Hypersequent Systems for the Admissible Rules of Modal and Intermediate Logics

Rosalie Iemhoff[1] and George Metcalfe[2]

[1] Department of Philosophy, Utrecht University
Bestuursgebouw, Heidelberglaan 6-8, 3584 CS Utrecht, The Netherlands
Rosalie.Iemhoff@phil.uu.nl
[2] Department of Mathematics, Vanderbilt University
1326 Stevenson Center, Nashville TN 37240, USA
george.metcalfe@vanderbilt.edu

Abstract. The admissible rules of a logic are those rules under which the set of theorems of the logic is closed. In a previous paper by the authors, formal systems for deriving the admissible rules of Intuitionistic Logic and a class of modal logics were defined in a proof-theoretic framework where the basic objects of the systems are sequent rules. Here, the framework is extended to cover derivability of the admissible rules of intermediate logics and a wider class of modal logics, in this case, by taking hypersequent rules as the basic objects.

1 Introduction

Investigations into logical systems typically focus on the derivability of formulas or other structures within the system. However, the admissibility of rules for the system may also play a key role. A rule is *admissible* for a logic (viewed as a consequence relation) if adding it to the logic produces no new theorems. Such a notion is of interest in Computer Science for (at least) two reasons. First, admissible rules show that the derivability of certain formulas implies the derivability of stronger formulas, in the sense that the latter derive the former but not vice versa, an example being the disjunction property, where the derivability of a disjunction implies that one of the disjuncts is derivable. Second, equational unification can be formulated in terms of admissible rules. A formula A is unifiable for a consistent logic L iff σA is a theorem of L for some substitution σ. But this is equivalent to the claim that the rule A/q is not admissible in L for any variable q not occurring in A. Finally, admissible rules are also interesting from an algebraic perspective: they correspond to quasi-equations holding in the free algebra with countably many generators (or the Lindenbaum algebra of the logic).

Classical Logic has no non-derivable admissible rules; that is, it is structurally complete. However, for non-classical logics, this is no longer the case, and it is an interesting and often quite challenging task to provide characterizations of admissibility for these logics. In the case of modal and intermediate logics, a wide range of results for admissible rules such as decidability and complexity have been obtained, in particular by Rybakov [13]. Axiomatic-style presentations have been provided for wide classes of intermediate logics by Iemhoff [7, 8] (the case of Intuitionistic Logic was considered independently by Rozière [12]) and modal logics by Jeřábek [10], both making crucial use of Ghilardi's work on unification and projective approximations [4, 5].

In [9], the current authors introduced a proof-theoretic framework for admissibility: analytic "Gentzen-style" proof systems for deriving the admissible rules of both Intuitionistic Logic and a class of "extensible" modal logics including $\mathsf{K4}$, $\mathsf{S4}$, and GL. The key idea of this approach is that just as calculi for derivability in these logics can often be presented using sequents, so the corresponding systems for admissibility can be presented using sequent rules as basic objects. Here, we extend this approach to both a class of intermediate logics, including De Morgan Logic KC and the bounded cardinality logics $\mathsf{BC}_1, \mathsf{BC}_2, \ldots$, and a wider class of "mono-extensible" modal logics, including logics such as $\mathsf{GL.3}$, $\mathsf{S4.2}$, etc. In this case, however, the natural home for derivability is not sequents, but the framework of *hypersequents* – intuitively, disjunctions of sequents – introduced by Avron in [1] and used to define calculi for families of both intermediate logics (see e.g. [3]) and fuzzy logics (see e.g. [11]). Hence, for admissibility in these logics, the basic objects of our systems will be hypersequent rules.

2 Admissible Rules

Let us assume for this paper that the logic L is treated as a consequence relation based on a propositional language with binary connectives $\wedge, \vee, \rightarrow$, a constant $\bot$, and sometimes also a modal connective $\Box$. Other connectives are then defined as:

$$\neg A =_{\text{def}} A \rightarrow \bot \quad \top =_{\text{def}} \neg\bot \quad A \leftrightarrow B =_{\text{def}} (A \rightarrow B) \wedge (B \rightarrow A) \quad \boxdot A =_{\text{def}} \Box A \wedge A$$

We denote (propositional) variables by $p, q, r, \ldots$, formulas by $A, B, C, \ldots$, and finite sets of formulas by $\Gamma, \Pi, \Sigma, \Delta, \Theta, \Psi$. Formulas $p \rightarrow q$ and $\Box p$ are called *variable implications* and *boxed variables*, respectively. We also write $\bigvee \Gamma$ and $\bigwedge \Gamma$ where $\bigvee \emptyset = \bot$ and $\bigwedge \emptyset = \top$ for iterated disjunctions and conjunctions of formulas in a finite set Γ, and make use of the abbreviations:

$$\Box\Gamma =_{\text{def}} \{\Box A : A \in \Gamma\} \quad \boxdot\,\Gamma =_{\text{def}} \Gamma \cup \Box\Gamma \quad (\Gamma \equiv \Box\Gamma) =_{\text{def}} \{A \leftrightarrow \Box A : A \in \Gamma\}$$

Typically, logical rules are asymmetric, having many premises but just one conclusion. However, for admissibility, it is convenient to treat instead *generalized rules* of the form $\Gamma \rhd \Delta$, where both Γ and Δ are sets of formulas. Intuitively, such a rule is admissible for a logic L if whenever a substitution makes all the premises theorems of L, it also makes one of the conclusions a theorem. More precisely, an L-*unifier* for a formula A is a substitution σ such that $\vdash_{\mathsf{L}} \sigma A$. Then a generalized rule $\Gamma \rhd \Delta$ is L-*admissible*, written $\Gamma \mathrel{|\!\!\sim}_{\mathsf{L}} \Delta$, if each L-unifier for all $A \in \Gamma$, is an L-unifier for some $B \in \Delta$.

Example 1. A nice example of an admissible generalized rule for Intuitionistic Logic is the *disjunction property*, formulated as $p \vee q \rhd p, q$. If $\vdash_{\mathsf{IPC}} \sigma(p) \vee \sigma(q)$, then either $\vdash_{\mathsf{IPC}} \sigma(p)$ or $\vdash_{\mathsf{IPC}} \sigma(q)$. However, this rule is not admissible for Classical Logic; e.g. for $\sigma(p) = p$ and $\sigma(q) = \neg p$, plainly $\vdash_{\mathsf{CPC}} p \vee \neg p$, but $\nvdash_{\mathsf{CPC}} p$ and $\nvdash_{\mathsf{CPC}} \neg p$.

Although admissibility and derivability do not coincide in general for non-classical logics, Ghilardi in [4, 5] identified classes of "projective" formulas A where the relationship "$A \mathrel{|\!\!\sim}_{\mathsf{L}} B$ iff $A \vdash_{\mathsf{L}} B$" holds for all formulas B. Let us make this precise. A formula

A is L-*projective* for a logic L if there exists a substitution σ, called an L-*projective unifier* for A, such that $\vdash_{\mathsf{L}} \sigma A$ and $A \vdash_{\mathsf{L}} \sigma(p) \leftrightarrow p$ for all variables p.

Lemma 1. *Let* L *be an intermediate logic or a normal extension of* $\mathsf{K4}$*:*

(a) If A is L-projective, then $A \mathrel{|\!\!\sim}_{\mathsf{L}} \Delta$ iff $A \vdash_{\mathsf{L}} B$ for some $B \in \Delta$.
(b) If $A_1, \ldots, A_n$ are L-projective, then $\bigvee_{i=1}^n A_i \mathrel{|\!\!\sim}_{\mathsf{L}} B$ iff $\bigvee_{i=1}^n A_i \vdash_{\mathsf{L}} B$.
(c) If L' extends L (as a consequence relation) and $A_1, \ldots, A_n$ are L-projective formulas, then $\bigvee_{i=1}^n A_i \mathrel{|\!\!\sim}_{\mathsf{L}'} B$ iff $\bigvee_{i=1}^n A_i \vdash_{\mathsf{L}'} B$.
(d) If L' extends a normal modal logic L (as a consequence relation) and $A_1, \ldots, A_n$ are L-projective formulas, then $\bigvee_{i=1}^n \Box A_i \mathrel{|\!\!\sim}_{\mathsf{L}'} \Box B$ iff $\bigvee_{i=1}^n \Box A_i \vdash_{\mathsf{L}'} \Box B$.

