv_{def} (R;\preccurlyeq).$$

For a relation R we define $wR = \{v \mid wRv\}$. For a set U , we write $u \preccurlyeq U$ if for all $x \in U$, $u \preccurlyeq x$. We write '$x \preccurlyeq y_1, \ldots, y_n$' for '$x \preccurlyeq y_1 \wedge x \preccurlyeq y_2 \wedge \ldots \wedge x \preccurlyeq y_n$'. Similarly for other relations. A node v in a frame is *above* w if $w \preccurlyeq v$. In this case w is called *below* v.

Remark 1 The condition $(\preccurlyeq;R) \subseteq R$, included to guarantee persistence for formulas $A \rhd B$, may be weakened to

$$(\preccurlyeq;R) \subseteq (R;\preccurlyeq) \qquad (w \preccurlyeq w'Rv' \Rightarrow \exists u(wRu \preccurlyeq v')).$$

However we prefer to work with the simple condition where possible. For more discussion on this topic, see [7].

A property P on frames *corresponds* to a set T of formulas if for all frames $\mathcal{F}$: $\mathcal{F} \models T$ iff F has property P. Note that in this case we have

if $\vdash_{\mathsf{iT}} A$ then A is valid on all frames with property P.

When a frame $\mathcal{F}$ has a property P we say that $\mathcal{F}$ *is a P-frame*. We call $\mathcal{F}$ a $P_1 \ldots P_n$-frame when it has the properties $P_1 \ldots P_n$. If $\mathcal{C}$ is a class of frames, a logic iT is called *complete with respect to* $\mathcal{C}$ if

for all A: $\vdash_{\mathsf{iT}} A$ iff A valid on all frames in $\mathcal{C}$.

The logic iT is called *complete* if $\mathcal{C}$ is the class of frames to which iT corresponds.

3.3 Canonicity

Canonical models are defined in a similar manner as in classical modal logic. A set of formulas X is called *adequate* if it is closed under subformulas and contains $\top$ and $\bot$. A set of formulas Γ is called *X-saturated* with respect to a logic T if it is a consistent subset of X such that

- $\Gamma \vdash_T A$ implies $A \in \Gamma$, for all $A \in X$,
- $\Gamma \vdash_T A \vee B$ implies $A \in \Gamma$ or $B \in \Gamma$, for all $A \vee B \in X$.

If X is the set of all formulas, an X-saturated set is just called *saturated*. It can be easily seen that for any (finite) adequate set X and for any A for which $\nvdash A$, there is an (finite) X-saturated set Γ such that $\Gamma \nvdash A$. Note also that any $\Delta \subseteq X$ for which $\Delta \nvdash A$, can be extended to an X-saturated Γ such that $\Gamma \nvdash A$.

For any logic T, for any adequate set X, the *T-canonical X-model* is the model $(W, \preccurlyeq, R, V)$ defined as follows:

$$
\begin{array}{lll}
\multicolumn{3}{l}{W \text{ consists of the } X\text{-saturated sets (with respect to } \vdash_T)} \\
w \preccurlyeq v & \equiv_{def} & w \subseteq v \\
wRv & \equiv_{def} & \text{if } A_1, \ldots, A_n, B \in X, w \vdash_T A_1, \ldots, A_n \rhd B \text{ and} \\
 & & A_1, \ldots, A_n \in v, \text{ then } B \in v \\
w \Vdash p & \equiv_{def} & p \in w, \text{ for propositional variables } p \in X.
\end{array}
$$

Recall that $A_1, \ldots, A_n \rhd B$ is short for $(\bigwedge A_i) \rhd B$. Note that in the definition of R we take formulas $A \rhd B$ into account which do not belong to X.

To see that this indeed defines a model, see the completeness proof for iP. When X is the set of all formulas, we call the canonical X-model the *canonical model of T*. We call a logic iT *canonical* if the canonical model has the frame property to which the logic corresponds.

Note that in the iT-canonical frame in general $(R;\preccurlyeq) \subseteq R$ does not hold. On the other hand, if we restrict our language to $\Box$ and the connectives, the canonical models do satisfy $(R;\preccurlyeq) \subseteq R$. That $(R;\preccurlyeq) \subseteq R$ is too strong a requirement in the context of preservativity logic follows from the fact that $A \rhd B \to \Box(A \to B)$ is valid on such frames. This principle is not in the preservativity logic of HA, as the following deduction shoes.

$$
\begin{array}{lll}
\mathsf{HA} \vdash & \neg\Box\bot \rhd \Box\neg\Box\bot & (4p) \\
 & \Box(\neg\Box\bot \to \Box\neg\Box\bot) & \\
 & \Box(\neg\Box\bot \to \Box\bot) & (L) \\
 & \Box(\neg\neg\Box\bot) &
\end{array}
$$

Thus HA would derive $\Box\neg\neg\Box\bot$, and hence it would derive its own iterated inconsistency $\Box\Box\bot$, quod non.

3.4 Extendible properties

In this section we introduce a general construction to make certain extensions of sets of formulas. In many proofs to come we will extend certain sets of formulas to saturated sets with certain properties. It turns out that the way these extensions are made follow the same pattern. Therefore, we choose to define a general notion of extension which covers this.

Let iT be a preservativity logic and X an adequate set. A property $*(\cdot)$ on sets of formulas such that we have both

$$\begin{array}{ll} \text{for all } A \in X\text{:} & \text{if } *(x) \text{ and } x \vdash_{\mathsf{iT}} A, \text{ then } *(x \cup \{A\}) \\ \text{for all } (A \vee B) \in X\text{:} & \text{if } *(x \cup \{A \vee B\}), \text{ then} \\ & *(x \cup \{A\}) \text{ or } *(x \cup \{B\}), \end{array}$$

is called an iT-*extendible property* (*w.r.t.* X). If in addition it holds that

$$\text{for all } A \in X\text{:} \quad \text{if } *(x) \text{ and } y \vdash_{\mathsf{iT}} x \rhd A, \text{ then } *(x \cup \{A\})$$

then it is called an iT-*extendible* y-*successor property*. For a property $*$ such that $*(\Gamma)$ holds, the $*$-*extension of* Γ is the union $x = \bigcup x_i$ of sets x_i which are constructed as follows. Given an enumeration $B_0, B_1, \ldots$ of all formulas in X, in which every formula occurs infinitely often, we define

$$\begin{array}{lcl} x_0 & = & \Gamma \\ x_{i+1} & = & \left\{ \begin{array}{lll} x_i & \text{if} & \text{not } *(x_i \cup \{B_i\}) \\ x_i \cup \{B_i\} & \text{if} & *(x_i \cup \{B_i\}), B_i \text{ no disjunction} \\ x_i \cup \{B_i, E\} & \text{if} & *(x_i \cup \{B_i\}), B_i = C \vee D, \\ & & E = C \text{ if } *(x_i \cup \{B_i, C\}), \\ & & E = D \text{ otherwise.} \end{array} \right. \end{array}$$

Observe that $x \supset \Gamma$ is X-saturated. Thus, x is a node in the canonical X-model, and if Γ is a node in the canonical X-model as well, then $\Gamma \preccurlyeq x$. If in addition $*$ is an iT-extendible y-successor property, then also yRx holds in the iT-canonical X-model.

Remark 2 Note that for an iT-extendible w-successor property, the first requirement is redundant, because it follows from the third one. Namely, if $x \vdash A$ holds we have $\vdash_{\mathsf{iT}} (x \to A)$, and hence by Preservation Rule $\vdash_{\mathsf{iT}} x \rhd A$. Thus clearly $w \vdash_{\mathsf{iT}} x \rhd A$.

In the completeness proofs in the next chapters we often use extendible properties in the following way. Given a set Δ with a certain property, we want to extend it to a *saturated* set with this property, i.e. to a node in the canonical model with this property. There are two particular properties which often occur in this setting. The following lemma shows that these properties are extendible w-successor properties.

Lemma 3 For any logic iT containing iP, for any formula C and for all nodes w, v in the iT-canonical model, the following two properties are extendible w-successor properties:

$*(x)$ $w \nvdash_{\mathsf{iT}} x \rhd C$.

$\star(x)$ for all D: $w \vdash_{\mathsf{iT}} x \rhd D$ implies $D \in v$.

Proof We write $\vdash$ for $\vdash_{\mathsf{iT}}$. First we consider the property $*(\cdot)$. We have to show that

for all $A \in X$:	if $w \nvdash x \rhd C$ and $x \vdash A$, then $w \nvdash x, A \rhd C$
for all $(A \vee B) \in X$:	if $w \nvdash x, (A \vee B) \rhd C$, then $w \nvdash x, A \rhd C$ or
	$w \nvdash x, B \rhd C$
for all $A \in X$:	if $w \nvdash x \rhd C$ and $w \vdash x \rhd A$, then $w \nvdash x, A \rhd C$.

Recall that we write $x, A \rhd C$ for $(\bigwedge x \wedge A) \rhd C$. By Remark 2 we know that if the third requirement holds, so does the first. Therefore, it suffices to show that the last two requirements hold.

For the second requirement, assume $w \vdash x, A \rhd C$ and $w \vdash x, B \rhd C$. To show that $*(\cdot)$ satisfies the second requirement we have to prove that $w \vdash x, (A \vee B) \rhd C$. This follows immediately from *Dp*.

For the third requirement assume $w \vdash x \rhd A$ and $w \vdash x, A \rhd C$. We show that $w \vdash x \rhd C$, and this will show that $*(\cdot)$ satisfies the third requirement. By the Preservation Rule we have $\vdash x \rhd \bigwedge x$, which is short for $\vdash \bigwedge x \rhd \bigwedge x$. Therefore, we certainly have $w \vdash x \rhd \bigwedge x$. Thus by *P2* we have $w \vdash x \rhd (\bigwedge x \wedge A)$. Together with $w \vdash x, A \rhd C$ and *P1* this leads to $w \vdash x \rhd C$.

