The unification type of Łukasiewicz logic

Based on a joint work with V. Marra

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E-unifiers

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$\mathcal{L}$ is a first order language and
E is an equational theory in $\mathcal{L}$, arbitrary but fixed.

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$$\text{Let } s, t \in Term(L)$$

A substitution σ is an E-unifier for the pair (s, t) if

$$E \models \sigma(t) \approx \sigma(s)$$

Jump to Łukasiewicz logic

Ordering the set of solutions

Given two unifiers σ, σ' for a pair (s,t) we say that

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if there is a substitution τ such that

$$E \models \sigma'(x) \approx \tau \circ \sigma(x) \text{ for all } x \in Var(s, t)$$

E-equivalence

The relation $\leq_E$ is a pre-order on the set of E-unifiers.

So it makes sense to say that two substitutions σ, σ' are

E-equivalent iff $\sigma \leq_E \sigma'$ and $\sigma' \leq_E \sigma$,
(written $\sigma \sim_E \sigma'$).

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2. if τ is a unifier for (s, t), $\sigma \leq_E \tau$ implies $\sigma \sim_E \tau$.

In other words, σ is a maximal element in the partial order induced by the equivalence $\sim_E$ on the preorder $\leq_E$.

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For any pair (s, t), there is one E-unifier μ which is more general than any E-unifier of (s, t).

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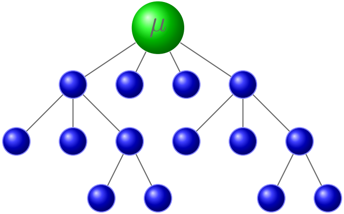

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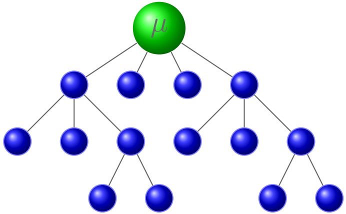

Then the unification type of E is called unary.

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For any pair (s, t), there are finitely many E-unifiers $\mu_1, \ldots, \mu_n$ such that for any unifier σ, some μ_i is more general than σ.

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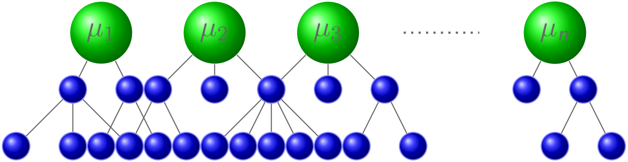

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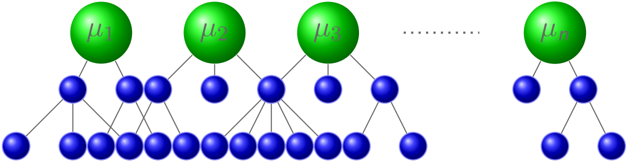

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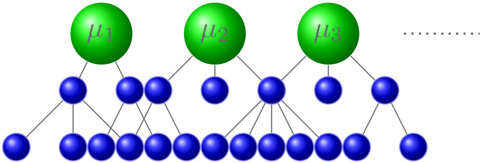

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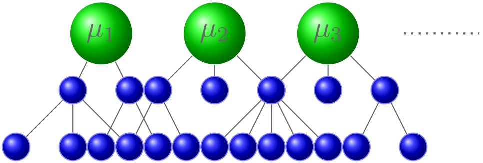

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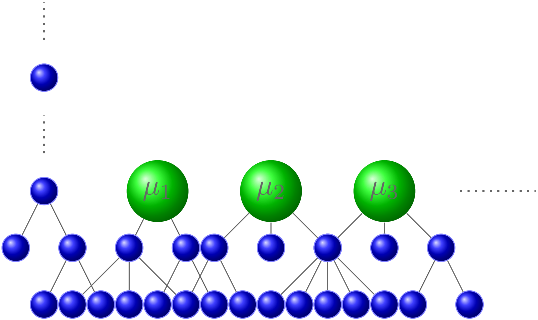