Proof. (a) The right-to-left direction is immediate. For the other direction, suppose that $A \mathrel{|\!\!\sim}_{\mathsf{L}} \Delta$ where A is L-projective. Then there exists an L-projective unifier σ of A, such that $\vdash_{\mathsf{L}} \sigma B$ for some $B \in \Delta$. Also $A \vdash_{\mathsf{L}} \sigma B \to B$, so by modus ponens, $A \vdash_{\mathsf{L}} B$. (b) Again, the right-to-left direction is immediate. For the other direction, suppose that $\bigvee_{i=1}^n A_i \mathrel{|\!\!\sim}_{\mathsf{L}} B$. Also then $A_i \mathrel{|\!\!\sim}_{\mathsf{L}} B$ for $i = 1 \ldots n$. By (a), $A_i \vdash_{\mathsf{L}} B$ for $i = 1 \ldots n$ and hence $\bigvee_{i=1}^n A_i \vdash_{\mathsf{L}} B$. For (c), let L' be an extension of L and let $A_1, \ldots, A_n$ be L-projective formulas. Since L' extends L, we get that $A_1, \ldots, A_n$ are also L'-projective. The result then follows from (b). For (d), as for (c), we get that $A_1, \ldots, A_n$ are L'-projective. Suppose that $\bigvee_{i=1}^n \Box A_i \mathrel{|\!\!\sim}_{\mathsf{L}'} \Box B$. Then also $\Box A_i \mathrel{|\!\!\sim}_{\mathsf{L}'} \Box B$ for $i = 1 \ldots n$. Hence $A_i \mathrel{|\!\!\sim}_{\mathsf{L}'} B$ and by (a), $A_i \vdash_{\mathsf{L}'} B$ for $i = 1 \ldots n$. But then $\Box A_i \vdash_{\mathsf{L}'} \Box B$ for $i = 1 \ldots n$ and hence $\bigvee_{i=1}^n \Box A_i \vdash_{\mathsf{L}'} \Box B$. The other direction is almost immediate. $\Box$

3 Modal Logics

In [9], formal systems were defined for deriving the admissible rules of *extensible modal logics* by taking sequent rules as basic objects. Below, we recall this characterization and show that it can be extended to a wider class of logics that we call *mono-extensible* by taking our basic objects to be hypersequent rules.

3.1 Extensible and Mono-Extensible Modal Logics

For a comprehensive account of modal logics, see e.g. [2]. Let us just recall that for any normal modal logic L, an L-frame is such that every model on that frame is a model of L, and an L-model is a model based on an L-frame. L has the finite model property FMP if every refutable formula is refutable on a finite L-frame. For a Kripke model K with accessibility relation R, the *root* of K is the cluster $\{k : \forall l \neq k(kRl)\}$, and K_k denotes the Kripke model K restricted to the domain $\{l : kRl \text{ or } k = l\}$. Two Kripke models K_1, K_2 are *variants* of one another if they have the same nodes and accessibility relation, and their forcing relations agree on all nodes except possibly the root.

Ghilardi [5] has given a characterization for projectivity for a wide range of modal logics (following here the terminology of [10]).

Theorem 1 (Ghilardi [5]). *A class of finite models $\mathcal{K}$ has the* modal extension property *if for any model K, whenever $K_k \in \mathcal{K}$ for all k not in the root of K, there is a variant of K in $\mathcal{K}$. For every normal extension L of $\mathsf{K4}$ with the* FMP*, a formula is L-projective iff its class of L-models has the modal extension property.*

To get a handle on the modal extension property, we recall two useful constructions.

Definition 1. *For frames $F_1, \ldots, F_n$, $(\sum F_j)^i$ and $(\sum F_j)^r$ are obtained by adding, respectively, an irreflexive or a reflexive node beneath (connected to all nodes of) the disjoint sum of $F_1 \ldots F_n$. A normal extension* L *of* $\mathsf{K4}$ *with the* FMP *is* extensible *if for all finite* L*-frames $F_1, \ldots, F_n$:*

(i) $(\sum F_j)^i$ *is an* L*-frame unless* L *is reflexive;*
(ii) $(\sum F_j)^r$ *is an* L*-frame unless* L *is irreflexive.*

L *is* mono-extensible *if it satisfies the above for $n = 1$; that is, for each finite* L*-frame F: (i) F^i is an* L*-frame, unless* L *is reflexive; (ii) F^r is an* L*-frame, unless* L *is irreflexive.*

L *is* linear-extensible *if it is mono-extensible and linear; i.e. all rooted* L*-frames are linear (or* L *proves $\Box(\boxdot A \to B) \vee \Box(\boxdot B \to A)$).*

Every extensible logic L obeys the modal disjunction property: if $\vdash_{\mathsf{L}} \Box A \vee \Box B$, then $\vdash_{\mathsf{L}} \Box A$ or $\vdash_{\mathsf{L}} \Box B$. I.e. $\Box p \vee \Box q \rhd \Box p, \Box q$ is L-admissible. Significant examples of these logics include $\mathsf{K4}$, $\mathsf{S4}$, Grz, and GL. Linear-extensible logics (treated in [10]), which include the logics $\mathsf{S4.3}$, $\mathsf{K4.3}$, and $\mathsf{GL.3}$, and clearly do not satisfy the modal disjunction property, are the most obvious examples of mono-extensible but not extensible logics. Other interesting examples include the logics $\mathsf{S4.2}$, $\mathsf{K4.2}$, and $\mathsf{GL.2}$ (also discussed in [10]) which are mono-extensible but neither extensible nor linear-extensible.

3.2 Sequent Systems for Extensible Logics

Gentzen systems for derivability in many non-classical logics, in particular core modal logics, can be obtained in the framework of sequents. Since order and multiplicity of formulas is unimportant in the context of modal logics, we define a *sequent* S here as an ordered pair of finite sets of formulas, written $\Gamma \Rightarrow \Delta$. Such a sequent is said to be L-*derivable*, written $\vdash_{\mathsf{L}} S$, iff $\vdash_{\mathsf{L}} I(S)$ where $I(\Gamma \Rightarrow \Delta) =_{\text{def}} \bigwedge \Gamma \to \bigvee \Delta$.

To obtain Gentzen-style proof systems for admissibility in extensible modal logics, it is convenient again to use sequents, but this time at the level of rules. A *generalized sequent rule* (gs-rule for short) $\mathtt{R}$ is an ordered pair of finite sets of sequents, written:

$$\{\Gamma_i \Rightarrow \Delta_i\}_{i=1}^n \rhd \{\Pi_j \Rightarrow \Sigma_j\}_{j=1}^m$$

- $\mathtt{R}$ is L-*admissible*, written $\mathrel{|\!\sim}_{\mathsf{L}} \mathtt{R}$, iff $\{I(\Gamma_i \Rightarrow \Delta_i)\}_{i=1}^n \mathrel{|\!\sim}_{\mathsf{L}} \{I(\Pi_j \Rightarrow \Sigma_j)\}_{j=1}^m$.
- $\mathtt{R}$ is L-*derivable*, written $\vdash_{\mathsf{L}} \mathtt{R}$, iff $\bigwedge_{i=1}^n I(\Gamma_i \Rightarrow \Delta_i) \vdash_{\mathsf{L}} \bigvee_{j=1}^m I(\Pi_j \Rightarrow \Sigma_j)$.

Note that $A_1, \ldots, A_n \mathrel{|\!\sim}_{\mathsf{L}} B_1, \ldots, B_m$ iff $\mathrel{|\!\sim}_{\mathsf{L}} (\Rightarrow A_1), \ldots, (\Rightarrow A_n) \rhd (\Rightarrow B_1), \ldots, (\Rightarrow B_m)$. Hence a proof system for the admissibility of gs-rules is also a proof system for the admissibility of generalized rules, and of course, rules in the usual sense.

Rules (now at the next level up) for gs-rules are sets of rule instances, each consisting of a set of premises $\mathtt{R}_1, \ldots, \mathtt{R}_n$ and a conclusion $\mathtt{R}$; instances with no premises being called *initial gs-rules*. They are defined here schematically, using p, q to stand for variables, A, B for formulas, $\Gamma, \Pi, \Sigma, \Delta, \Theta, \Psi$ for sets of formulas, S for sequents, and $\mathcal{G}, \mathcal{H}$ for sets of sequents. We call all sequents not in $\mathcal{G}$ or $\mathcal{H}$ for instances of such rules, *principal sequents*. Such rules are L-*sound* if whenever $\mathrel{|\!\sim}_{\mathsf{L}} \mathtt{R}_i$ for $i = 1 \ldots n$, then $\mathrel{|\!\sim}_{\mathsf{L}} \mathtt{R}$, and L-*invertible*, if whenever $\mathrel{|\!\sim}_{\mathsf{L}} \mathtt{R}$, then $\mathrel{|\!\sim}_{\mathsf{L}} \mathtt{R}_i$ for $i = 1 \ldots n$. Calculi for extensible modal logics are defined in this framework as follows:

Initial GS-Rules and Structural Rules

$$\frac{}{\mathcal{G} \,\triangleright\, (\Gamma, A \Rightarrow A, \Delta), \mathcal{H}}\ (\text{ID}) \qquad \frac{\mathcal{G} \,\triangleright\, \mathcal{H}}{\mathcal{G}, S \,\triangleright\, \mathcal{H}}\ (\text{W})\triangleright \qquad \frac{\mathcal{G} \,\triangleright\, \mathcal{H}}{\mathcal{G} \,\triangleright\, S, \mathcal{H}}\ \triangleright(\text{W})$$