Consider the property $\star$. To show that $\star$ is an extendible w-successor property we have to prove that

for all $A \in X$:	if $\star(x)$ and $x \vdash A$, then
	(for all D: $w \vdash x, A \rhd D$ implies $D \in v$)
for all $(A \vee B) \in X$:	if $\star(x \cup \{A \vee B\})$, then
	(for all D: $w \vdash x, A \rhd D$ implies $D \in v$) or
	(for all D: $w \vdash x, B \rhd D$ implies $D \in v$)
for all $A \in X$:	if $\star(x)$ and $w \vdash x \rhd A$, then
	(for all D: $w \vdash x, A \rhd D$ implies $D \in v$).

By Remark 2, it suffices to show that the last two requirements hold.

For the second requirement, assume that neither $\star(x \cup \{A\})$ nor $\star(x \cup \{B\})$ holds. We prove that $\star(x \cup \{A \vee B\})$ does not hold. By assumption there are formulas C and D such that $C \notin v$ and $D \notin v$, and both $w \vdash x, A \rhd C$ and $w \vdash x, B \rhd D$. Clearly, both C and D imply $(C \vee D)$. Hence by Preservation Rule we have $\vdash C \rhd (C \vee D)$ and $\vdash D \rhd (C \vee D)$. Applying *P1* gives $w \vdash x, A \rhd (C \vee D)$ and $w \vdash x, B \rhd (C \vee D)$. Thus by *Dp* we have $w \vdash x, (A \vee B) \rhd (C \vee D)$. If $\star(x \cup \{A \vee B\})$ would hold, this would imply that $(C \vee D) \in v$. Since v is a node in the canonical model it is a saturated set. Therefore, this would imply that $C \in v$ or $D \in v$, which contradicts our assumption.

We show that the third requirement holds. Assume that $\star(x)$ and $w \vdash x \rhd A$ hold, and that we have $w \vdash x, A \rhd D$, for some D. We have to show that $D \in v$. The same reasoning as above for $*(\cdot)$, shows that we have $w \vdash x \rhd (\bigwedge x \wedge A)$. Therefore, $w \vdash x, A \rhd D$ implies $w \vdash x \rhd D$ by *P1*. The fact that $\star(x)$ holds, gives $D \in v$. **QED**

3.5 The Construction Method

We define a method, *the construction method*, to obtain from a given model a new one. This method is similar to the construction method in classical modal logic. The construction method is often used to obtain a completeness result with respect to some class of finite frames. Let $\mathcal{M} = (W, \preccurlyeq, R, V)$ be some canonical model, let X be an adequate set for which $A \rhd B \in X$ implies $\Box B \in X$. The method allows us to construct for any $w \in W$ a model $\mathcal{M}' = (W', \preccurlyeq', R', V')$ the domain of which consists of (copies of) nodes in W, which intuitively is the minimal set of nodes required to have w forcing the same formulae in X in the models $\mathcal{M}$ and $\mathcal{M}'$. We will restrict ourselves to a construction method for models that besides iP also satisfy *Lp* and *Mp*.

The construction proceeds as follows. We choose step by step, starting with w, a subset of W which will be the domain W' of our new model $\mathcal{M}'$. Note that the elements of W are sets of formulas. First, define

$$w^X_{\rhd} = \{A \rhd B \in X \mid A \rhd B \in w\}$$

$$w^X_{\not\rhd} = \{A \rhd B \in X \mid A \rhd B \notin w\}.$$

Similarly for $\to$ and $\Box$. We omit the superscript X when possible. Let $*$ denote the concatenation function on strings:

$$\langle x_1, \ldots, x_n \rangle * \langle y_1, \ldots, y_m \rangle = \langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle.$$

Put $\alpha_{\langle\rangle} = w$. Suppose $v = \alpha_\sigma$ is defined. We choose elements $\alpha_{\sigma * \langle A \to B \rangle}$ and $\alpha_{\sigma * \langle A \rhd B \rangle}$ in W, for all elements $(A \to B) \in \sigma_{\not\to}$, $A \rhd B \in \sigma_{\not\rhd}$.

The node $\alpha_{\sigma * \langle A \to B \rangle}$ is an element $u \in W$ such that $v \preccurlyeq u$, $A \in u$ and $B \notin u$. Note that such elements can always be found. The node $\alpha_{\sigma * \langle A \rhd B \rangle}$ is an element $u \in W$ such that vRu, $A \in u$, $B \notin u$ and $\Box B \in u$. Observe that u contains more boxed formulas than v, for in the presence of *Lp*, and hence of *4p* and *4*, vRu and $\Box C \in v$ implies that $\Box C \in u$. To prove that such a node u exists it suffices to show that in any canonical model for a logic containing iPLM, if $A \rhd B \notin v$ there exists a v-successor extension of $\{A, \Box B\}$ omitting $\{B\}$. Thus we have to see that $v \nvdash A, \Box B \rhd B$. Suppose not. Then we have, using Lp and Mp:

$$\begin{array}{lll} v & \vdash & A, \Box B \rhd B \\ & & (\Box B \to A \wedge \Box B) \rhd (\Box B \to B) \\ & & A \rhd (\Box B \to B) \\ & & A \rhd B. \end{array}$$

Define $W' = \{\sigma \mid \sigma \text{ is defined }\}$, and V via

$$\sigma \Vdash p \equiv_{def} \alpha_\sigma \Vdash p, \text{ for p} \in X.$$

We define the intuitionistic and the modal relation such that

$$\text{for all } A \in X, \text{ for all } \sigma \in W': \quad \alpha_\sigma \Vdash A \text{ iff } \sigma \Vdash A.$$

As the choice of the relations will differ from case to case we do not give any specific examples here besides the obvious one;

$$\begin{array}{lll} \sigma \preccurlyeq' \tau & \equiv_{def} & \alpha_\sigma \preccurlyeq \alpha_\tau \\ \sigma R' \tau & \equiv_{def} & \alpha_\sigma R \alpha_\tau. \end{array}$$

It is not difficult to see that this choice gives a model with the desired property, be it not always on a frame with the desired properties.

Remark 4 It is easy to see that W' is finite if X is. First note that by construction, a node (saturated set) $\sigma * \langle B \rhd C \rangle$ contains more boxed formulas (formulas of the form $\Box C$) that belong to X than σ. A node $\sigma * \langle B \to C \rangle$ contains more implications that belong to X than σ. Moreover, for a node $\tau = \sigma * \langle B \to C \rangle$ we have that $\alpha_\sigma \preccurlyeq \alpha_\tau$ holds in the canonical model, i.e. $\alpha_\sigma \subseteq \alpha_\tau$. Clearly, all the implications that have to be treated, i.e. all implications for which we possibly have to add a new node in the construction, belong to X. And similarly for boxed formulas and preservations. Therefore, in going from σ to $\sigma * \langle B \rhd C \rangle$ or $\sigma * \langle B \to C \rangle$ either the number of boxed formulas that have to be treated decreases, or it stays the same and the number of implications that have to be treated decreases. Finally, if there are no more boxed formulas to be treated this means that for all $\Box B \in X$, it holds that $\Box B \in \alpha_\sigma$. Hence for all $B \rhd C \in X$, we have $\Box C \in \alpha_\sigma$ and thus $B \rhd C \in \alpha_\sigma$. Therefore, if there are no more boxed formulas to be treated there are no formulas of the form $B \rhd C$ to be treated either. Since the preservations and implications that belong to X are the only formulas that have to be treated in the construction method, this shows that the method is finite if X is.

3.6 Useful lemma's

In this section we prove two lemma's that state basic properties of preservativity logic. We will often use these properties in the rest of the paper without mentioning it.

Lemma 5

(*i*) for any logic iT containing iP^-: $\vdash_{\mathsf{iT}} A$ implies $\vdash_{\mathsf{iT}} \Box A$.

(*ii*) $\vdash_{\mathsf{iP}^-} \Box(A \to B) \to A \rhd B$ and $\vdash_{\mathsf{iP}^-} A \rhd B \to (\Box A \to \Box B)$.

Proof (*i*) Observe that $\vdash_{\mathsf{iT}} (A \to B)$ implies $\vdash_{\mathsf{iT}} \top \to (A \to B)$. Hence by the Preservation Rule $\vdash_{\mathsf{iT}} \top \rhd (A \to B)$, which is equivalent to $\Box(A \to B)$.

(*ii*) The second implication follows immediately from $P1$, using the fact that $\Box A$ is defined as $\top \rhd A$. The following derivation proofs the first implication.

We have

$\vdash_{\mathsf{iP}^-}$	$\Box(A \to B) \leftrightarrow \top \rhd (A \to B)$		(1)
	$A \rhd \top$	(Preservation Rule)	(2)
	$\Box(A \to B) \to A \rhd (A \to B)$	(1)($P1$)	(3)
	$A \rhd A$	(Preservation Rule)	(4)
	$\Box(A \to B) \to A \rhd (A \wedge (A \to B))$	(3)(4)($P2$)	(5)
	$(A \wedge (A \to B)) \rhd B$	(Preservation Rule)	(6)
	$\Box(A \to B) \to A \rhd B.$	(5)(6)($P1$)	

This completes the proof. **QED**

The next lemma will only be used when we treat Visser' Scheme in Section 8

Lemma 6

(*i*) $(A)(B)$ implies $(A \to B)$, and $(A)(B) \vee (A)(C)$ implies $(A)(B \vee C)$.

(*ii*) For $A = (\bigwedge_{i=1}^{n} (A_i \to B_i))$, for all m, we have

$$\vdash_{\mathsf{iPV}} (A \to A_{n+1} \vee \ldots \vee A_{n+m}) \rhd (A)(A_1, \ldots, A_{n+m}).$$

Proof (*i*) Left to the reader; for the first statement, use induction on B, for the second statement, use the first one.