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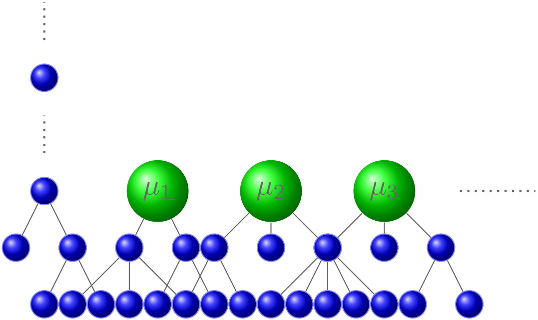

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Łukasiewicz logic

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Łukasiewicz (infinite-valued propositional) logic is a non-classical system going back to the 1920's which may be axiomatised using the primitive connectives $\to$ (implication) and $\neg$ (negation) by the four axiom schemata:

(A1) $\alpha \to (\beta \to \alpha)$,

(A2) $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$,

(A3) $(\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$,

(A4) $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$,

with *modus ponens* as the only deduction rule.

Semantics of Łukasiewicz logic

The semantics of Łukasiewicz logic is many-valued: assignments μ to atomic formulæ range in the unit interval $[0,1] \subseteq \mathbb{R}$; they are extended compositionally to compound formulæ via

$$\mu(\alpha \to \beta) = \min\{1, 1 - \mu(\alpha) + \mu(\beta)\}\,,$$
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Tautologies are defined as those formulæ that evaluate to 1 under every such assignment.
Chang first considered the Tarski-Lindenbaum algebras of Łukasiewicz logic, and called them MV-algebras. This allowed him to obtain an algebraic proof of the completeness theorem.

MV-algebras

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- * is an involution
- the interaction between those two operations is described by the following two axioms:
  - $x \oplus 0^* = 0^*$
  - $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$

Some small technical considerations

In MV-alegrbas (as in several other cases) the problem of unifying two terms reduces to finding a substitution that identifies a term with a constant (in our case either 0 or 1).

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$$\begin{cases} t_1(\bar{x}) = s_1(\bar{x}) \\ t_1(\bar{x}) = s_2(\bar{x}) \\ \ldots \\ t_n(\bar{x}) = s_n(\bar{x}) \end{cases} \qquad \Longleftrightarrow \qquad u(\bar{x}) = 1.$$

Projective objects

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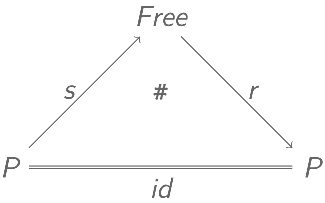

Algebraic unifiers

Let us think of a generic E-unification problem (s, t) as a finitely presented E-algebra $A = \mathcal{F}_n / \langle (s, t) \rangle$.

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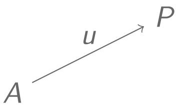
An algebraic E-unifier for the problem $A = \mathcal{F}_n / \langle (s, t) \rangle$ is a pair (P, u) where

1. P is a finitely presented projective E-algebra and
2. u is an arrow from A to P, $u \colon A \longrightarrow P$.

Most general algebraic unifiers

An algebraic E-unifier (P, u)



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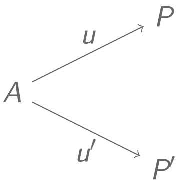

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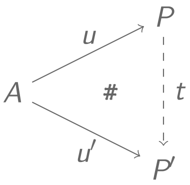

Equivalence of the two approaches

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Theorem (1997 Ghilardi)

The syntactic approach and the algebraic are equivalent (as categories).

Jump to the proof

Unitarity of finite-valued Łukasiewicz logic

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This was re-proved explicitly and generalised to any finite-valued extension of Basic Logic by Dzik.