Anti-Cut and Projection Rules

$$\frac{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Pi \Rightarrow A, \Sigma), (\Gamma, \Pi \Rightarrow \Sigma, \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Pi \Rightarrow A, \Sigma) \,\triangleright\, \mathcal{H}}\ (\text{AC}) \qquad \frac{\mathcal{G}, S \,\triangleright\, (\Gamma, \boxdot I(S) \Rightarrow \Delta), \mathcal{H}}{\mathcal{G}, S \,\triangleright\, \mathcal{H}}\ (\text{PJ})$$

where $(\Gamma \Rightarrow \Delta) \in \mathcal{H} \cup \{\Rightarrow\}$

Right Logical Rules

$$\frac{}{\mathcal{G} \,\triangleright\, (\Gamma, \bot \Rightarrow \Delta), \mathcal{H}}\ \triangleright(\bot\Rightarrow) \qquad \frac{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow \bot, \Delta), \mathcal{H}}\ \triangleright(\Rightarrow\bot)$$

$$\frac{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow A, \Delta), \mathcal{H} \quad \mathcal{G} \,\triangleright\, (\Gamma \Rightarrow B, \Delta), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow A \wedge B, \Delta), \mathcal{H}}\ \triangleright(\Rightarrow\wedge) \qquad \frac{\mathcal{G} \,\triangleright\, (\Gamma, A, B \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma, A \wedge B \Rightarrow \Delta), \mathcal{H}}\ \triangleright(\wedge\Rightarrow)$$

$$\frac{\mathcal{G} \,\triangleright\, (\Gamma, A \Rightarrow \Delta), \mathcal{H} \quad \mathcal{G} \,\triangleright\, (\Gamma, B \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma, A \vee B \Rightarrow \Delta), \mathcal{H}}\ \triangleright(\vee\Rightarrow) \qquad \frac{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow A, B, \Delta), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow A \vee B, \Delta), \mathcal{H}}\ \triangleright(\Rightarrow\vee)$$

$$\frac{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow A, \Delta), \mathcal{H} \quad \mathcal{G} \,\triangleright\, (\Gamma, B \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma, A \rightarrow B \Rightarrow \Delta), \mathcal{H}}\ \triangleright(\rightarrow\Rightarrow) \qquad \frac{\mathcal{G} \,\triangleright\, (\Gamma, A \Rightarrow B, \Delta), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow A \rightarrow B, \Delta), \mathcal{H}}\ \triangleright(\Rightarrow\rightarrow)$$

Left Logical Rules

$$\frac{\mathcal{G} \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma, \bot \Rightarrow \Delta) \,\triangleright\, \mathcal{H}}\ (\bot\Rightarrow)\triangleright \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow \bot, \Delta) \,\triangleright\, \mathcal{H}}\ (\Rightarrow\bot)\triangleright$$

$$\frac{\mathcal{G}, (\Gamma, A, B \Rightarrow \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma, A \wedge B \Rightarrow \Delta) \,\triangleright\, \mathcal{H}}\ (\wedge\Rightarrow)\triangleright \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow A, \Delta), (\Gamma \Rightarrow B, \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \wedge B, \Delta) \,\triangleright\, \mathcal{H}}\ (\Rightarrow\wedge)\triangleright$$

$$\frac{\mathcal{G}, (\Gamma \Rightarrow A, B, \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \vee B, \Delta) \,\triangleright\, \mathcal{H}}\ (\Rightarrow\vee)\triangleright \qquad \frac{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Gamma, B \Rightarrow \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma, A \vee B \Rightarrow \Delta) \,\triangleright\, \mathcal{H}}\ (\vee\Rightarrow)\triangleright$$

$$\frac{\mathcal{G}, (\Gamma, B \Rightarrow \Delta), (\Gamma \Rightarrow A, \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma, A \rightarrow B \Rightarrow \Delta) \,\triangleright\, \mathcal{H}}\ (\rightarrow\Rightarrow)\triangleright \qquad \frac{\mathcal{G}, (\Gamma, A \Rightarrow B, \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \rightarrow B, \Delta) \,\triangleright\, \mathcal{H}}\ (\Rightarrow\rightarrow)\triangleright$$

$$\frac{\mathcal{G}, (\Gamma, \Box p \Rightarrow \Delta), (A \Rightarrow p) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma, \Box A \Rightarrow \Delta) \,\triangleright\, \mathcal{H}}\ (\Box\Rightarrow)\triangleright \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow \Box p, \Delta), (p \Rightarrow A) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow \Box A, \Delta) \,\triangleright\, \mathcal{H}}\ (\Rightarrow\Box)\triangleright$$

where p does not occur in $\mathcal{G}, \mathcal{H}, \Gamma, \Delta, A$ in $(\Box\Rightarrow)\triangleright$ and $(\Rightarrow\Box)\triangleright$.

Fig. 1. Core Modal Rules

$$\frac{[\mathcal{G}, (\Box\Gamma \Rightarrow \Box\Delta), (\boxdot\Gamma \Rightarrow A) \,\triangleright\, \mathcal{H}]_{A \in \Delta}}{\mathcal{G}, (\Box\Gamma \Rightarrow \Box\Delta) \,\triangleright\, \mathcal{H}}\ (\text{V}^{i})$$

$$\frac{[\mathcal{G}, (\Gamma \equiv \Box\Gamma \Rightarrow \Box\Delta), (\boxdot\Gamma \Rightarrow A) \,\triangleright\, \mathcal{H}]_{A \in \Delta}}{\mathcal{G}, (\Gamma \equiv \Box\Gamma \Rightarrow \Box\Delta) \,\triangleright\, \mathcal{H}}\ (\text{V}^{r})$$

$$\frac{\mathcal{G}, (\Gamma, \Theta \Rightarrow \Delta), (\Pi \Rightarrow \Psi, \Sigma), (\Gamma, \Pi, A \leftrightarrow \Box A \Rightarrow \Sigma, \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma, \Theta \Rightarrow \Delta), (\Pi \Rightarrow \Psi, \Sigma) \,\triangleright\, \mathcal{H}}\ (\text{AC}_{\Box})$$

where $\Theta \cup \Psi \subseteq \{A, \Box A\}$ and $\Theta, \Psi \neq \emptyset$.

Fig. 2. Additional Extensible Modal Rules

Theorem 2 ([9]). *For an extensible modal logic* L *and gs-rule calculus* GAML *where:*

(1) GAML *extends the core modal rules of Figure 1.*
(2) If L *is not reflexive, then* (V^i) *is a rule of* GAML.
(3) If L *is not irreflexive, then* (V^r) *and* $(\mathrm{AC}_\Box)$ *are rules of* GAML.
(4) If $\vdash_{\mathsf{L}} S$*, then* $\vdash_{\mathsf{GAML}} \ \rhd S$.
(5) If $\vdash_{\mathsf{GAML}} \mathtt{R}$*, then* $\mathrel{|\!\sim}_{\mathsf{L}} \mathtt{R}$.

$\mathrel{|\!\sim}_{\mathsf{L}} \mathtt{R}$ *iff* $\vdash_{\mathsf{GAML}} \mathtt{R}$ *for any gs-rule* $\mathtt{R}$.

Note that the right logical rules of Figure 1 are just usual rules for modal logics embedded into the gs-rule framework, and that the non-modal left logical rules are obtained from these rules by replacing the conclusion sequent with the premise sequents (including $\rhd(\bot \Rightarrow)$, an instance of $(\mathrm{W})\rhd$ but included here for uniformity). The (non-invertible) modal rules, $(\Box\Rightarrow)\rhd$ and $(\Rightarrow\Box)\rhd$, decompose modal formulas on the left by replacing the formula A in $\Box A$ by a new variable p, soundness following from the fact that any substitution for the conclusion can be extended (since p does not occur there) by subsituting A for p. The "projection rule" (PJ) allows sequents on the left to be used as modal implications on the right, corresponding to the fact that derivability implies admissibility, while the "anti-cut" rule (AC) corresponds directly to the fact that the usual cut rule is admissible in the logic. The more complicated "Visser rules" (V^i) and (V^r) reflect the existence of non-derivable admissible rules for irreflexive and reflexive logics.

Example 2. In particular, we can obtain calculi for $\mathsf{K4}$, GL, and $\mathsf{S4}$ by adding (from left to right) the first rule for $\mathsf{K4}$, the second rule for GL, and the first and the third for $\mathsf{S4}$:

$$\frac{\mathcal{G} \ \rhd \ (\boxdot\Gamma \Rightarrow A), \mathcal{H}}{\mathcal{G} \ \rhd \ (\Box\Gamma, \Pi \Rightarrow \Box A, \Delta), \mathcal{H}} \qquad \frac{\mathcal{G} \ \rhd \ (\boxdot\Gamma, \Box A \Rightarrow A), \mathcal{H}}{\mathcal{G} \ \rhd \ (\Box\Gamma, \Pi \Rightarrow \Box A, \Delta), \mathcal{H}} \qquad \frac{\mathcal{G} \ \rhd \ (\boxdot\Gamma, \Pi \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \ \rhd \ (\Box\Gamma, \Pi \Rightarrow \Delta), \mathcal{H}}$$

to the core modal rules, with (V^i) for GL, (V^r) and $(\mathrm{AC}_\Box)$ for $\mathsf{S4}$, and all three for $\mathsf{K4}$.

3.3 Hypersequent Systems for Mono-Extensible Logics

To deal with mono-extensible modal logics, we move beyond the sequent level. In particular, adapting slightly the usual definition (see e.g. Avron [1]), we define a *hypersequent* to be a finite non-empty set of sequents, written $S_1 \mid \ldots \mid S_n$, and let $\vdash_{\mathsf{L}} G$ iff $\vdash_{\mathsf{L}} I^\Box(G)$ where $I^\Box(G) = \bigvee_{i=1}^{n} \Box I(S_i)$. Hypersequent calculi are particularly useful for characterizing intermediate logics (see e.g. [3]) and logics characterized by linearly ordered structures [11]. An example of both is a calculus for the intermediate and fuzzy Gödel-Dummett Logic LC, defined by adding a single rule to a hypersequent version of Gentzen's calculus LJ for Intuitionistic Logic.