(*ii*) Use induction on m. For $m = 1$, observe that $(A \to A_{n+1})$ is equivalent to $(A \to A_{n+1} \vee \bot)$. We leave the rest of this case to the reader. For $m = 2$

the statement holds by the definition of Visser's Scheme. For $m > 2$, we let $C = A_{n+2} \vee \ldots \vee A_{n+m}$. It is clear that

$$\vdash_{\mathsf{iPV}} (A \to A_{n+1} \vee \ldots \vee A_{n+m}) \rhd (A \to A_{n+1} \vee C).$$

By the definition of Visser's Scheme we have that

$$\vdash_{\mathsf{iPV}} (A \to A_{n+1} \vee C) \rhd (A)(A_1, \ldots, A_{n+1}, C).$$

Note that because C is a disjunction it holds that $(A)(C) = (A \to C)$. By induction hypothesis we have

$$\vdash_{\mathsf{iPV}} (A \to C) \rhd (A)(A_1, \ldots, A_n, A_{n+2}, \ldots, A_{n+m}).$$

We leave it to the reader to check that, using the Disjunctive Principle and $P1$, all this leads to the desired result,

$$\vdash_{\mathsf{iPV}} (A \to A_{n+1} \vee \ldots \vee A_{n+m}) \rhd (A)(A_1, \ldots, A_{n+m}).$$

QED

4 The semantical base preservativity logic

In this section we show that the frames defined in Subsection 3.2 are exactly the frames we need for the semantical base preservativity logic iP.

Proposition 7 $\vdash_{\mathsf{iP}} A$ iff A is valid on all finite frames.

Proof We treat the direction from right to left. Suppose $\mathsf{iP} \nvdash A$. We have to show that there is a model for iP which does not force A. Let X be a finite adequate set containing A. We prove that the canonical X-model is such a model. Observe that the canonical X-model is indeed a model, i.e. $(\preccurlyeq;R) \subseteq R$, and that every model satisfies the axioms of iP. It is easy to see that there is an X-saturated set (hence a node in this model) which does not contain A. Therefore, to see that A is not valid on this model it suffices to show that

$$\forall B \in X \forall w: \ B \in w \text{ iff } w \Vdash B.$$

This can be easily shown by formula induction. We only treat implication and preservation for the direction from right to left. Suppose $B = (C \to D)$ and $B \notin w$. If $w \cup \{C\}$ would derive D, then also $w \vdash (C \to D)$. Thus $w \cup \{C\} \nvdash D$. This implies that $w \cup \{C\}$ is consistent. Let v be an X-saturated extension of $w \cup \{C\}$ which does not derive D. Then $w \preccurlyeq v$, $v \Vdash C$ and $v \nVdash D$ hold, hence $w \nVdash (C \to D)$.

Now suppose $B = C \rhd D \notin w$. It suffices to construct an X-saturated set v such that wRv and $C \in v$ while $D \notin v$. Consider the property

$$*(x) \qquad w \nvdash x \rhd D.$$

By Lemma 3, $*(\cdot)$ is an iP-extendible w-successor property. Note that $*(C)$ holds. Any $*$-extension of $\{C\}$ can be taken for v. The fact that v does not contain D follows from the definition of a $*$-extension. **QED**

5 The principle $4p$

We show that iP4 is complete with respect to the class of gathering frames. We call a model or a frame *gathering* if it satisfies

$(gathering) \qquad wRvRu \to v \preccurlyeq u.$

Proposition 8

- (i) The principle $4p$ corresponds to gatheringness.
- (ii) The logic iP4 is canonical.
- (iii) $\vdash_{\mathsf{iP4}} A$ iff A valid on all finite gathering frames.

Proof The three statements are easy to prove. We leave (i), (ii) and the direction from left to right of (iii) to the reader. For the the direction from right to left of the last statement it suffices to observe that for any finite adequate set which contains $\Box B$ for any nonboxed $B \in X$, the iP4-canonical X-model is gathering. **QED**

6 Löb's Preservativity Principle

We show that Löb's Preservativity Principle *Lp* corresponds to the gathering conversely well-founded frames. We call a frame *conversely well-founded* if the modal relation on the frame is conversely well-founded. We do not know if iPL is also complete with respect to these frames. If we restrict ourselves to the language without $\rhd$ but with $\Box$, then Löb's Principle is complete with respect to the gathering conversely well-founded frames [3]. However, the 'trick' used in this completeness proof for iL breaks down for iPL in the absence of the principle *Mp*. The completeness proof for iL is similar to the one in classical logic. We have included it for completeness' sake.

Classically as well as intuitionistically we have that the principles 4 is derivable from Löb's Principle. In analogy with that we have

$$\vdash_{\mathsf{iPL}} L \text{ and } \vdash_{\mathsf{iPL}} 4p \text{ and } \vdash_{\mathsf{iP(4p\oplus L)}} Lp.$$

The first deduction is trivial. The second one has a similar proof as the above mentioned analogue:

$$\begin{array}{ll} \vdash_{\mathsf{iPL}} & A \to (\Box(\Box A \wedge A) \to \Box A \wedge A)) \\ & A \rhd (\Box(\Box A \wedge A) \to \Box A \wedge A)) \\ & A \rhd (\Box A \wedge A) \\ & A \rhd \Box A \end{array}$$

The third derivation runs as follows.

$$\begin{array}{ll}
\vdash_{\mathsf{iL}} & \Box(\Box(\Box A \to A) \to \Box A) \\
\vdash_{\mathsf{iL}} & \Box(\Box A \to A) \rhd \Box A \\
\vdash_{\mathsf{iP4}} & (\Box A \to A) \rhd \Box(\Box A \to A) \wedge (\Box A \to A) \\
\vdash_{\mathsf{iP(4p \oplus L)}} & (\Box A \to A) \rhd (\Box A \wedge (\Box A \to A)) \\
\vdash_{\mathsf{iP(4p \oplus L)}} & (\Box A \to A) \rhd A
\end{array}$$

Lemma 9 The principle *Lp* corresponds to gatheringness plus converse well-foundedness of the modal relation.

Proof Left to the reader. **QED**

7 Montagna's Principle

We show that Montagna's Principle *Mp* corresponds to the *Mp*-property defined as

(*Mp-property*) $\quad wRv \preccurlyeq u \to \exists x(wRx \wedge v \preccurlyeq x \preccurlyeq u \wedge x\tilde{R} \subseteq u\tilde{R}).$

Then we prove that iPM is canonical.

If a principle corresponds to a frame property in which expressions like $x\tilde{R} \subseteq y\tilde{R}$ occur, like Montagna's Principle, then for a proof of its canonicity we need to know what $x\tilde{R} \subseteq y\tilde{R}$ means on the canonical model, i.e. in terms of saturated sets. This is the content of the following lemma. For the definition of $w_\Box$ see Section 3.5.

Lemma 10 In any canonical model: $v\tilde{R} \subseteq w\tilde{R}$ iff $w_\Box \subseteq v_\Box$.

Proof First the direction from left to right. Suppose $\Box A \in w$ while $\Box A \notin v$. By Lemma 3, the property

$*(x) \qquad v \nvdash x \rhd A,$

is an extendible v-successor property. Note that $*(\{\top\})$ holds, and let u be any $*$-extension of $\{\top\}$. Clearly, vRu, and $A \notin u$ hence $w(R;\preccurlyeq)u$ cannot hold.

For the other direction, assume $w_\Box \subseteq v_\Box$ and vRu. We have to construct a node u' such that $wRu' \subseteq u$. By Lemma 3, the property

$*(x) \qquad$ for all A: $w \vdash x \rhd A$ implies $A \in u$,

is an extendible w-successor property. Clearly, $*(\{\top\})$ holds. Therefore, any $*$-extension of $\{\top\}$ will do for u'. **QED**

Proposition 11

(i) The principle Mp corresponds to the the Mp-property.

(ii) The logic iPM is canonical.

Proof We prove part (ii) of the proposition and leave (i) to the reader. Consider $wRv \preccurlyeq u$ in the iPM-canonical model. Define the property

*(x) for all A: $w \vdash x \rhd A$ implies $A \in u$.

It is easy to see that $*(\cdot)$ is an iPM-extendible w-successor property. Thus if $*(v \cup u_\Box)$ holds, then any $*$-extension of $v \cup u_\Box$ is a node x such that wRx (Section 3.4) and $v \preccurlyeq x \preccurlyeq u$ and $x\tilde{R} \subseteq u\tilde{R}$ hold (Lemma 10). Thus it remains to show that $*(v \cup u_\Box)$ holds. This follows from the fact that for all finite subsets $\Gamma \subseteq v$ and $\Delta \subseteq u_\Box$, and for all B we have that $w \vdash \Gamma, \Delta \rhd B$ implies $B \in u$. Therefore, suppose that for some such Γ, Δ, B it does hold that $w \vdash \Gamma, \Delta \rhd B$. Replace Δ by the equivalent $\Box A$ where $A = (\bigwedge\{C \mid \Box C \in \Delta\})$. Then

$$\begin{array}{lll} w & \vdash & \Gamma, \Box A \rhd B \\ & & (\Box A \to \bigwedge \Gamma \wedge \Box A) \rhd (\Box A \to B) \\ & & \Gamma \rhd (\Box A \to B). \end{array}$$

This implies that $(\Box A \to B) \in v$, whence that $B \in u$. **QED**

8 Visser's Scheme

For the frame characterization of Visser's Scheme we need the notion of a tight predecessor. We will first give the intuition behind it. Let $\bar{v}, \bar{u}$ range over finite sets of nodes, and write e.g. $x \preccurlyeq \bar{v}$ for 'for all $v \in \bar{v}(x \preccurlyeq v)$', and similarly for R and $\tilde{R}$. Consider two main instances of Visser's Scheme:

$$(2) \qquad (\bigwedge_{i=1}^{n} (A_i \to B_i) \to A_{n+1} \vee A_{n+2}) \rhd (\bigvee_{j=1}^{n+2} (\bigwedge_{i=1}^{n} (A_i \to B_i) \to A_j))$$

$$(3) \qquad (\bigvee_{i=1}^{n} \neg\neg\Box A_i) \rhd (\bigvee_{i=1}^{n} \Box A).$$

The first principle arises when we restrict Visser's Scheme to pure propositional variables, the second one if we restrict it to boxed formulas and $\bot$. These two principles are related to two parts of the frame characterization of Visser's Scheme. It is easy to see that (3) is valid on frames which satisfy

$$(4) \quad wRvR\bar{u} \to \exists x\ (v \preccurlyeq x \wedge x\tilde{R}\bar{u} \wedge \neg\exists y(x \prec y)).$$

Formula (2) holds on frames which satisfy

$$(5) \quad wRv \preccurlyeq \bar{v} \to \exists x(v \preccurlyeq x \preccurlyeq \bar{v} \wedge \forall y \succ x \exists z \in \bar{v}(z \preccurlyeq y)).$$

We show this for $n = 3$. If for nodes wRv in such a frame we have $v \Vdash ((p_1 \to q) \to p_2 \vee p_3)$, and not $v \Vdash (p_1 \to q) \to p_j$ then there are nodes $u_1, u_2, u_3 \succcurlyeq v$ that force $(p_1 \to q)$ and such that u_i does not force p_i. Let $\bar{v} = \{u_1, u_2, u_3\}$ and let x be the node such that $v \preccurlyeq x \preccurlyeq \bar{v}$ and such that for all $y \succ x$, it holds that $u_i \preccurlyeq y$ for some i. Observe that x forces $(p_1 \to q)$ but that it does not force $(p_2 \vee p_3)$, contradicting the assumption that v forces $((p_1 \to q) \to p_2 \vee p_3)$. For arbitrary n the reasoning is the same.