Non unitarity of the unification in Łukasiewicz logic

Łukasiewicz logic has a weak disjunction property. Namely:

if $\varphi \vee \neg\varphi$ is derivable then either φ or $\neg\varphi$ must be derivable. (in other words the multiple-conclusion rule $\varphi \vee \neg\varphi / \varphi, \neg\varphi$ is admissible.)

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Indeed if σ is a unifier for $x \vee \neg x$, then it must unify either x (hence it is the substitution $x \mapsto 1$) or $\neg x$ (hence it must be the substitution $x \mapsto 0$).

Intermezzo: Commutative ℓ-groups...

Theorem (Mundici 1986)

The category of MV-alegrbas is equivalent to the category of Abelian ℓ-groups with strong unit *(with ℓ-morphisms preserving the strong unit).*

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In 1975, Beynon, expanding previous results by Baker, established a categorical duality which enabled a geometrical study of finitely presented ℓ-groups.

This duality led to the following purely algebraic result.

Theorem (1977 Beynon)

Finitely generated projective ℓ-groups are exactly the finitely presented ℓ-groups.

...and their unification type.

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In a forthcoming paper with V. Marra, we exploit Beynon's geometrical duality to give an algorithm that, taken any (system of) term in the language of ℓ-groups, outputs its most general unifier.

Finitarity result

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The unification type of the 1-variable fragment of Łukasiewicz logic is finitary.

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Proposition

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In particular the proof shows that there are at most two most general unifiers, for any given formula.

Jump to the proof

A geometrical view on unification

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Rational polyhedra and MV-algebras

Let $\mathcal{MV}_{fp}$ be the category of finitely presented MV-algebras with their homomorphisms.

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Let $\mathcal{MV}_{fp}$ be the category of finitely presented MV-algebras with their homomorphisms.
Let $\mathcal{P}_{\mathbb{Z}}$ be the category of rational polyhedra and $\mathbb{Z}$-maps between them.
I will define a pair of (contravariant) functors:

$$\mathcal{I}\colon \mathcal{MV}_{fp} \to \mathcal{P}_{\mathbb{Z}} \qquad \text{and} \qquad \mathcal{V}\colon \mathcal{P}_{\mathbb{Z}} \to \mathcal{MV}_{fp}\,.$$

These functors operate very similarly to the classical ones in algebraic geometry that associate ideals with varieties.

Interpreting MV-algebras inside polyhedra: objects

Let $P \in \mathcal{P}_{\mathbb{Z}}$.

Let us write $\imath(P)$ for the collection of all pair MV-terms (s, t) such that

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The set $\imath(P)$ is a congruence of the free MV-algebras on n generators, so it makes sense to set

$$\mathcal{I}(P) = \frac{Free_n}{\imath(P)}.$$

Interpreting polyhedra inside MV-algebras: arrows

Let $\zeta\colon P \to Q$ be a diagram in $\mathcal{P}_{\mathbb{Z}}$.

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Then, the function $\mathcal{I}(\zeta)\colon \mathcal{I}(Q) \to \mathcal{I}(P)$ is a homomorphism of MV-algebras.

Interpreting polyhedra inside MV-algebras: objects

$$\text{Let } A = \frac{Free_n}{\theta} \in \mathcal{MV}_{fp}.$$

Let us write $V(\theta)$ for the collection of all real points p in $[0,1]^n$ such that

$$s(p) = t(p) \text{ for all } (s,t) \in \theta$$

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The set $V(\theta)$ is a rational polyhedron, so we set

$$\mathcal{V}(A) = V(\theta)\,.$$

Interpreting MV-algebras inside polyhedra: arrows

Let $h\colon A \to B$ be a diagram in $\mathcal{MV}_{fp}$.

Suppose that h sends the generators of A into the elements $\{t_i\}_{i \in I}$ of B, then define

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Then, the function $\mathcal{V}(h)\colon \mathcal{V}(B) \to \mathcal{V}(A)$ is a $\mathbb{Z}$-map.