Extending gs-rules to the hypersequent case, a *generalized hypersequent rule* (gh-rule for short) $\mathtt{R}$ is an ordered pair of sets of hypersequents, written:

$$G_1, \ldots, G_n \rhd H_1, \ldots, H_m$$

If $m \leq 1$, then $\mathtt{R}$ is called a *single-conclusion* gh-rule (sgh-rule for short).

$$\frac{\mathcal{G}, G \ \triangleright\ (\boxdot I^{\Box}(G) \Rightarrow I^{\Box}(H)), \mathcal{H}}{\mathcal{G}, G \ \triangleright\ \mathcal{H}}\ (\textsc{pj})^h \quad \text{where } H \in \mathcal{H} \cup \{\Rightarrow\}$$

$$\frac{\mathcal{G}, (G \mid \Box\Gamma \Rightarrow \Box\Delta), (G \mid \{\boxdot\Gamma \Rightarrow C : C \in \Delta\}) \ \triangleright\ \mathcal{H}}{\mathcal{G}, (G \mid \Box\Gamma \Rightarrow \Box\Delta) \ \triangleright\ \mathcal{H}}\ (\textsc{v}^i)^h$$

$$\frac{\mathcal{G}, (G \mid \Gamma \equiv \Box\Gamma \Rightarrow \Box\Delta), (G \mid \{\boxdot\Gamma \Rightarrow C : C \in \Delta\}) \ \triangleright\ \mathcal{H}}{\mathcal{G}, (G \mid \Gamma \equiv \Box\Gamma \Rightarrow \Box\Delta) \ \triangleright\ \mathcal{H}}\ (\textsc{v}^r)^h$$

$$\frac{\mathcal{G}, (G \mid \Gamma, A \Rightarrow \Delta), (H \mid \Pi \Rightarrow A, \Sigma)(G \mid H \mid \Gamma, \Pi \Rightarrow \Sigma, \Delta) \ \triangleright\ \mathcal{H}}{\mathcal{G}, (G \mid \Gamma, A \Rightarrow \Delta), (H \mid \Pi \Rightarrow A, \Sigma) \ \triangleright\ \mathcal{H}}\ (\textsc{ac})^h$$

$$\frac{\mathcal{G}, (G \mid \Gamma, \Theta \Rightarrow \Delta), (H \mid \Pi \Rightarrow \Psi, \Sigma), (G \mid H \mid \Gamma, \Pi, A \leftrightarrow \Box A \Rightarrow \Sigma, \Delta) \ \triangleright\ \mathcal{H}}{\mathcal{G}, (G \mid \Gamma, \Theta \Rightarrow \Delta), (H \mid \Pi \Rightarrow \Psi, \Sigma) \ \triangleright\ \mathcal{H}}\ (\textsc{ac}_{\Box})^h$$

$$\text{where } (\Theta \cup \Psi) \subseteq \{A, \Box A\} \text{ and } \Theta, \Psi \neq \emptyset$$

Fig. 3. Additional Mono-Extensible Modal Rules

- R is L-*admissible*, written $\mid\!\sim_{\mathsf{L}} \mathtt{R}$, iff $\{I^{\Box}(G_i)\}_{i=1}^n \mid\!\sim_{\mathsf{L}} \{I^{\Box}(H_j)\}_{j=1}^m$.
- R is L-*derivable*, written $\vdash_{\mathsf{L}} \mathtt{R}$, iff $\bigwedge_{i=1}^n I^{\Box}(G_i) \vdash_{\mathsf{L}} \bigvee_{j=1}^m I^{\Box}(H_j)$.

A core set of rules for mono-extensible logics is obtained from the core modal rules by adding a context variable G in the premises and conclusion standing for an arbitrary context hypersequent; e.g. $(\Rightarrow\wedge)\triangleright$ becomes:

$$\frac{\mathcal{G}, (G \mid \Gamma \Rightarrow A, \Delta), (G \mid \Gamma \Rightarrow B, \Delta) \ \triangleright\ \mathcal{H}}{\mathcal{G}, (G \mid \Gamma \Rightarrow A \wedge B, \Delta) \ \triangleright\ \mathcal{H}}\ (\Rightarrow\wedge)\triangleright^h$$

Hypersequent versions of the projection and anti-cut rules of Fig. 2 are given in Fig 3.

Definition 2. *The* gh-version $\mathtt{R}^h$ *of a rule schema* R *for gs-rules is obtained by replacing each principal sequent S in* R *by $G \mid S$ for a fixed hypersequent variable G.*

Our starting point for a calculus for a mono-extensible modal logic L is rules on the right of the $\triangleright$ symbol that provide a sound and complete calculus for L-derivability. We then expand this calculus with hypersequent versions of the core modal rules, and versions of the appropriate Visser rules $(\textsc{v}^i)$ and $(\textsc{v}^r)$.

Definition 3. *A calculus* GAML *is* L*-fitting for a mono-extensible modal logic* L *if:*

(1) GAML *extends the gh-versions of the core modal rules.*
(2) If L *is not reflexive, then* $(\textsc{v}^i)^h$ *of Fig. 3 is a rule of* GAML.
(3) If L *is not irreflexive, then* $(\textsc{v}^r)^h$ *and* $(\textsc{ac}_{\Box})^h$ *of Fig. 3 are rules of* GAML.
(4) If $\vdash_{\mathsf{L}} G$*, then* $\vdash_{\mathsf{GAML}} \ \triangleright G$.
(5) If $\vdash_{\mathsf{GAML}} \mathtt{R}$*, then* $\mid\!\sim_{\mathsf{L}} \mathtt{R}$.

Example 3. For non-reflexive logics, we can use $(\textsc{v}^i)$ as follows, noting that the top gh-rule is easily seen to be derivable using rules for K4 on the right:

$$\dfrac{\dfrac{\dfrac{\triangleright\ \boxdot(\Box(\boxdot A \to B) \vee \Box(\boxdot A \Rightarrow C)) \Rightarrow \Box(\boxdot A \to (B \vee C))}{(\boxdot A \Rightarrow B \mid \boxdot A \Rightarrow C) \ \triangleright\ \boxdot(\Box(\boxdot A \to B) \vee \Box(\boxdot A \Rightarrow C)) \Rightarrow \Box(\boxdot A \to (B \vee C))}\ (\textsc{w})\triangleright}{(\boxdot A \Rightarrow B \mid \boxdot A \Rightarrow C) \ \triangleright\ (\boxdot A \Rightarrow B \vee C)}\ (\textsc{pj})^h}{\dfrac{(\Box A \Rightarrow \Box B, \Box C), (\boxdot A \Rightarrow B \mid \boxdot A \Rightarrow C) \ \triangleright\ (\boxdot A \Rightarrow B \vee C)}{(\Box A \Rightarrow \Box B, \Box C) \ \triangleright\ (\boxdot A \Rightarrow B \vee C)}\ (\textsc{v}^i)^h}\ (\textsc{w})\triangleright$$

Proposition 1. *Let* L *be a mono-extensible modal logic.*

(a) All the core modal rules are L*-sound.*
(b) If L *is irreflexive (in particular, a* GL*-extension), then* $(\mathrm{V}^i)^h$ *is* L*-sound.*
(c) If L *is reflexive (equivalently, an* $\mathsf{S4}$*-extension), then* $(\mathrm{V}^r)^h$ *and* $(\mathrm{AC}_\Box)$ *are* L*-sound.*

Proof. Many parts of this proof are exactly as in the extensible case considered in [9]. It remains only to check the soundness of the Visser rules $(\mathrm{V}^i)^h$ and $(\mathrm{V}^r)^h$. For the former, it is sufficient to show $I^\Box(G \mid \Box\Gamma \Rightarrow \Box\Delta) \mathrel{|\!\sim}_{\mathsf{L}} I^\Box(G \mid \{\boxdot\Gamma \Rightarrow C : C \in \Delta\})$. Suppose that $\nvdash_{\mathsf{L}} \sigma I^\Box(G \mid \{\boxdot\Gamma \Rightarrow C : C \in \Delta\})$ for some substitution σ. If $\Delta = \emptyset$, then $\nvdash_{\mathsf{L}} \sigma I^\Box(G)$ and since $\vdash_{\mathsf{L}} \Box\neg\Box A \leftrightarrow \Box\bot$ for any formula A, easily $\nvdash_{\mathsf{L}} \sigma I^\Box(G \mid \Box\Gamma \Rightarrow)$. Suppose then that $\Delta \neq \emptyset$. Since L has the FMP, let K be a finite L-model refuting $\sigma I^\Box(G \mid \{\boxdot\Gamma \Rightarrow C : C \in \Delta\})$ and let F be the frame of K. L is mono-extensible and irreflexive, so F^i is also an L-frame. Consider a model on the frame F^i for which the forcing in all nodes except the root is the same as in K, and no variables are forced at the root. $\Box I(\Box\sigma\Gamma \Rightarrow \Box\sigma\Delta)$ and $\Box I(\sigma S)$ are refuted at the root for all $S \in G$, so $\nvdash_{\mathsf{L}} \sigma I^\Box(G \mid \Box\Gamma \Rightarrow \Box\Delta)$ and we are done. Note that the extra boxes in the interpretation $I^\Box$ of hypersequents is essential here, since we cannot conclude that G is not forced at the root, but we can for $I^\Box(G)$.