The combination of the two frame properties above leads to the frame property with respect to which Vp is complete. However, Vp does correspond to a weaker property, which will be called the Vp^∞-property. This is best illustrated by the discussion on formula (2) above. Namely, one can weaken (5) by requiring that all nodes y above x are either below all nodes in $\bar{v}$ or above at least one node in $\bar{v}$:

$$wRv \preccurlyeq \bar{v} \to \exists x(v \preccurlyeq x \preccurlyeq \bar{v} \wedge \forall y \succcurlyeq x(y \preccurlyeq \bar{v} \vee \exists z \in \bar{v}(z \preccurlyeq y))).$$

The same reasoning as above shows that Vp is still valid on frames with this property.

8.0.1 Tight predecessors (in modal logic)

We say that a node x in K is a *semi-tight predecessor of* $\bar{v}$ *holding* $\bar{u}$, if

$$x \preccurlyeq \bar{v} \wedge x\tilde{R}\bar{u} \wedge \forall y \succ x(\exists z \in \bar{v}(z \preccurlyeq y) \vee (y \preccurlyeq \bar{v} \wedge y\tilde{R}\bar{u})).$$

It is called a *tight predecessor* if in addition there holds the stronger

$$v\tilde{R} \subseteq x\tilde{R} \wedge \forall y \succ x \exists z \in \bar{v}(z \preccurlyeq y).$$

We call a frame (model) a Vp^∞*-frame (model)* if it has the Vp^∞-property:

(Vp^∞*-property*) for all finite sets of nodes $\bar{v}, \bar{u}$: $wRv \wedge v \preccurlyeq \bar{v} \wedge vR\bar{u} \to$
$\exists x \succcurlyeq v$(x is a semi-tight predecessor of $\bar{v}$ holding $\bar{u}$).

An inspection of the Vp^∞-property will convince the reader that there are hardly any finite models that have this property.

Observe that if one reads tight for semi-tight in the Vp^∞-property, it expresses (4) if $\bar{v}$ is empty, and (5) if $\bar{u}$ is empty.

Proposition 12

- (i) Visser's Scheme corresponds to the Vp^∞-property.
- (ii) The logic iPV is canonical.
- (iii) The canonical model iPV satisfies the following property which is stronger than the Vp^∞-property:

  (Vp*-property*) $wRv \preccurlyeq v_1, \ldots, v_m \to$
  $\exists x(v \preccurlyeq x \preccurlyeq v_1, \ldots, v_m \wedge v\tilde{R} \subseteq x\tilde{R} \wedge \forall y \succ x \exists i(v_i \preccurlyeq y))$.

  (Recall that in this case x is called *a tight predecessor of* $v_1, \ldots, v_n$ *for* v.)

Proof We often use part (i) of Lemma 6 without mentioning. (i) First we show that Vp holds on a Vp^∞-frame. Suppose wRv and $v \nVdash (A)(D_1,\ldots,D_{n+2})$ hold, for some $A = \bigwedge_{i=1}^n (D_i \to E_i)$, on some Vp^∞-frame. We show that $v \nVdash (A \to D_{n+1} \vee D_{n+2})$. Assume $D_i = B_i \wedge \Box C_i$, where B_i is not of the form $\Box C$. From the assumption it follows that $v \nVdash (A \to B_i) \wedge \Box C_i$, whence either $v \nVdash \Box C_i$ or $v \nVdash (A \to B_i)$. Therefore, there are finite sets of nodes $\bar{v}$ and $\bar{u}$ such that for all i we have that *either* there is a node $x \in \bar{u}$ with vRx and $x \nVdash C_i$ *or* there is a node $x \in \bar{v}$ with $v \preccurlyeq x$, $x \Vdash A$ and $x \nVdash B_i$. Let $\bar{v}$ and $\bar{u}$ be a smallest pair of sets with these properties. Let $u \succcurlyeq v$ be a semi-tight predecessor of $\bar{v}$ holding $\bar{u}$. We show that $u \Vdash A$ and $u \nVdash (D_{n+1} \vee D_{n+2})$. This will prove that $v \nVdash (A \to D_{n+1} \vee D_{n+2})$.

To see that that $u \nVdash (D_{n+1} \vee D_{n+2})$, note that for $i = n+1, n+2$ we have that either there is node $x \in \bar{u}$ with $x \nVdash C_i$ *or* there is a node $x \in \bar{v}$ with $x \Vdash A$ and $x \nVdash B_i$. In the first case we have that $u\tilde{R}x$, and hence $u \nVdash \Box C_i$. In the second case we have that $u \preccurlyeq x$ and thus $u \nVdash (A \to B_i)$. Hence in both cases we can conclude $u \nVdash D_i$. To see that $u \Vdash A$, consider $y \succcurlyeq u$. Then either $y \preccurlyeq \bar{v}$ and $y\tilde{R}\bar{u}$, or $z \preccurlyeq y$ for some $z \in \bar{v}$. In the last case y forces A because all nodes in $\bar{v}$ force A. In the first case, it suffices to show that for all $i \leq n$, we have that $y \nVdash B_i \wedge \Box C_i$. Note that for all $i \leq n$ either there is node $x \in \bar{u}$ with $x \nVdash C_i$ *or* there is a node $x \in \bar{v}$ with $x \Vdash A$ and $x \nVdash B_i$. In the first case we have that $y\tilde{R}x$ holds, and whence $y \nVdash \Box C_i$. In the second case we have $y \preccurlyeq x$, and therefore $y \nVdash B_i$. Hence in both cases we can conclude $y \nVdash D_i$.

For the other part of (i), assume that a frame $\mathcal{F}$ does not have the Vp^∞-property. Thus there are nodes wRv and finite sets $v \preccurlyeq \bar{v}$ and $v\tilde{R}\bar{u}$ such that for all $x \succcurlyeq v$, x is no semi-tight predecessor of $\bar{v}$ holding $\bar{u}$. Thus

(6) $\quad \forall x \succcurlyeq v(\neg(x\tilde{R}\bar{u}) \vee \forall y \succ x(\forall z \in \bar{v}(z \not\preccurlyeq v) \wedge (y \not\preccurlyeq \bar{v} \vee \neg(y\tilde{R}\bar{u}))).$

First, the case that $\bar{v}$ is empty. Then (6) becomes $\forall x \succcurlyeq v \neg(x\tilde{R}\bar{u})$. Suppose $\bar{u} = u_1,\ldots,u_m$. Define the valuation

$$x \Vdash r_i \equiv_{def} x \not\preccurlyeq u_i.$$

Let $A = \bigwedge_{i=1}^m \neg\Box r_i$. We leave it to the reader to verify that $v \Vdash (A \to \bot)$, but $v \nVdash (A)(\Box r_1,\ldots,\Box r_m,\bot)$.

Now we consider the case that $\bar{v}$ contains at least one node. Note that in that case it has to contain at least two nodes, otherwise this single node in $\bar{v}$ is a semi-tight predecessor of $\bar{v}$ holding $\bar{u}$. Suppose $\bar{v} = v_1,\ldots,v_{n+2}$. For the same reason, there have to be at least two nodes among them, say v_{n+1}, v_{n+2}, such that neither $v_{n+1} \preccurlyeq v_{n+2}$ nor $v_{n+2} \preccurlyeq v_{n+1}$. Suppose $\bar{u} = u_1,\ldots,u_m$. Define the following valuation,

$$\begin{array}{lll} x \Vdash p_i & \equiv_{def} & x \not\preccurlyeq v_i \\ x \Vdash q & \equiv_{def} & v_i \preccurlyeq x, \text{ for some } i \leq n+2 \\ x \Vdash r_i & \equiv_{def} & x \not\preccurlyeq u_i. \end{array}$$

Let

$$A = \bigwedge_{i=1}^n (p_i \to q) \wedge \bigwedge_{i=1}^m \neg\Box r_i.$$

We leave it to the reader to verify that under this valuation we have $v \not\Vdash (A)(p_1, \ldots, p_{n+2}, \Box r_1, \ldots, \Box r_m)$, since $v \not\Vdash \Box r_i$ and $v_i \Vdash A$ but $v_i \not\Vdash p_i$. We show that $v \Vdash (A \to p_{n+1} \vee p_{n+2})$. Therefore, consider $v \preccurlyeq x \Vdash A$. Thus $x\tilde{R}\bar{u}$. By (6) there exists a node $y \succ x$ such that $\forall z \in \bar{v}(z \not\preccurlyeq v) \wedge (y \not\preccurlyeq \bar{v} \vee \neg(y\tilde{R}\bar{u}))$. Since $y \Vdash A$, in particular $y \Vdash \neg\Box r_i$. Thus $y\tilde{R}\bar{u}$. This implies $\forall z \in \bar{v}(z \not\preccurlyeq v) \wedge y \not\preccurlyeq \bar{v}$. If $y \not\preccurlyeq v_i$, for some $i \leq n$, then $y \Vdash p_i$. Since $y \Vdash A$, then $y \Vdash q$. But this contrtadicts the fact that $\forall z \in \bar{v}(z \not\preccurlyeq v)$. Therefore, $y \not\preccurlyeq \bar{v}$ implies that $y \not\preccurlyeq v_{n+1}$ or $y \not\preccurlyeq v_{n+2}$. Thus the same holds for x, and this shows that $x \Vdash p_{n+1} \vee p_{n+2}$.