Duality for finitely presented MV-algebras

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Theorem (Folklore)

The pair of functors

$$\mathcal{I}\colon \mathcal{MV}_{fp} \to \mathcal{P}_{\mathbb{Z}} \quad \text{and} \quad \mathcal{V}\colon \mathcal{P}_{\mathbb{Z}} \to \mathcal{MV}_{fp}\,.$$

constitutes a contravariant equivalence between the two categories.

Co-unification problems

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A co-unifier for A is a pair (P, u), where

1. P is a $\mathbb{Z}$-retract of $[0,1]^n$ (i.e. a retract by $\mathbb{Z}$-maps) for some n, and
2. u is a $\mathbb{Z}$-map from P to A.

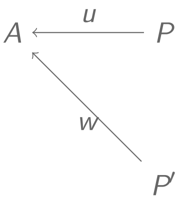
Co-unification type

A co-unifier (P, u)

$$A \xleftarrow{\ u\ } P$$

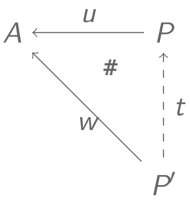
Co-unification type

A co-unifier $(P, u)$is more general than (Q, w)



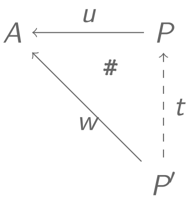
Co-unification type

A co-unifier $(P, u)$is **more general** than (Q, w) if there exists t such that:



Co-unification type

A co-unifier $(P, u)$is more general than (Q, w) if there exists t such that:



Remark

The unification type of Łukasiewicz logic and the co-unification type of rational prolyhedra coincide.

Nullarity of Łukasiewicz logic

Theorem

The full Łukasiewicz logic has nullary unification type.

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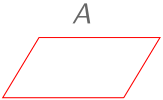
The rational polyhedron $\mathcal{V}(\theta)$ associated to the finitely presented MV-algebra $Free_2/\langle\theta\rangle$ is the following union of four rational polytopes:



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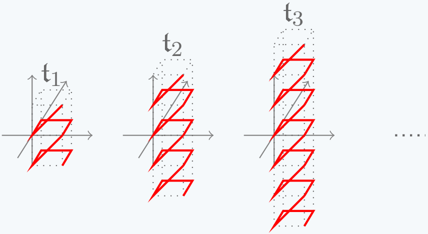

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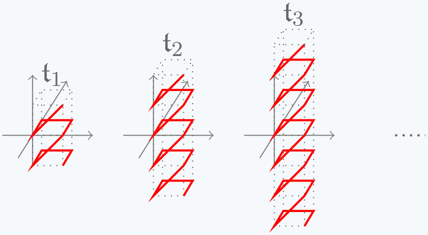

Together with the projections $\zeta_i \colon \mathfrak{t}_i \to A$ into A. It can be proved (cfr. Cabrer and Mundici) that each $\mathfrak{t}_i$ is a retract of $[0,1]^m$ for some m,

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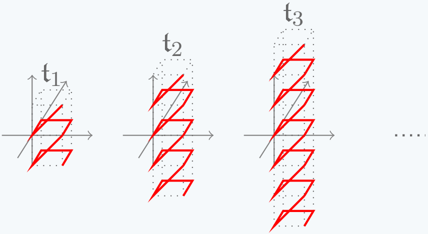

Together with the projections $\zeta_i\colon \mathfrak{t}_i \to A$ into A. It can be proved (cfr. Cabrer and Mundici) that each $\mathfrak{t}_i$ is a retract of $[0,1]^m$ for some m, so the pairs $(\mathfrak{t}_i, \zeta_i)$ are co-unifiers for A.

Proof Cont.'d

Step 2.

The sequence is increasing, i.e.