For (V^r), we show $I^\Box(G \mid \Gamma \equiv \Box\Gamma \Rightarrow \Box\Delta) \mathrel{|\!\sim}_{\mathsf{L}} I^\Box(G \mid \{\boxdot\Gamma \Rightarrow C : C \in \Delta\})$. Suppose that $\nvdash_{\mathsf{L}} \sigma I^\Box(G \mid \{\boxdot\Gamma \Rightarrow C : C \in \Delta\})$ for some substitution σ. If $\Delta = \emptyset$, then $\nvdash_{\mathsf{L}} \sigma I^\Box(G)$ and since $\vdash_{\mathsf{L}} \Box\neg(A \leftrightarrow \Box A) \leftrightarrow \Box\bot$ for any formula A, easily $\nvdash_{\mathsf{L}} \sigma I^\Box(G \mid \Gamma \equiv \Box\Gamma \Rightarrow)$. Suppose then that $\Delta \neq \emptyset$. Since L has the FMP, let K be a finite L-model refuting $\sigma I^\Box(G \mid \{\boxdot\Gamma \Rightarrow C : C \in \Delta\})$ and let F be the frame of K. L is mono-extensible and reflexive, so F^r is an L-frame with a reflexive root r. By the reflexivity of r and since K forces $\sigma(\boxdot\Gamma)$, r forces $\sigma(A) \leftrightarrow \sigma(\Box A)$ for all $A \in \Gamma$. Hence r forces $\bigwedge\{\sigma(A) \leftrightarrow \sigma(\Box A) : A \in \Gamma\}$ but refutes $\sigma(\bigvee \Box\Delta)$. So $\nvdash_{\mathsf{L}} \sigma I^\Box(G \mid \Gamma \equiv \Box\Gamma \Rightarrow \Box\Delta)$. $\Box$

Notice that we have considered here only logics that are either reflexive or irreflexive, meaning that logics such as $\mathsf{K4.2}$ lacking these properties are currently beyond our scope (although we believe that very similar methods should suffice for such cases).

We now turn our attention to establishing completeness for fitting calculi, restricting our attention (at least to start with) to the single-conclusion case. First, we show that L-derivable sgh-rules are also GAML-derivable.

Lemma 2. *Let* L *be a mono-extensible modal logic and let* GAML *be* L*-fitting. If* $\vdash_{\mathsf{L}} \mathtt{R}$*, then* $\vdash_{\mathsf{GAML}} \mathtt{R}$ *for any sgh-rule* $\mathtt{R}$*.*

Proof. Let $\mathtt{R} = (\mathcal{G} \rhd \mathcal{H})$ where $|\mathcal{H}| \leq 1$ and suppose that $\vdash_{\mathsf{L}} \mathtt{R}$. If $\mathcal{H}$ is $\{H\}$, or taking H to be $(\Rightarrow)$ if $\mathcal{H} = \emptyset$, then $\bigwedge_{G \in \mathcal{G}} I^\Box(G) \vdash_{\mathsf{L}} I^\Box(H)$. But then since we are above $\mathsf{K4}$, using the modal deduction theorem:

$$\vdash_{\mathsf{L}} \bigwedge_{G \in \mathcal{G}} \boxdot I^\Box(G) \to I^\Box(H)$$

GAML is L-fitting, so by repeated applications of $(\mathrm{PJ})^h$, $\vdash_{\mathsf{GAML}} \mathcal{G} \rhd \mathcal{H}$ as required. $\Box$

Our task now is to reduce the L-admissibility of a gh-rule to the admissibility of more manageable gh-rules. We introduce the following notions:

Definition 4. *A gh-rule* $\mathtt{R} = (\mathcal{G} \rhd \mathcal{H})$ *is:*

- modal-irreducible *if* $\mathcal{G}$ *contains only variables and boxed variables.*
- modal-semi-irreducible *if* $\mathcal{G}$ *contains only variables and boxed variables, and on the left of sequents possibly also equivalences of the form* $p \leftrightarrow \Box p$.
- full *with respect to a set of rules* X *if whenever* $\mathtt{R}_1, \ldots, \mathtt{R}_n / \mathtt{R}$ *is an instance of a rule in* X, *then* $\mathtt{R}_i \subseteq \mathtt{R}$ *for some* $i \in \{1, \ldots, n\}$ *(i.e. applying a rule in* X *backwards to* $\mathtt{R}$ *adds no new sequents to the gh-rule).*

It is an easy task (see e.g. [9]) – essentially following from the soundness of the usual logical rules for modal logics – to show that the left logical rules are all L-invertible. Moreover, each such rule (working upwards) removes an occurrence of a logical connective from a sequent in a hypersequent on the left. Hence, if we define the complexity of a sequent as the multiset of complexities (number of symbols) of its formulas, and the complexity of a gh-rule as the multiset of complexities of its sequents, then it is a standard inductive proof to show the following:

Lemma 3. *Each* L*-admissible gh-rule can be derived from an* L*-admissible modal-irreducible gh-rule using the left logical rules.*

But now notice that there is a finite number of different semi-modal-irreducible sequents built from a fixed set of variables. Hence applying any number of rules with the subformula property backwards to a modal-irreducible gh-rule will terminate with gh-rules full with respect to that set.

Lemma 4. *Let* $X \subseteq \{(\mathrm{V}^i)^h, (\mathrm{V}^r)^h, (\mathrm{AC}), (\mathrm{AC}_\Box), (\wedge\Rightarrow)\rhd, (\rightarrow\Rightarrow)\rhd\}$. *Then every modal-irreducible* L*-admissible gh-rule can be derived using* X *from a set of semi-modal-irreducible* L*-admissible gh-rules that are full with respect to* X. *If* X *does not contain* $(\mathrm{AC}_\Box)$*, then these gh-rules are modal-irreducible.*

For the main part of the proof, we again consider only irreflexive and reflexive logics (and single-conclusion gh-rules), treated by the following two theorems:

Theorem 3. *If* GAML *is* L*-fitting for a mono-extensible irreflexive modal logic* L*, then* $\mid\!\sim_{\mathsf{L}} \mathtt{R}$ *iff* $\vdash_{\mathsf{GAML}} \mathtt{R}$ *for any sgh-rule* $\mathtt{R}$.

Proof. The right-to-left direction follows from the definition of L-fitting and Proposition 1. For the left-to-right direction, it is sufficient by Lemma 4 to assume that $\mathtt{R} = (\mathcal{G} \rhd \mathcal{H})$ is an L-admissible modal-irreducible sgh-rule that is full with respect to (V^i) and (AC). Suppose now that:

$$\mathcal{G} = (G_1, \ldots, G_n) \quad \text{and} \quad G_i = (S^i_1 \mid \ldots \mid S^i_{m_i}) \quad \text{where} \quad S^i_j = (\Gamma^i_j \Rightarrow \Delta^i_j).$$

Let $A = \bigwedge_{i=1}^n I^\Box(G_i) = \bigwedge_{i=1}^n \bigvee_{j=1}^{m_i} \Box I(S^i_j)$. If A is inconsistent, then $\vdash_{\mathsf{GAML}} \mathcal{G} \rhd \mathcal{H}$ follows immediately by Lemma 2. We define:

$$X_{j_1,\ldots,j_n} = \{S^1_{j_1}, \ldots, S^n_{j_n}\} \quad \text{and} \quad C = \bigvee_{j_1 \leq m_1, \ldots, j_n \leq m_n} \; \bigwedge_{S \in X_{j_1,\ldots,j_n}} \Box I(S).$$

and observe that by distributivity: $\vdash_{\mathsf{L}} C \leftrightarrow A$.

Now we come to the crucial point of the proof. Each irreflexive logic with the FMP contains GL. Suppose that we can show that $\bigwedge_{S\in X_{j_1,\ldots,j_n}} I(S)$ is GL-projective or GL-inconsistent for each $j_1 \leq m_1, \ldots, j_n \leq m_n$. For $\mathcal{H} = \{H\}$ or taking H as $\Rightarrow$ if $\mathcal{H} = \emptyset$, it follows by Lemma 1 (d), that $C \vdash_{\mathsf{L}} I^{\Box}(H)$ iff $C \mathrel{|\!\sim}_{\mathsf{L}} I^{\Box}(H)$. So since $\vdash_{\mathsf{L}} C \leftrightarrow A$ and $A \mathrel{|\!\sim}_{\mathsf{L}} I^{\Box}(H)$, we get $\vdash_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H}$. But then by Lemma 2, $\vdash_{\mathsf{GAML}} \mathcal{G} \rhd \mathcal{H}$ as required. Note that the fact that $\mathcal{H}$ consists of at most one hypersequent plays a crucial role here.

Hence we have proved the theorem once we have shown that each $\bigwedge_{S\in X_{j_1,\ldots,j_n}} I(S)$ is either GL-projective or GL-inconsistent. To achieve this we make use of the result established in [9], that a modal irreducible *gs-rule* that is full with respect to (V^i) and (AC) is either projective or inconsistent. I.e. it is sufficient to show that $X_{j_1,\ldots,j_n} \rhd \mathcal{H}$ is a modal irreducible gs-rule that is full with respect to (V^i) and (AC). The rest of this proof will consist of a proof of this fact.