(ii) This follows from (iii).

(iii) Consider nodes wRv, $v \preccurlyeq v_1, \ldots, v_m$ in the iPV-canonical model. Let $\hat{v}$ denotes $v_1 \cap \ldots \cap v_m$. First note that in general $\hat{v}$ is not saturated. Therefore, it is not necessarily a node in the canonical model. Let

$$\Delta = \{(E \wedge \Box E' \to F) \mid F \in \hat{v} \wedge (E \notin \hat{v} \vee \Box E' \notin v)\}.$$

(Thus in particular the implications $(E \to F)$ and $(\Box E \to F)$, for which $F \in \hat{v}$ and respectively $E \notin \hat{v}$ and $\Box E \notin v$, are in Δ.) Note that $\Delta \subseteq \hat{v}$. Let $*(\cdot)$ be the property

$$*(x) \quad x \vdash A_1 \vee \ldots \vee A_m \vee \Box B_1 \vee \ldots \vee \Box B_n \text{ implies } \exists i\ (A_i \in \hat{v} \text{ or } \Box B_i \in v).$$

Clearly, $*(\cdot)$ is an extendible property (Section 3.4). We show that $*(v \cup \Delta)$ holds. Let $C = A_1 \vee \ldots \vee A_m \vee \Box B_1 \vee \ldots \vee \Box B_n$ and suppose $v \cup \Delta \vdash C$. This implies that there is a conjunct $D = \bigwedge_{i=1}^{k}(E_i \to F_i)$ of implications in Δ, such that $v \vdash (D \to C)$. Thus $(D \to C) \in v$, because v is saturated. Since

$$(D \to C) \rhd (D)(E_1, \ldots E_k, A_1, \ldots, A_m, \Box B_1, \ldots, \Box B_n),$$

also $(D)(E_1, \ldots E_k, A_1, \ldots, A_m, \Box B_1, \ldots, \Box B_n) \in v$. From the construction of Δ it follows that v does not contain any of $(D \to E) \wedge \Box E'$, for $E_i = E \wedge \Box E'$. Therefore v contains either $(D \to A_i)$ or $\Box B_i$ for some i. This proofs that $*(v \cup \Delta)$ holds. Let u be the $*$-extension of $v \cup \Delta$. As described in Section 3.4, u is saturated. We show that u is a semi-tight predecessor of $\bar{v}$ holding $\bar{u}$. Clearly, $v \preccurlyeq u \preccurlyeq v_1, \ldots, v_n$ holds, and by Lemma 10 we have $v\tilde{R} \subseteq u\tilde{R}$.

It remains to show that

$$\forall y \succ u \exists i (v_i \preccurlyeq y).$$

Arguing by contradiction, suppose $u \prec u'$ for some saturated set u' and assume that no v_i is contained in u'. For all $i \leq m$, we choose a formula $A_i \in v_i$ outside u'. Then the formula $(A_1 \vee \ldots \vee A_m)$ is in $\hat{v}$ but not in u'. From the construction of u, and the fact that u' is a superset of u, it follows that there is a formula $(E \wedge \Box E') \in u'$ such that either $E \notin \hat{v}$ or $E' \notin \hat{u}$. Now $(E \wedge \Box E' \to A_1 \vee \ldots \vee A_m)$ is an element of Δ, thus also of u. Hence $(A_1 \vee \ldots \vee A_n)$ should be in u', a contradiction. This proves that iPV is canonical. **QED**

9 The completeness iPH

First we sketch the idea of the completeness proof for iPH and then we treat the proof in full detail. Recall that iPH (Section 2) is the logic we conjecture to be the preservativity logic of HA.

9.0.2 Proof sketch

For formulas A that are not derivable in iPH we have to show that there is a model that refutes A and which has a gathering conversely well-founded *Mp Vp*-frame. To construct such a model we use the construction method (Section 3.5) with respect to a certain finite adequate set X. As expected, the resulting model will in general be infinite, since most of the frames which validate Visser's Scheme are not finite (Section 8). We use four subconstructions $\beta, \delta, \zeta, \xi$. Each of them expands a frame by adding nodes from the canonical model to it in an adequate way. We will explain how they select these nodes. To ensure that the new nodes x have certain properties we require that α_x has the corresponding properties in the canonical model. For example, if we demand $x \preccurlyeq \sigma$, then we choose α_x in such a way that $\alpha_x \preccurlyeq \alpha_\sigma$ holds in the canonical model. If $x\tilde{R} \subseteq \sigma\tilde{R}$ is the desired property, we demand that $(\alpha_\sigma)_\Box \subseteq (\alpha_x)_\Box$. Note that by Lemma 10 this is equivalent with $\alpha_x\tilde{R} \subseteq \alpha_\sigma\tilde{R}$.

The construction β choses nodes $\sigma * \langle B \to C\rangle$ and $\sigma * \langle B \rhd C\rangle$ as is usual in the construction method. In combination with δ, the construction ζ ensures that the final frame has the *Mp*-property (Section 7): for nodes $\sigma R \tau \preccurlyeq \tau'$ it constructs a node $a = \sigma * \langle m, \tau, \tau'\rangle$ such that in the final model $\sigma R a$ and $\tau \preccurlyeq a \preccurlyeq \tau'$ and $a\tilde{R} \subseteq \tau'\tilde{R}$ hold.

In combination with δ, the construction ξ ensures that the final frame has the *Vp*-property (Section 8): for nodes $\sigma R \tau \preccurlyeq \tau_1, \ldots, \tau_n$ it constructs a node $a = \tau * \langle v, \tau_1, \ldots, \tau_n\rangle$ such that in the final model $v \preccurlyeq a$ and a is a tight predecessor of $\tau_1, \ldots, \tau_n$ for τ.

The construction δ is an addition to both ζ and ξ. If we want $\pi\tilde{R} \subseteq \pi'\tilde{R}$ to hold in the final model and we add a node $\pi\tilde{R}\pi''$, then δ constructs a node $a = \pi' * \langle m, \pi''\rangle$ such that $\pi' R a \preccurlyeq \pi''$. Therefore, $\pi'\tilde{R}\pi''$ will hold. The discussion above shows that we have to ensure that $\pi\tilde{R} \subseteq \pi'\tilde{R}$ holds in the following cases: $\pi = \sigma * \langle m, \tau, \pi'\rangle$ or $\pi = \sigma * \langle m, \pi'\rangle$ or $\pi' = \pi * \langle v, \tau_1, \ldots, \tau_n\rangle$.

The following tricks are used in the construction in order to guarantee that no unnecessary nodes are selected. The reader interested in the construction but not in the complications which arise from this attempt for efficiency can skip these details.

If we want to guarantee that $\pi\tilde{R} \subseteq \pi'\tilde{R}$ we do not have to add a node $\pi' * \langle m, \pi''\rangle$ for all the nodes π'' with $\pi\tilde{R}\pi''$. For example, it could be that $\pi'\tilde{R}\pi$ already holds. Therefore, we will define a property $*(\pi, \pi', \pi'')$ that holds exactly when we want $\pi\tilde{R} \subseteq \pi'\tilde{R}$ to hold, and $\pi\tilde{R}\pi''$ holds but not $\pi'\tilde{R}\pi''$. For a similar reason, we define properties $\star(\sigma, \tau, \tau')$ and $\circ(\tau, \tau_1, \ldots, \tau_n)$ which holds exactly when we have to add nodes $\sigma * \langle m, \tau, \tau'\rangle$ or $\tau * \langle v, \tau_1, \ldots, \tau_n\rangle$ respectively. To recognize if $*$ or $\star$ hold we use a function γ. If for example $\sigma R\tau \preccurlyeq \tau'$ holds but $\tau = \sigma * \langle m, \tau' \rangle$, then we do not have to add a node $\sigma * \langle m, \tau, \tau' \rangle$ since τ itself will have the desired properties. The same holds for instance in the case that $\tau = \sigma * \langle m, \pi, \pi' \rangle$ and $\pi' = \sigma' * \langle m, \tau' \rangle$. We let the function γ cover all these cases by defining $\gamma(\sigma)$ inductively as: if $\sigma = \sigma'' * \langle m, \tau, \sigma' \rangle$ or $\sigma = \sigma'' * \langle m, \sigma' \rangle$, then $\gamma(\sigma) = \gamma(\sigma')$, and $\gamma(\sigma) = \sigma$ otherwise.

We use one device more to lower the number of nodes we have to construct. We let R and $\preccurlyeq''$ be one-step relations: intuitively, we have $\sigma R\tau$ if there is no $\sigma R\tau' R\tau$, and similarly for $\preccurlyeq''$. We let $\preccurlyeq$ be the transitive closure of $\preccurlyeq''$ and define R^* and $\preccurlyeq^*$ as the minimal extensions of R and $\preccurlyeq$ for which R^* is gathering and $\preccurlyeq^*$ is a partial order and $(\preccurlyeq^*;R^*) \subseteq R^*$ holds (Lemma 15). For example, when $\sigma R\sigma'(\preccurlyeq'';R)\tau$ we put $\sigma R^*\sigma' R^*\tau$ and $\sigma' \preccurlyeq^* \tau$. The relations R^* and $\preccurlyeq^*$ will be the modal and intuitionistic relation in our final model. The use of R and $\preccurlyeq''$ is best illustrated by an example. Suppose we have to define a node $\sigma * \langle m, \tau, \tau' \rangle$, and $\sigma R^*\tau$ holds and $\sigma R\tau$ does not hold. It follows from the definition of R^* that there is a node $\sigma' R\tau$. We only construct the node $\sigma' * \langle m, \tau, \tau' \rangle$ and observe that also $\sigma R^*\sigma' * \langle m, \tau, \tau' \rangle$. Hence $\sigma' * \langle m, \tau, \tau' \rangle$ has the desired properties of $\sigma * \langle m, \tau, \tau' \rangle$ in the final model. Therefore, the latter node does not have to be constructed.