Proof Cont.'d

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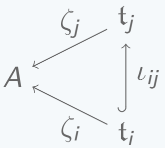
The sequence is increasing, i.e.for any $i < j$,
there exists ι_{ij} such that the following diagram commutes.



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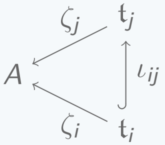

Indeed ι_{ij} is the embedding of $\mathfrak{t}_i$ in $\mathfrak{t}_j$

Proof Cont.'d

Step 3: The lifting of functions Lemma.

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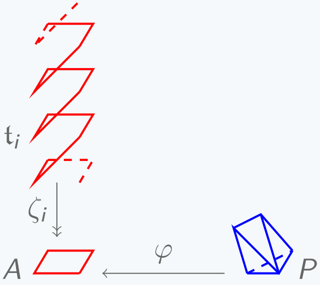
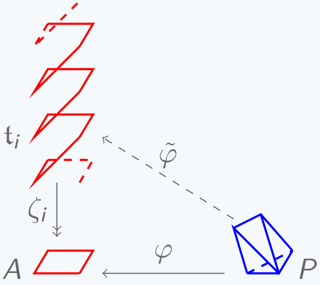


Proof Cont.'d

Step 3: The lifting of functions Lemma.

For any $\mathbb{Z}$-retract P of some cube $[0,1]^n$ and for any arrow $\varphi\colon P \to A$, there exists some $\mathfrak{t}_i$ and an arrow $\tilde{\varphi}$ (called the lift of φ) making the following diagram commute.



The universal cover of the circle

Intermezzo 2

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The reason why the above lemma works is that the infinite spiral $\mathfrak{t}_\infty = \bigcup_{i\in\omega} \mathfrak{t}_i$ is the piecewise linear correspondent of the universal cover of the circle, a space that, indeed, enjoys this factorisation property for any continuous maps from a simply connected spaces into the circle.

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Such a lift is known to be unique even in the rather general case of continuous maps, so the fact that in our setting such a map is actually a $\mathbb{Z}$-map is a quite pleasant discovery.

Proof Cont.'d

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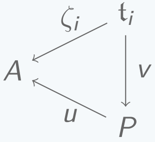
Step 4: Corollary 1.

If (P, u) is a co-unifier for A with strictly fewer lattice points than $\mathfrak{t}_i$, then there is no arrow $v\colon \mathfrak{t}_i \to P$ making the following diagram commute.



Proof Cont.'d

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As a consequence of the previous corollary we obtain:

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As a consequence of the previous corollary we obtain:

1. The sequence of $\mathfrak{t}_i$ is strict; i.e. $\mathfrak{t}_i$ is not more general than $\mathfrak{t}_j$ if $i < j$.
2. The sequence admits no bound with a finite number of lattice elements. Therefore, no rational polyhedra can bound the sequence of $\mathfrak{t}_i$.

Proof Cont.'d

Step 6: Corollary 2.

Given any co-unifier (P, u) for A, there exists a co-unifier of the form $(\mathfrak{t}_i, \zeta_i)$ such that

$$(P, u) \leq_{\mathcal{P}_{\mathbb{Z}}} (\mathfrak{t}_i, \zeta_i) .$$

End of the Proof

Conclusion

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Summing up, we have found a strictly linearly ordered, cofinal sequence of unifiers for A.
Furthermore the sequence is unbounded, hence the co-unification type of rational polyhedra is nullary (in a stronger sense).