We call a set $X_{j_1,\ldots,j_n}$ *minimal* if it does not contain a proper subset $X_{h_1,\ldots,h_n}$. It suffices to establish the modal irreducibility and fullness with respect to (V^i) and (AC) only for the gs-rules $X_{j_1,\ldots,j_n} \rhd \mathcal{H}$ for which $X_{j_1,\ldots,j_n}$ is minimal. For suppose that there is a $X_{h_1,\ldots,h_n} \subset X_{j_1,\ldots,j_n}$ for which $I(X_{h_1,\ldots,h_n})$ is GL-projective or GL-inconsistent. Then $\vdash_{\mathsf{GAML}} X_{h_1,\ldots,h_n} \rhd \mathcal{H}$, and, since $X_{h_1,\ldots,h_n} \subset X_{j_1,\ldots,j_n}$, $\vdash_{\mathsf{GAML}} X_{j_1,\ldots,j_n} \rhd \mathcal{H}$.

Let us fix a minimal $\mathcal{D} = X_{j_1,\ldots,j_n}$ for some $j_1 \leq m_1, \ldots, j_n \leq m_n$. Clearly $\mathcal{D} \rhd \mathcal{H}$ is modal-irreducible. The following two claims establish the fullness of $\mathcal{D} \rhd \mathcal{H}$ with respect to (V^i) and (AC), which completes the proof.

Claim. $\mathcal{D} \rhd \mathcal{H}$ is full with respect to (V^i).

Proof. Suppose that $\mathcal{D}$ contains a sequent $(\Box\Gamma \Rightarrow \Box\Delta)$. Then $\mathcal{G}$ contains a hypersequent $(G \mid \Box\Gamma \Rightarrow \Box\Delta)$. We have to show that it contains a sequent $(\boxdot\Gamma \Rightarrow A)$ for some $A \in \Delta$. By the fullness of $\mathcal{G} \rhd \mathcal{H}$ with respect to $(\mathrm{V}^i)^h$, it follows that $\mathcal{G}$ contains the hypersequent $(G \mid \{\boxdot\Gamma \Rightarrow A \mid A \in \Delta\})$ (just G if $\Delta = \emptyset$). By the definition of $\mathcal{D} = X_{j_1,\ldots,j_n}$ it follows that either $(\boxdot\Gamma \Rightarrow A)$ belongs to $\mathcal{D}$, in which case we are done, or there is a sequent S in G that belongs to $\mathcal{D}$. But it is not hard to see that in this case there is a set $X_{h_1,\ldots,h_n}$, corresponding to a disjunct of C, that is the result of replacing $(\Box\Gamma \Rightarrow \Box\Delta)$ in $\mathcal{D}$ by S. But then $X_{h_1,\ldots,h_n}$ is a proper subset of $\mathcal{D}$, contradicting the minimality of $\mathcal{D}$. Observe that we use here the fact that no hypersequent in $\mathcal{G}$, being just a set of sequents, can contain the same sequent twice.

Claim. $\mathcal{D} \rhd \mathcal{H}$ is full with respect to (AC).

Proof. The proof is similar to the proof of the claim above, but let us spell it out nevertheless. Suppose that $\mathcal{D}$ contains the sequents $(\Gamma, A \Rightarrow \Delta)$ and $(\Pi \Rightarrow A, \Sigma)$. We have to show that it contains the sequent $(\Gamma, \Pi \Rightarrow \Sigma, \Delta)$. Observe that $\mathcal{G}$ has to contain hypersequents of the form $(G \mid \Gamma, A \Rightarrow \Delta)$ and $(H \mid \Pi \Rightarrow A, \Sigma)$ for some hypersequents G and H. By the fullness of $\mathcal{G} \rhd \mathcal{H}$ with respect to $(\mathrm{AC})^h$, it follows that if $\mathcal{D}$ does not contain the sequent $(\Gamma, \Pi \Rightarrow \Sigma, \Delta)$ it has to contain a sequent S from G or H. In this case, replacing the $(\Gamma, A \Rightarrow \Delta)$ in $\mathcal{D}$ by S in case S occurs in G and replacing $(\Pi \Rightarrow A, \Sigma)$ by S otherwise, we obtain a set $X_{h_1,\ldots,h_n}$, corresponding to a disjunct of C, that is a proper subset of $\mathcal{D}$, contradicting the minimality of $\mathcal{D}$. $\Box$

Theorem 4. *If* GAML *is* L*-fitting for a mono-extensible reflexive modal logic* L*, then* $\mathrel{|\!\sim}_{\mathsf{L}} \mathtt{R}$ *iff* $\vdash_{\mathsf{GAML}} \mathtt{R}$ *for any sgh-rule* $\mathtt{R}$*.*

Proof. Since the reasoning is similar to the completeness proof for irreflexive logics given above, we just explain the points of divergence and leave the details to the reader. In this case, for the left-to-right direction, it is sufficient by Lemma 4 to assume that $\mathtt{R} = (\mathcal{G} \rhd \mathcal{H})$ is a modal-semi-irreducible L-admissible sgh-rule that is full with respect to $(\mathrm{V}^r)^h$, $(\mathrm{AC})^h$, $(\mathrm{AC}_\Box)^h$, $(\wedge\Rightarrow)_\rhd{}^h$, and $(\to\Rightarrow)_\rhd{}^h$, and obtained by applying these rules (backwards) to a modal-irreducible gh-rule. We then define $X_{j_1,\ldots,j_n}$, C, and $\mathcal{D}$ as in the irreflexive case. This time, since each reflexive extension of $\mathsf{K4}$ contains $\mathsf{S4}$, it is sufficient to show that each $\bigwedge_{S \in X_{j_1,\ldots,j_n}} I(S)$ is either $\mathsf{S4}$-projective or $\mathsf{S4}$-inconsistent.

To achieve this we make use of the result established in [9], that a modal-semi-irreducible *gs-rule* that is full with respect to $(\mathrm{V}^r)^h$, $(\mathrm{AC})^h$, $(\mathrm{AC}_\Box)^h$, $(\wedge\Rightarrow)_\rhd{}^h$, and $(\to\Rightarrow)_\rhd{}^h$, is either $\mathsf{S4}$-projective or $\mathsf{S4}$-inconsistent. I.e. it is sufficient to show that $\mathcal{D} \rhd \mathcal{H}$ is a modal-semi-irreducible gs-rule that is full with respect to $(\mathrm{V}^r)^h$, $(\mathrm{AC})^h$, $(\mathrm{AC}_\Box)^h$, $(\wedge\Rightarrow)_\rhd{}^h$, and $(\to\Rightarrow)_\rhd{}^h$. The proofs of these facts are similar to the proofs of the claims in the previous proof, and are left to the reader. $\Box$

The obvious question remaining here (apart from extending beyond the reflexive and irreflexive cases) is what happens in the case of multiple-conclusion rules. We just give a partial answer here, leaving the general case for further investigation. First, we recall from [10] that L has *essentially single-conclusion* admissible rules if whenever $\Gamma \mathrel{|\!\sim}_{\mathsf{L}} \Delta$, there exists $A \in \Delta \cup \{\bot\}$ such that $\Gamma \mathrel{|\!\sim}_{\mathsf{L}} A$. Jeřábek has shown the following using the notion of filtering unification investigated by Ghilardi and Sacchetti in [6].

Theorem 5 ([10]). *Every extension of* $\mathsf{K4.2}$ *has essentially single-conclusion admissible rules. Moreover, any normal extension of* $\mathsf{K4.1}$ *with essentially single-conclusion admissible rules is an extension of* $\mathsf{K4.2}$*.*

Corollary 1. *If* GAML *is* L*-fitting for a mono-extensible reflexive or irreflexive extension* L *of* $\mathsf{K4.2}$*, then* $\mathrel{|\!\sim}_{\mathsf{L}} \mathtt{R}$ *iff* $\vdash_{\mathsf{GAML}} \mathtt{R}$ *for any gh-rule* $\mathtt{R}$*.*

Let us note finally for this section that concrete systems for admissiblity can be defined for mono-extensible logics such as $\mathsf{S4.2}$, $\mathsf{GL.3}$, etc. by adding to our core set of rules any kind of calculus for derivability in these logics. In particular, hypersequent calculi can be developed for many of these cases, but we omit the details here for space reasons.

4 Intermediate Logics

We turn our attention now to intermediate logics, recalling first the result of [8] that if an intermediate logic L admits the following *Visser rules*, then they form a basis for the admissible rules of L:

$$(V_n) \;\; (C \to (A_{n+1} \vee A_{n+2})) \vee D \;/\; (\bigvee_{j=1}^{n+2} C \to A_j) \vee D$$

Initial GS-Rules, Structural Rules, Anti-Cut Rule, Projection Rule: *as in the core modal rules*.