Finally, in construction δ we select the nodes $a = \pi' * \langle m, \pi'' \rangle$ in such a way that $(\alpha_{\pi''})_\Box = (\alpha_a)_\Box$. This allows us to ensure that $\pi''\tilde{R} = a\tilde{R}$ in the final model. And that guarantees that for the situation $\pi' Ra \preccurlyeq \pi''$, which arises from the definition of a, we do not again have to add a node $\pi' * \langle m, a, \pi'' \rangle$, since the node a has the same properties. Lemma 13 shows that we can choose α_a as desired. This completes the informal discussion of the completeness proof for iPH.

Lemma 13 In any canonical model of a logic containing *Mp* it holds that

$$\text{if } w_\Box \subseteq v_\Box \wedge vRu \text{ then } \exists u'(wRu' \preccurlyeq u \wedge u'_\Box = u_\Box).$$

(By Lemma 10 this is equivalent with the property that if $w_\Box \subseteq v_\Box$ and vRu, then there exists a node u' such that $wRu' \preccurlyeq u$ and $u'\tilde{R} = u\tilde{R}$.)

Proof Let $*(\cdot)$ be the property

*(x) for all A: $w \vdash x \rhd A$ implies $A \in u$.

In Lemma 3 we have shown that $*$ is an extendible w-successor property (in the lemma it is denoted with $\star$) and that $*(u_\Box)$ holds. Let u' be the $*$-extension of $u_\Box$. Clearly, u' has the desired properties. **QED**

Remark 14 In any gathering model,

$$\text{if } w'Rw \preccurlyeq v_1 Rv_2 \preccurlyeq v_3 Rv_4 \preccurlyeq \ldots v_n \text{ then, for all } i,\ w \preccurlyeq v_i.$$

This can be easily seen, using the gatheringness and the fact that $(\preccurlyeq;R) \subseteq R$.

9.0.3 The relations R^* and $\preccurlyeq^*$

Let R and $\preccurlyeq$ respectively be a binary relation and a partial order on a finite set W. We define relations $\preccurlyeq^*$ and R^* which are the minimal extensions of $\preccurlyeq$ and R such that R^* is gathering and $(\preccurlyeq^*;R^*) \subseteq R^*$ holds. The idea behind these extensions is given by Remark 14. We define $\preccurlyeq^*$ and R^* via

$$\begin{aligned} wR^*v \quad &\equiv_{def} \quad \exists x(w \preccurlyeq xRv) \vee \exists x_1 \ldots x_n yy'(w \preccurlyeq y \preccurlyeq x_1 \wedge \\ &\qquad \wedge y'Ry \wedge w \preccurlyeq x_1 Rx_2 \preccurlyeq x_3 \ldots x_n Rv) \\ w \preccurlyeq^* v \quad &\equiv_{def} \quad w \preccurlyeq v \vee \exists xyzz'(w \preccurlyeq z \preccurlyeq x \wedge z'Rz \wedge w \preccurlyeq xR^*y \preccurlyeq v). \end{aligned}$$

We write $\tilde{R}^*$ for $(R^*;\preccurlyeq^*)$. The first disjunct in the definition of R^* arises from the fact that we want to have $(\preccurlyeq^*;R^*) \subseteq R^*$. The second disjunct arises from the fact that we want R^* to be gathering. Namely, by Remark 14, $y'Ry$ and $y \preccurlyeq x_1Rx_2 \preccurlyeq x_3 \ldots x_n$ implies $y \preccurlyeq^* x_n$, since we construct $\preccurlyeq^*$ and R^* in such a way that R^* is gathering. Thus we have $w \preccurlyeq y \preccurlyeq^* x_nRv$, hence $w(\preccurlyeq^*;R)v$. As we want to have $(\preccurlyeq^*;R^*) \subseteq R^*$, we have to demand wR^*v. Similar explanations apply to the definition of $\preccurlyeq^*$.

Lemma 15 Let R and $\preccurlyeq$ respectively be a binary relation and a partial order on a finite set W. If both

$$wRv \preccurlyeq^* u \to \exists x(wR^*x \wedge v \preccurlyeq^* x \preccurlyeq^* u \wedge x(\preccurlyeq;R) \subseteq u\tilde{R}^*)$$

$$\begin{aligned} &wRv \preccurlyeq^* u_1, \ldots, u_n \to \exists x(v \preccurlyeq^* x \preccurlyeq^* u_1, \ldots, u_n \wedge \\ &\quad v(\preccurlyeq;R) \subseteq x\tilde{R}^* \wedge \forall y \succ^* x(u_i \preccurlyeq^* y, \text{ for some } i)) \end{aligned}$$

then $(W, R^*, \preccurlyeq^*)$ is a gathering *Mp Vp*-frame.

Proof Although one have to check many cases, it is not difficult to see that $(W, R^*, \preccurlyeq^*)$ is indeed a frame, ($\preccurlyeq^*$ is a partial order and $(\preccurlyeq^*;R^*) \subseteq R^*$ holds), and that it is gathering. We show that

$$wR^*v \preccurlyeq^* u \to \exists x(wR^*x \wedge v \preccurlyeq^* x \preccurlyeq^* u \wedge x\tilde{R}^* \subseteq u\tilde{R}^*)$$

$$\begin{aligned} &wR^*v \preccurlyeq^* u_1, \ldots, u_n \to \exists x(v \preccurlyeq^* x \preccurlyeq^* u_1, \ldots, u_n \wedge \\ &\quad x\tilde{R}^* \subseteq u\tilde{R}^* \wedge \forall y \succ^* x(u_i \preccurlyeq^* y, \text{ for some } i)) \end{aligned}$$

hold, that is, that $(W, R^*, \preccurlyeq^*)$ is an *Mp Vp*-frame. The following two Claims suffice.

Claim 1 If wR^*v then there exists a node w' such that $w'Rv$ and for all $w'R^*v'$, also wR^*v'.

Proof of Claim 1 Suppose wR^*v. This implies that there are $x_1, y_1, \ldots x_n, y_n$, such that

$$w \preccurlyeq x_1Ry_1 \preccurlyeq x_2R \ldots \preccurlyeq x_nRy_n = v,$$

and *either* $n = 1$, so $v = y_1$, *or* $\exists zz'(w \preccurlyeq z \preccurlyeq x_1 \wedge z'Rz)$. In both cases, $w' = x_n$ has the desired properties. This proves Claim 1.
Claim 2 If $x(\preccurlyeq;R) \subseteq u\tilde{R}^*$, then $x\tilde{R}^* \subseteq u\tilde{R}^*$.
Proof of Claim 2 Suppose xR^*a. This implies there are $x \preccurlyeq a_1Ra_2 \preccurlyeq \ldots Ra$ such that *either* $a_2 = a$ *or* $\exists bb'(x \preccurlyeq b \preccurlyeq a_1 \wedge b'Rb)$. By assumption $u\tilde{R}^*a_2$, say $uR^*u' \preccurlyeq a_2$. In the first case, we clearly have $u\tilde{R}^*a$. In the second case, since uR^*u' there is $u''Ru'$, from which it follows that $u' \preccurlyeq^* a$. Hence also $u\tilde{R}^*a$. This proves Claim 2. **QED**

Theorem 16 $\vdash_{\mathsf{iPH}} A$ iff A is valid on all gathering *Mp Vp*-frames for which the modal relation is conversely well-founded.

Proof We only treat the direction from right to left. Suppose $\nvdash_{\mathsf{iPH}} A$. We construct a gathering *Mp Vp*-model for which the modal relation is conversely well-founded by the construction method (see Subsection 3.5). Let X be a finite adequate set, containing A, such that $B \rhd C \in X$ implies $\Box C \in X$. Consider the iPH-canonical model and let R' and $\preccurlyeq'$ be the relations on this model. Let $\alpha_{\langle\rangle}$ be a node at which A is not valid. With $W^*, R^*, \preccurlyeq^*$ we denote respectively the domain and the relations of the model M^* we are going to construct.

Using the construction method, we construct binary relations R and $\preccurlyeq''$ along with a set W^*. We denote the reflexive transitive closure of $\preccurlyeq''$ by $\preccurlyeq$. Then we define R^* and $\preccurlyeq^*$ as explained above, and show that in $(W^*, \preccurlyeq^*, R^*)$,

(7) $\quad \sigma R\tau \preccurlyeq^* \tau' \to \exists x(\sigma R^* x \wedge \tau \preccurlyeq^* x \preccurlyeq^* \tau' \wedge x(\preccurlyeq;R) \subseteq \tau'\tilde{R}^*)$

(8) $\quad \sigma R\tau \preccurlyeq^* \tau_1, \ldots, \tau_n \to \exists x(\tau \preccurlyeq^* x \preccurlyeq^* \tau_1, \ldots, \tau_n \wedge$
$\qquad \tau(\preccurlyeq;R) \subseteq x\tilde{R}^* \wedge \forall y \succ^* x(\tau_i \preccurlyeq^* y, \text{ for some } i))$

and apply Lemma 15 to conclude that $(W^*, R^*, \preccurlyeq^*)$ is a gathering *Mp Vp*-frame. Finally, we show that R^* is conversely well-founded.

During the construction we guarantee that

(9) $\quad$ if $\sigma R\tau$ respectively $\sigma \preccurlyeq \tau$, then $\alpha_\sigma R'\alpha_\tau$ respectively $\alpha_\sigma \preccurlyeq' \alpha_\tau$.

We will often use (9) without mentioning it.

First some notations and conventions. We write $\sigma * A$ for $\sigma * \langle A\rangle$ and $\sigma_\Box$ for $(\alpha_\sigma)_\Box$. For a sequence σ, we define $\gamma(\sigma)$ inductively via: if $\sigma = \sigma'' * \langle m, \tau, \sigma'\rangle$ or $\sigma = \sigma'' * \langle m, \sigma'\rangle$, then $\gamma(\sigma) = \gamma(\sigma')$, and $\gamma(\sigma) = \sigma$ otherwise.

For $B \rhd C \in X \backslash \alpha_\sigma$, the node $\alpha_{\sigma*\langle B\rhd C\rangle}$ is a node such that $\alpha_\sigma R'\alpha_{\sigma*\langle B\rhd C\rangle}$, and $B \in \alpha_{\sigma*\langle B\rhd C\rangle}$ while $C \notin \alpha_{\sigma*\langle B\rhd C\rangle}$. For $(B \to C) \in X\backslash\alpha_\sigma$ the node $\alpha_{\sigma*\langle B\to C\rangle}$ is a node for which $\alpha_\sigma \preccurlyeq' \alpha_{\sigma*\langle B\to C\rangle}$, and $B \in \alpha_{\sigma*\langle B\to C\rangle}$ while $C \notin \alpha_{\sigma*\langle B\to C\rangle}$ (Section 3.5).

For $\sigma R\tau \preccurlyeq^* \tau'$, the node α_a, where $a = \sigma * \langle m, \tau, \tau'\rangle$, is a node with the following properties: $\alpha_\sigma R'\alpha_a$, $(\alpha_a)_\Box = (\alpha_{\tau'})_\Box$ and $\alpha_\tau \preccurlyeq' \alpha_a \preccurlyeq' \alpha_\tau$. Note that the existence of α_a is guaranteed by Proposition 11, using (9). Observe that, by Lemma 10, $(\alpha_a)_\Box = (\alpha_{\tau'})_\Box$ implies $\alpha_a \tilde{R}' \subseteq \alpha_{\tau'} \tilde{R}'$ ($\tilde{R}' = (R';\preccurlyeq')$).

For $\sigma R \tau \preccurlyeq^* \tau_1, \ldots, \tau_n$, the node α_a, where $a = \tau * \langle v, \tau_1, \ldots, \tau_n \rangle$, is a node such that $\alpha_\tau \preccurlyeq' \alpha_a$ and α_a is a tight predecessor of $\alpha_{\tau_1}, \ldots, \alpha_{\tau_n}$ for α_τ, that is: $\alpha_a \preccurlyeq' \alpha_{\tau_1}, \ldots, \alpha_{\tau_n}$, $\alpha_\tau \tilde{R}' \subseteq \alpha_a \tilde{R}'$ and for all $x \succ' \alpha_a$, $\alpha_{\tau_i} \preccurlyeq' x$ for some i. Note that such a node exists by Corollary 12.

If $\sigma'(\preccurlyeq;R)\pi$, and σ is either $\gamma(\sigma')$ or $\sigma' * \langle v, \tau_1, \ldots, \tau_n \rangle$, the node $\alpha_{\sigma * \langle m, \pi \rangle}$ is a node such that $\alpha_\sigma R' \alpha_{\sigma * \langle m, \pi \rangle} \preccurlyeq' \alpha_\pi$ and $(\alpha_{\sigma * \langle m, \pi \rangle})_\Box = \pi_\Box$. Note that such nodes exist by Lemma 13, using the fact that $\sigma'_\Box = \sigma_\Box$.

We define properties $i(\cdot), p(\cdot)$ and $*(\cdot), \circ(\cdot), \star(\cdot)$ on respectively pairs of nodes and formulas, and sequences of nodes:

$$
\begin{array}{ll}
i(\sigma, B \to C) & (B \to C) \in X \backslash \sigma \wedge \neg \exists \sigma' (\sigma \preccurlyeq \sigma' \wedge B \in \sigma' \wedge C \notin \sigma') \\
p(\sigma, B \rhd C) & B \rhd C \in X \backslash \sigma \wedge \neg \exists \sigma' (\sigma R^* \sigma' \wedge B \in \sigma' \wedge C \notin \sigma')
\end{array}
$$

$$
\begin{array}{ll}
\star(\sigma, \tau, \tau') & \sigma R \tau \preccurlyeq^* \tau' \wedge \gamma(\tau) \neq \gamma(\tau') \\
\circ(\tau, \tau_1, \ldots, \tau_n) & \exists \sigma (\sigma R \tau) \wedge \tau \preccurlyeq^* \tau_1, \ldots, \tau_n \\
*(\sigma, \sigma', \tau) & \sigma (\preccurlyeq;R) \tau \wedge \neg (\sigma' (R^*;\preccurlyeq^*) \tau) \wedge \\
& (\sigma' = \gamma(\sigma) \vee \sigma' = \sigma * \langle \tau_1, \ldots, \tau_n \rangle, \text{ for some } \tau_i).
\end{array}
$$

Note that these properties can change during the construction. For example, if Y, Y' are two distinct sets of constructed nodes containing σ, σ', τ, $*(\sigma, \sigma', \tau)$ can hold in Y but not in Y'.

The construction of $(W^*, R, \preccurlyeq)$ uses four subconstructions, $\beta, \delta, \zeta, \xi$, which we will apply in a certain order. Every subconstruction consists of making an extension of the frame constructed so far by constructing some new nodes. The result, $\beta(Y)$, of the application of β to a frame $Y = (W_Y, \preccurlyeq_Y, R_Y)$ results in a frame $(W_{\beta(Y)}, \preccurlyeq_{\beta(Y)}, R_{\beta(Y)})$. Similarly for δ, ζ and ξ. When we say that for some nodes σ, τ, τ' in Y, $\star(\sigma, \tau, \tau')$ does (not) hold in Y, we read $\preccurlyeq_Y$ for $\preccurlyeq$, and similarly for the other relations. Similarly for the other properties. Again, $\preccurlyeq_Y$ is the transitive closure of $\preccurlyeq''_Y$, thus to define $\preccurlyeq_Y$ it suffices to define $\preccurlyeq''_Y$. The definitions of $\beta, \delta, \zeta, \xi$ run as follows.

$$
\begin{array}{rcl}
W_{\beta(Y)} & = & W_y \cup \{\sigma * \langle B \to C \rangle \mid i(\sigma, B \to C) \text{ holds in } Y\} \cup \\
& & \{\sigma * \langle B \rhd C \rangle \mid p(\sigma, B \to C) \text{ holds in } Y\} \\
\preccurlyeq''_{\beta(Y)} & = & \preccurlyeq''_Y \cup \{(\sigma, \sigma * \langle B \to C \rangle) \mid \sigma * \langle B \to C \rangle \notin Y\} \\
R_{\beta(Y)} & = & R_Y \cup \{(\sigma, \sigma * \langle B \rhd C \rangle) \mid \sigma * \langle B \rhd C \rangle \notin Y\}
\end{array}
$$

$$
\begin{aligned}
W_{\zeta(Y)} &= W_y \cup \{\sigma * \langle m,\tau,\tau'\rangle \mid \star(\sigma,\tau,\tau') \text{ holds in } Y\} \\
\preccurlyeq''_{\zeta(Y)} &= \preccurlyeq''_Y \cup \\
&\quad \{(\tau,\sigma * \langle m,\tau,\tau'\rangle),(\sigma * \langle m,\tau,\tau'\rangle,\tau') \mid \sigma * \langle m,\tau,\tau'\rangle \notin Y\} \\
R_{\zeta(Y)} &= R_Y \cup \{(\sigma,\sigma * \langle m,\tau,\tau'\rangle) \mid \sigma * \langle m,\tau,\tau'\rangle \notin Y
\end{aligned}
$$

$$
\begin{aligned}
W_{\xi(Y)} &= W_y \cup \{\tau * \langle v,\tau_1,\ldots,\tau_n\rangle \mid \circ(\tau,\tau_1,\ldots,\tau_n) \text{ holds in } Y\} \\
\preccurlyeq''_{\xi(Y)} &= \preccurlyeq''_Y \cup \{(\tau,\tau * \langle v,\tau_1,\ldots,\tau_n\rangle),(\tau * \langle v,\tau_1,\ldots,\tau_n\rangle,\tau_i) \mid \\
&\quad \tau * \langle v,\tau_1,\ldots,\tau_n\rangle \notin Y, i \leq n\} \\
R_{\xi(Y)} &= R_Y
\end{aligned}
$$

$$
\begin{aligned}
W_{\delta(Y)} &= W_y \cup \{\sigma' * \langle m,\tau\rangle \mid *(\sigma,\sigma',\tau) \text{ holds in } Y\} \\
\preccurlyeq''_{\delta(Y)} &= \preccurlyeq''_Y \cup \{(\sigma' * \langle m,\tau\rangle,\tau) \mid \sigma' * \langle m,\tau\rangle \notin Y\} \\
R_{\delta(Y)} &= R_Y \cup \{(\sigma',\sigma' * \langle m,\tau\rangle) \mid \sigma' * \langle m,\tau\rangle \notin Y\}.
\end{aligned}
$$

Let $k = (|\{B \to C|(B \to C) \in X\}|+1)\cdot|\{\Box B|\Box B \in X\}|$. We define an iterated version of $\beta(Y)$, $\bar{\beta}(Y)$, to be the frame $(\bigcup_{i=0}^k W_{Y_i}, \bigcup_{i=0}^k \preccurlyeq_{Y_i}, \bigcup_{i=0}^k R_{Y_i})$, where $Y_0 = Y$ and $Y_{i+1} = \beta(Y_i)$. It is easy to see that $i(\sigma, B \to C)$ or $p(\sigma, B \rhd C)$ can never hold in $\bar{\beta}(Y)$.