Logical Rules: *as in the core modal rules for* $\bot$, $\land$, *and* $\lor$, *plus*:

$$\frac{\mathcal{G} \,\rhd\, (\Gamma, A \to B \Rightarrow A, \Delta), \mathcal{H} \quad \mathcal{G} \,\rhd\, (\Gamma, B \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \,\rhd\, (\Gamma, A \to B \Rightarrow \Delta), \mathcal{H}} \,\rhd(\to\Rightarrow)^i \qquad \frac{\mathcal{G} \,\rhd\, (\Gamma, A \Rightarrow B), \mathcal{H}}{\mathcal{G} \,\rhd\, (\Gamma \Rightarrow A \to B, \Delta), \mathcal{H}} \,\rhd(\Rightarrow\to)^i$$

$$\frac{\mathcal{G}, (\Gamma, B \Rightarrow \Delta), (\Gamma, A \to B \Rightarrow A, \Delta) \,\rhd\, \mathcal{H}}{\mathcal{G}, (\Gamma, A \to B \Rightarrow \Delta) \,\rhd\, \mathcal{H}} \,(\to)\rhd$$

$$\frac{\mathcal{G}, (\Gamma \Rightarrow p, \Delta), (p, A \Rightarrow B) \,\rhd\, \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \to B, \Delta) \,\rhd\, \mathcal{H}} \,(\Rightarrow\to)\rhd^i \quad \frac{\mathcal{G}, (\Gamma, p \to q \Rightarrow \Delta), (p \Rightarrow A), (B \Rightarrow q) \,\rhd\, \mathcal{H}}{\mathcal{G}, (\Gamma, A \to B \Rightarrow \Delta) \,\rhd\, \mathcal{H}} \,(\to\Rightarrow)\rhd^i$$

where p and q do not occur in $\mathcal{G}, \mathcal{H}, \Gamma$, and Δ in $(\to\Rightarrow)\rhd^i$, $(\Rightarrow\to)\rhd^i$.

Visser Rule

$$\frac{[\mathcal{G}, (\Gamma \Rightarrow \Delta), (\Gamma \Rightarrow A) \,\rhd\, \mathcal{H}]_{A \in \Delta} \quad [\mathcal{G}, (\Gamma \Rightarrow \Delta) \,\rhd\, (\Gamma^{\Pi}, \Pi \Rightarrow \Delta), \mathcal{H}]_{\emptyset \neq \Pi \subseteq \Gamma_{\Delta}}}{\mathcal{G}, (\Gamma \Rightarrow \Delta) \,\rhd\, \mathcal{H}} \,(\mathrm{V})$$

where Γ contains only implications, and:

1. $\Gamma^{\Pi} = \{A \to B \in \Gamma : A \notin \Pi\}$. 2. $\Gamma_{\Delta} = \{A \notin \Delta : \exists B\, (A \to B) \in \Gamma\}$.

Fig. 4. The Calculus GAMI

for $n = 1, 2, \ldots$, where $C = \bigwedge_{i=1}^{n}(A_i \to B_i)$. In some cases, such as Gödel-Dummett logic LC, the Visser rules (and hence all admissible rules) are derivable. Here we consider some logics where this does not happen: in particular, de Morgan (or Jankov) logic KC, axiomatized by adding the axiom $\neg A \lor \neg\neg A$ to IPC, and the family of logics with Kripke models of bounded cardinality BC_n for $n = 1, 2, \ldots$ (noting that for $n = 1, 2$, the Visser rules are in fact derivable).

We also recall Ghilardi's useful characterization of IPC-projective formulas.

Theorem 6 (Ghilardi [4]). *For Kripke models $K_1, \ldots, K_n$, let $(\sum_i K_i)'$ denote the Kripke model obtained by attaching one new node below all nodes in $K_1, \ldots, K_n$ where no variables are forced. A class of Kripke models $\mathcal{K}$ has the* extension property *if for every finite family of models $K_1, \ldots, K_n \in \mathcal{K}$, there is a variant of $(\sum_i K_i)'$ in $\mathcal{K}$. A formula is* IPC *-projective iff its class of Kripke models has the extension property.*

Figure 4 displays the gs-rule calculus GAMI for Intuitionistic Logic of [9]. In this case it is the (non-invertible on the right) implication rules that use new variables on the left.

Theorem 7 ([9]). $\vdash_{\mathsf{GAMI}} \mathtt{R}$ *iff* $\mathrel{|\!\sim}_{\mathsf{IPC}} \mathtt{R}$ *for any gs-rule* $\mathtt{R}$.

Just as we stepped from extensible to mono-extensible modal logics, so we can step here from a calculus for IPC to calculi for intermediate logics admitting the Visser rules. Note, however, that we require a slightly different (more usual) interpretation of hypersequents. For an intermediate logic L and hypersequent G, we write $\vdash_{\mathsf{L}} G$ iff $\vdash_{\mathsf{L}} I(G)$ where $I(G) = \bigvee_{S \in G} I(S)$. Also, a gh-rule $\mathtt{R} = (G_1, \ldots, G_n \rhd H_1, \ldots, H_m)$ is L-*admissible*, written $\mathrel{|\!\sim}_{\mathsf{L}} \mathtt{R}$, iff $\{I(G_i)\}_{i=1}^{n} \mathrel{|\!\sim}_{\mathsf{L}} \{I(H_j)\}_{j=1}^{m}$ and L-*derivable*, written $\vdash_{\mathsf{L}} \mathtt{R}$, iff $\bigwedge_{i=1}^{n} I(G_i) \vdash_{\mathsf{L}} \bigvee_{j=1}^{m} I(H_j)$. It will also be helpful to restrict the notion of

$$\frac{\mathcal{G}, (G \mid \Gamma, A \Rightarrow \Delta), (H \mid \Pi \Rightarrow A, \Sigma)(G \mid H \mid \Gamma, \Pi \Rightarrow \Sigma, \Delta) \rhd \mathcal{H}}{\mathcal{G}, (G \mid \Gamma, A \Rightarrow \Delta), (H \mid \Pi \Rightarrow A, \Sigma) \rhd \mathcal{H}} \ (\text{AC})^h$$

$$\frac{\mathcal{G}, G \rhd (I(G) \Rightarrow I(H)), \mathcal{H}}{\mathcal{G}, G \rhd \mathcal{H}} \ (\text{IPJ}) \quad \text{where } H \in \mathcal{H} \cup \{\Rightarrow\}$$

$$\frac{\mathcal{G}, (G \mid \{\Gamma \Rightarrow A\}_{A \in \Delta}) \rhd \mathcal{H} \quad [\mathcal{G} \rhd (\Gamma^{\Pi}, \Pi \Rightarrow \Delta), \mathcal{H}]_{\Pi \subseteq \Gamma_{\Delta}}}{\mathcal{G}, (G \mid \Gamma \Rightarrow \Delta) \rhd \mathcal{H}} \ (\text{V})^h$$

where Γ contains only implications, and:
1. $\Gamma^{\Pi} = \{A \to B \in \Gamma : A \notin \Pi\}$. 2. $\Gamma_{\Delta} = \{A \notin \Delta : \exists B\, (A \to B) \in \Gamma\}$.

Fig. 5. Additional Rules for Intermediate Logics

an sgh-rule a little further to an *ssgh-rule*: an sgh rule where not only is there at most one hypersequent on the right, but also this hypersequent consists of just one sequent. Essentially, the reason for this is that completeness results for hypersequent calculi for intermediate logics given in the literature (see e.g. [3]) are typically restricted to sequents rather than hypersequents.

Definition 5. *A calculus* GAML *is* L*-fitting for an intermediate logic* L *if:*

(1) GAML *extends the* core intermediate rules*: gh-versions of the initial gs-rules, structural rules, and logical rules of* GAMI*, and the additional rules of Fig. 5.*
(2) If $\vdash_{\mathsf{L}} S$*, then* $\vdash_{\mathsf{GAML}} \rhd S$ *for any sequent* S*.*
(3) If $\vdash_{\mathsf{GAML}} \mathtt{R}$*, then* $\mid\!\sim_{\mathsf{L}} \mathtt{R}$*.*

Lemma 5. *The core intermediate rules are* L*-sound for every intermediate logic* L *admitting the Visser rules.*

Proof. Let L be an intermediate logic admitting the Visser rules. We just consider $(\text{V})^h$ since other proofs are very similar to those for GAMI in [9]. Suppose that σ is an L-unifier for $I(H)$ for all $H \in \mathcal{G}$ and $I(G \mid \Gamma \Rightarrow \Delta)$, where $\Delta = \{A_1, \ldots, A_n\}$. Using the right set of premises, σ is an L-unifier for $I(\Gamma^{\Pi}, \Pi \Rightarrow \Delta)$ for all $\Pi \subseteq \Gamma_{\Delta}$. It suffices now to show that σ is an L-unifier for $I(G \mid \{\Gamma \Rightarrow A\}_{A \in \Delta})$. Suppose, arguing contrapositively, that this is not the case. Then there exists a countermodel of L for $I(\sigma(G)) \vee \bigvee_{A \in \Delta} I(\sigma(\Gamma) \Rightarrow \sigma(A))$. This implies that for every $A \in \Delta$ there are countermodels K_A such that K_A is a model of L, $K_A \Vdash \sigma(\bigwedge \Gamma)$, and $K_A \nVdash \sigma(A)$. Since L admits the Visser rules, there is a variant K of $(\sum_{A \in \Delta} K_A)'$ that is a model of L. Let $\Pi = \{D \in \Gamma_{\Delta} : K \Vdash \sigma(D)\}$. Observe that for all $B \to C \in \Gamma$ such that $B \notin \Pi$, either $B \in \Delta$ or $K \nVdash \sigma(B)$. Note also that $B \in \Delta$ implies $K \nVdash \sigma(B)$. Hence for all $B \notin \Pi$ it follows that $K \nVdash \sigma(B)$, and so $K \Vdash \sigma(B \to C)$. It follows that $K \Vdash \sigma(\bigwedge(\Gamma^{\Pi} \cup \Pi))$. Thus $K \Vdash \sigma(\bigvee \Delta)$, a contradiction. □

The difference between the rules $(\text{V})^h$ and (V) is essentially due to the fact that IPC has the disjunction property, while intermediate logics in general do not.

The core set of rules given above can be extended on the right to obtain proof systems for admissibility in various intermediate logics. In particular, we can make use of hypersequent calculi provided for KC and BC_n $(n = 1, 2, \ldots)$ in [3], to obtain:

– GAMKC consists of the core intermediate rules plus:

$$\frac{\mathcal{G} \,\triangleright\, (G \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \Gamma_1, \Gamma_2 \Rightarrow)}{\mathcal{G} \,\triangleright\, (G \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2)} \text{ (J)}$$

– $\mathsf{GAMBC_n}$ for $n = 1, 2, \ldots$ consists of the core intermediate rules plus:

$$\frac{[\mathcal{G} \,\triangleright\, (G \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_{n+1} \Rightarrow \Delta_{n+1} \mid \Gamma_i, \Gamma_j \Rightarrow \Delta_i)]_{1 \leq i < j \leq n+1}}{\mathcal{G} \,\triangleright\, (G \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_{n+1} \Rightarrow \Delta_{n+1})} \text{ (BC}_n\text{)}$$

Corollary 2. GAMKC *is* KC*-fitting and* $\mathsf{GAMBC_n}$ *is* BC_n*-fitting for* $n = 1, 2, \ldots$.

To prove completeness for L-fitting systems GAML for intermediate logics L admitting the Visser rules, we proceed similarly to the case of mono-extensible modal logics. First we can show, exactly as in Lemma 2 (except replacing the application of the modal deduction theorem with the usual deduction theorem), that L-derivable ssgh-rules are also GAML-derivable.

Lemma 6. *Let* L *be an intermediate logic admitting the Visser rules and let* GAML *be* L*-fitting. If* $\vdash_{\mathsf{L}} \mathtt{R}$*, then* $\vdash_{\mathsf{GAML}} \mathtt{R}$ *for any ssgh-rule* $\mathtt{R}$.

As for Lemmas 3 and 4, applying the invertible left logical rules backwards reduces any gh-rule to a gh-rule of a certain form (in this case with variable implications on the left), and then applying the rules $(\mathrm{V})^h$, $(\rightarrow)\triangleright^h$, and $(\mathrm{AC})^h$ exhaustively backwards terminates with a set of gh-rules full with respect to these rules.

Lemma 7. *A gh-rule* $\mathcal{G} \triangleright \mathcal{H}$ *is* implication-irreducible *if all sequents in* $\mathcal{G}$ *contain only variables on the right and variables and variable implications on the left. Every admissible gh-rule is* GAML*-derivable from admissible implication-irreducible gh-rules that are full with respect to* $(\mathrm{V})^h$*,* $(\rightarrow)\triangleright^h$*, and* $(\mathrm{AC})^h$.

The completeness theorem is then established similarly to the proof for IPC in [9], the main complication being that (as in the mono-extensible modal case) we now have to take care of all the different disjuncts occurring in hypersequents on the left.

Theorem 8. *For any intermediate logic* L *admitting the Visser rules and* L*-fitting calculus* GAML*,* $\mid\!\sim_{\mathsf{L}} \mathtt{R}$ *iff* $\vdash_{\mathsf{GAML}} \mathtt{R}$ *for any ssgh-rule* $\mathtt{R}$.

Proof. The right-to-left direction follows directly from Lemma 5. For the left-to-right direction, it is sufficient to assume (proceeding exactly as in the IPC-case) that $\mathtt{R} = (\mathcal{G} \triangleright \mathcal{H})$ is an L-admissible implication-irreducible gh-rule that is full with respect to $(\mathrm{V})^h$, $(\rightarrow)\triangleright^h$, and $(\mathrm{AC})^h$. The proof is similar to the completeness proofs for the modal logics given above, but since some details are essentially different we will sketch the proof for intermediate logics briefly.

Define C and $X_{j_1,\ldots,j_n}$ as in the modal completeness proofs above, but without boxes. By Lemma 1 (d), if each $I(X_{j_1,\ldots,j_n})$ is IPC-projective or IPC-inconsistent, then $\vdash_{\mathsf{L}} \mathcal{G} \triangleright \mathcal{H}$. But then by Lemma 6, $\vdash_{\mathsf{GAML}} \mathcal{G} \triangleright \mathcal{H}$. It is then sufficient to show that each consistent $\bigwedge_{X_{j_1,\ldots,j_n}} I(S)$ is IPC-projective or derives $I(\mathcal{H})$. Recall that $X_{j_1,\ldots,j_n}$ is called minimal if there is no $X_{h_1,\ldots,h_n}$ that is a proper subset of $X_{j_1,\ldots,j_n}$. Reasoning as in the modal case, it suffices to consider only minimal sets. Let $\mathcal{D}$ denote a minimal set $X_{j_1,\ldots,j_n}$. To show that $\mathcal{D} \triangleright \mathcal{H}$ has the mentioned properties, we make use of the result established in [9], that an implication-irreducible *gs-rule* that is full with respect to $(\vee)^h$, $(\rightarrow)\triangleright^h$, and $(\mathrm{AC})^h$ is either projective or derives $I(\mathcal{H})$. I.e. it is sufficient to show that $\mathcal{D} \triangleright \mathcal{H}$ is an implication-irreducible gs-rule that is full with respect to $(\vee)^h$, $(\rightarrow)\triangleright^h$, and $(\mathrm{AC})^h$. Proofs of these facts are similar to the claims in the completeness proofs for modal logics. We will only treat $(\vee)^h$, leaving the other cases to the reader.

Claim. $\mathcal{D} \triangleright \mathcal{H}$ is full with respect to the rule $(\vee)^h$.

Proof. Suppose that $\mathcal{D}$ contains a sequent $(\Gamma \Rightarrow \Delta)$, where Γ contains only implications. Thus $\mathcal{G}$ contains a hypersequent $(G \mid \Gamma \Rightarrow \Delta)$ for some hypersequent G. By the fullness of $(\mathcal{G} \triangleright \mathcal{H})$ with respect to $(\vee)^h$, it follows that either $\mathcal{H}$ contains $(\Gamma^{\Pi}, \Pi \Rightarrow \Delta)$ for some $\Pi \subseteq \Gamma_{\Delta}$, or $\mathcal{D}$ contains a sequent $(\Gamma \Rightarrow A)$ for some $A \in \Delta$, or $\mathcal{D}$ contains a sequent S of G. In the first two cases we are done. In the last case, by replacing $(\Gamma \Rightarrow \Delta)$ by S in $\mathcal{D}$, we obtain a set $X_{h_1,\ldots,h_n}$, corresponding to a disjunct of C, that is a proper subset of $\mathcal{D}$, contradicting the minimality of $\mathcal{D}$. $\square$

Corollary 3. *For* $\mathsf{L} \in \{\mathsf{KC}, \mathsf{BC}_1, \mathsf{BC}_2, \ldots\}$*:* $\mid\!\sim_{\mathsf{L}} \mathtt{R}$ *iff* $\vdash_{\mathsf{GAML}} \mathtt{R}$ *for any ssgh-rule* $\mathtt{R}$.

As in the modal case, there exist essentially single-conclusion logics where the preceding corollary extends to multiple-conclusion rules; indeed, we conjecture that this is the case for all extensions of KC admitting the Visser rules.

References

1. A. Avron. A constructive analysis of RM. *Journal of Symbolic Logic*, 52(4):939–951, 1987.
2. A. Chagrov and M. Zakharyaschev. *Modal Logic*. Oxford University Press, 1996.
3. A. Ciabattoni and M. Ferrari. Hypersequent calculi for some intermediate logics with bounded Kripke models. *Journal of Logic and Computation*, 11(2):283–294, 2001.
4. S. Ghilardi. Unification in intuitionistic logic. *Journal of Symbolic Logic*, 64(2):859–880, 1999.
5. S. Ghilardi. Best solving modal equations. *Annals of Pure and Applied Logic*, 102(3):184–198, 2000.
6. S. Ghilardi and L. Sacchetti. Filtering unification and most general unifiers in modal logic. *Journal of Symbolic Logic*, 69(3):879–906, 2004.
7. R. Iemhoff. On the admissible rules of intuitionistic propositional logic. *Journal of Symbolic Logic*, 66(1):281–294, 2001.
8. R. Iemhoff. Intermediate logics and Visser's rules. *Notre Dame Journal of Formal Logic*, 46(1):65–81, 2005.
9. R. Iemhoff and G. Metcalfe. Proof theory for admissible rules. Submitted. Available online at http://www.math.vanderbilt.edu/people/metcalfe/publications.
10. E. Jerábek. Admissible rules of modal logics. *Journal of Logic and Computation*, 15:411–431, 2005.
11. G. Metcalfe, N. Olivetti, and D. Gabbay. *Proof Theory for Fuzzy Logics*. Springer Series in Applied Logic Vol. 36, to appear 2009.
12. P. Rozière. *Regles Admissibles en calcul propositionnel intuitionniste*. PhD thesis, Université Paris VII, 1992.
13. V. Rybakov. *Admissibility of Logical Inference Rules*. Elsevier, 1997.