Now define frames $Y_0, Y_1, \ldots$ via: $Y_0 = (W_{Y_0}, \preccurlyeq_{Y_0}, R_{Y_0},)$, where $W_{Y_0} = \{\langle\rangle\}$, $\preccurlyeq_{Y_0} = \{(\langle\rangle,\langle\rangle)\}$ and R_{Y_0} is empty, and

$$
\begin{array}{lll}
Y_{6n+1} = \bar{\beta}(Y_{6n}) & Y_{6n+3} = \bar{\beta}(Y_{6n+2}) & Y_{6n+5} = \bar{\beta}(Y_{6n+4}) \\
Y_{6n+2} = \zeta(Y_{6n+1}) & Y_{6n+4} = \xi(Y_{6n+3}) & Y_{6n+6} = \delta(Y_{6n+5}).
\end{array}
$$

Let $W^* = \bigcup_i W_{Y_i}$, and let $\preccurlyeq$ be the transitive closure of $\bigcup_i \preccurlyeq_{Y_i}$, and let $R = \bigcup R_{Y_i}$. We show that (7) holds in W^*: if $\sigma R \tau \preccurlyeq^* \tau'$ and $\gamma(\tau) \neq \gamma(\tau')$ it is clear that there will be a node $x = \sigma * \langle m,\tau,\tau'\rangle$ such that $\sigma R^* x$ and $\tau \preccurlyeq^* x \preccurlyeq^* \tau'$ and $x(\preccurlyeq;R) \subseteq \tau' \tilde{R}^*$. We show that also in the case that $\sigma R \tau \preccurlyeq^* \tau'$ but $\gamma(\tau) = \gamma(\tau')$, there exists such a node x, namely $x = \tau$. It suffices to show that $\tau(\preccurlyeq;R) \subseteq \tau' \tilde{R}^*$. Therefore, assume $\tau(\preccurlyeq;R)\pi$. Thus, by construction, $\gamma(\tau') = \gamma(\tau)\tilde{R}^*\pi$. If $\gamma(\tau) = \tau'$ this gives $\tau'\tilde{R}^*\pi$. If $\gamma(\tau') \neq \tau'$, there exists $\sigma' R \tau'$. Because also $\tau'(\preccurlyeq;\tilde{R}^*)\pi$, since $\tau' \preccurlyeq \gamma(\tau')$, we can again conclude $\tau'\tilde{R}^*\pi$.

To see that (8) also holds, first observe that the construction is such that if $\sigma R \tau \preccurlyeq^* \tau_1, \ldots, \tau_n$, there exists a node $x = \tau * \langle v,\tau_1,\ldots,\tau_n\rangle$ such that $\tau \preccurlyeq^* x \preccurlyeq^* \tau_i$ and $\tau(\preccurlyeq;R) \subseteq x\tilde{R}^*$. Let Y_m be the first Y_i in which x occurs. It is clear that in Y_m we have $\forall y \succ^* x(\tau_i \preccurlyeq^* y)$, for some i. We show that this remains the case during the construction. We show this by induction on Y_n. The case Y_m is done. The case $(n > m)$. Assume $x \prec^* y$ holds in Y_n but not in Y_{n-1}. Without loss of generality assume that there is no $x \prec^* y' \prec^* y$. First, observe that the construction is such that $\sigma' R \tau'$ implies that $\tau' = \sigma' * D$, for some D of the form $B \rhd C, \langle m, \pi, \pi' \rangle$ or $\langle m, \pi \rangle$. Therefore, there is no x' with $x' R x$. Hence $x \prec^* y$ implies $x \prec y$. And since there is no $x \prec^* y' \prec^* y$, $x \prec'' y$. If $Y_n = \bar{\beta}(Y_{n-1})$, this implies $y = x * \langle B \to C \rangle$. We show that this cannot be, by showing that $i(x, B \to C)$ can never hold. It suffices to show that $i(x, B \to C)$ does not hold in Y_m. Note that all τ_i are already elements of some Y_j with $j < m$. This implies that $i(\tau_i, B \to C)$ does not hold in Y_m. Consider $(B \to C) \notin x$. Either $B \in x$ and $C \notin x$, in which case $i(x, B \to C)$ does not hold, or $B \notin x$. In the last case, $B \to C, B \notin \alpha_x$. Since α_x is a tight predecessor of $\alpha_{\tau_1}, \ldots, \alpha_{\tau_n}$ in the canonical model, this implies that $(B \to C) \notin \alpha_{\tau_i}$, for some i. Because $i(\tau_i, B \to C)$ does not hold in Y_m, this implies that there exists τ' such that $\tau_i \preccurlyeq \tau'$ and $B \in \tau'$ while $C \notin \tau'$. Clearly, this implies that $i(x, B \to C)$ does not hold in Y_m. Now consider the case in which $Y_n = \zeta(Y_{n-1})$. The fact that $x \prec'' y$ holds in Y_n but not in Y_{n-1}, implies that $x = \pi \preccurlyeq \sigma' * \langle m, \pi, \pi' \rangle = y$. Hence $\sigma' R x$. But we concluded before that there is no x' with $x' R x$, a contradiction. In the case that $Y_n = \xi(Y_{n-1})$, we have $x \preccurlyeq x * \langle v, \tau_1', \ldots, \tau_m' \rangle = y$. But this implies that there exists x' with $x' R x$, contradicting our previous observation that there is no x' with $x' R x$. We leave the remaining case, $Y_n = \delta(Y_{n-1})$, to the reader. This completes the proof that (8) holds.

Since (7) and (8) hold, we can apply Lemma 15 to conclude that the frame $(W^*, R^*, \preccurlyeq^*)$ is a gathering *Mp Vp*-frame. To show that R^* is conversely well-founded, it suffices to show that

$$\sigma R^* \tau \text{ implies } |\{\Box B \in X | \Box B \notin \tau\}| < |\{\Box B \in X | \Box B \notin \sigma\}|,$$

a proof which we leave to the reader. The valuation

$$\sigma \Vdash p \equiv_{def} \alpha_\sigma \Vdash p, \text{ for p} \in X.$$

(see Subsection 3.5) makes the frame into a model on which A is not valid, which completes the proof. **QED**

10 Admissible rules of preservativity logic

In this section we treat two admissible rules of iPH. If iPH would be the preservativity logic of HA it should certainly satisfy the

Reflection Rule $\quad \Box A / A$,

as (the arithmetical version of) $\Box A/A$ is an admissible rule of HA. The next lemma shows that this is indeed the case.

Proposition 17 The Reflection Rule holds: if $\vdash_{\mathsf{iPH}} \Box A$ then $\vdash_{\mathsf{iPH}} A$.

Proof We transform a model $\mathcal{M} = (W, \preccurlyeq, R, V)$ for iPH in which A is refuted to a model $\mathcal{M}'$ for iPH in which $\Box A$ is refuted. We can assume that A is not valid in the root w of $\mathcal{M}$. The first idea would be to extend the model in such a way that $w'Rw$ for some new node w'. However, this is not always possible. Namely, it can be the case that wRv but not $w \preccurlyeq v$, for some node v. If we add $w'Rw$ then we should also require $w \preccurlyeq v$ since we have to construct a gathering model. Therefore, we cannot guarantee that w forces the same formulas in both models. To overcome this problem we extend $\mathcal{M}$ in such a way that $w'Rw'' \preccurlyeq w$ for some new nodes w', w''.

We do not spell out the construction but only sketch the idea. We start with $W \cup \{w', w''\}$ and require $w'Rw'' \preccurlyeq w$. Then in every even step we add, in the notation of Theorem 16, nodes $v * \langle m, u, u' \rangle$, $v * \langle m, u \rangle$ and $v * \langle v, u_1, \ldots, u_n \rangle$ if respectively $\star(v, u, u')$, $*(v', v, u)$ or $\circ(v, u_1, \ldots, u_n)$ holds. It is not difficult to see that we will end up with a conversely well-founded, gathering *Mp Vp*-frame, and that for all nodes v which are not in $\mathcal{M}$, there is no u in $\mathcal{M}$ such that $u \preccurlyeq v$. Therefore, we can extend the valuation of $\mathcal{M}$ to nodes in $\mathcal{M}'$ by not forcing any propositional variable in a new node. Nodes in $\mathcal{M}$ force the same formulas in both models. Hence $w' \nVdash \Box A$. **QED**

It has been shown in [2] that if a c.e. extension of HA has the Disjunction Property then so does its provability logic. We have the following.

Proposition 18 The logic iPH has the Disjunction Property.

Proof Using the completeness result for iPH, Theorem 16, this is straightforward. **QED**

Recall that for all arithmetical realization *, HA proves $A^* \rhd B^*$ for all its propositional admissible rules A/B (see the Introduction). Hence for propositional formulas A, B,

$$(\Box A \to \Box B)/A \rhd B$$

is an admissible rule of the preservativity logic of HA. This rule does no longer hold when A, B range over arithmetical formulae. Consider for example the Rosser sentence R. Since, in HA, $(\Box R \to \Box \bot)$ is derivable, this rule would imply $R \rhd \bot$. Thus by the definition of preservativity and the fact that R is a Σ_1-formula, $\Box(R \to \bot)$ is derivable, quod non. However, $(\Box A \to \Box B)/A \rhd B$ is an admissible rule of iPH as the next lemma shows. Note that this is not in conflict with the possibility of iPH being the preservativity logic of HA.

Theorem 19 $\vdash_{\mathsf{iPH}} A \rhd B$ iff $\vdash_{\mathsf{iPH}} (\Box A \to \Box B)$.

Proof It suffices to show the following. For any model $\mathcal{M}$ with root w for which there is a node wRv such that $v \Vdash A$ and $v \nVdash B$, there is a submodel $\mathcal{M}'$ such that nodes in $\mathcal{M}$ above v force the same formulas in both models, and such that all nodes in $\mathcal{M}'$ are either equal to w of above v. Hence $w' \nVdash \Box B$ and $w \Vdash \Box A$. The proof is left to the reader. **QED**

11 Conclusions

In this paper we have given a summary of what, to the authors knowledge, is known so far about the provability and preservativity logic of HA, and we have presented some of our own results on this topic. In Section 2 we introduce a modal logic iPH the main principles of which were first presented in [10]. The logic iPH is conjectured to be the preservativity logic of HA. We have shown that this modal logic is complete with respect to a certain well-behaved class of frames. In the last section we have used this result to show that iPH has the Reflection and Disjunction property, properties that hold for the preservativity logic of HA (therefore, a logic that does not have these properties cannot be the preservativity logic of HA).

A lot of open questions remain. We end this paper with the most important ones: Is iPH the preservativity logic of HA? Is there a better modal completeness result for iPH, i.e. is there a class of frames or models with respect to which iPH is complete and which frames are easier to handle than gathering conversely well-founded *Mp Vp*-frames? What is the provability logic of HA? Let iH be the logic consisting of all formulas in the language of provability logic that are valid in iPH (thus iPH is conservative w.r.t. these formulas over iH). Does there exist a simple axiomatization of iH?

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