This proves that the Łukasiewicz calculus (as well as the theory of MV-algebras and ℓ-groups with strong unit) has nullary unification type. □

▸ Bibliography

Equivalence of the syntactical and algebraic approaches

The correspondence e works as follows. ▸ Back

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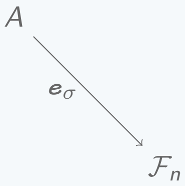

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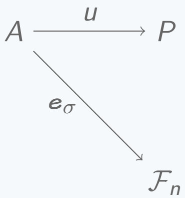

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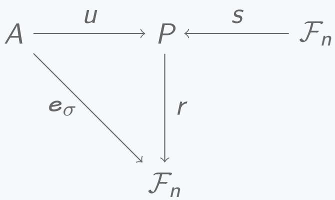

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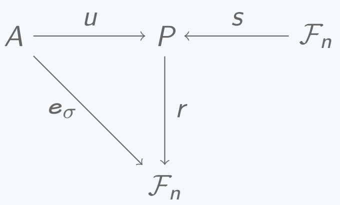

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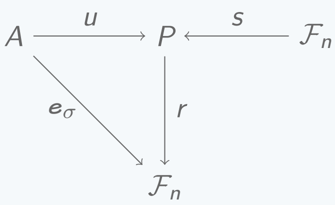

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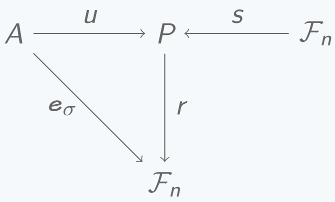

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Proof of finitarity

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In the 1-variable case, substitutions are just McNaughton functions.

Proof cont'd

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Step 1

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Proof cont'd

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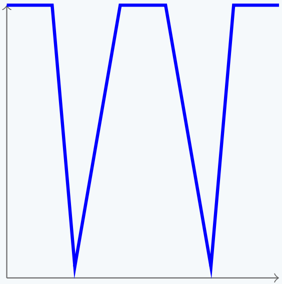

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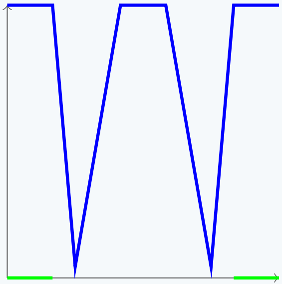

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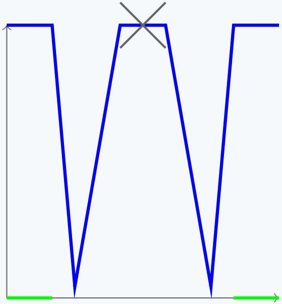

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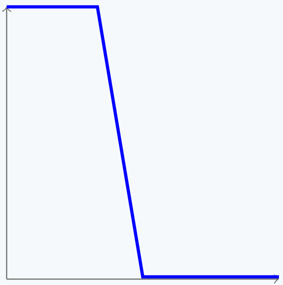

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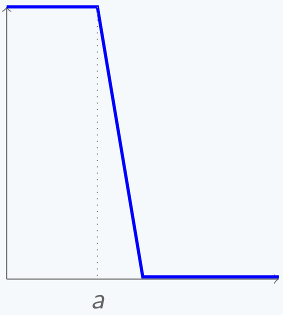

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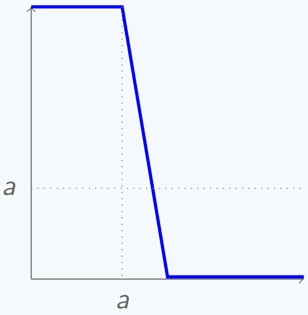

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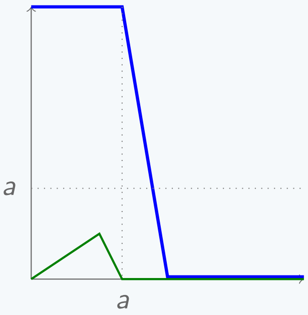

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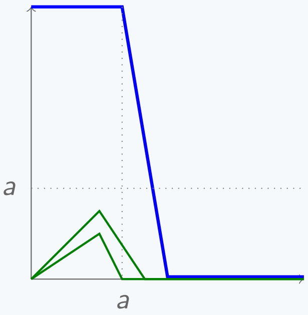

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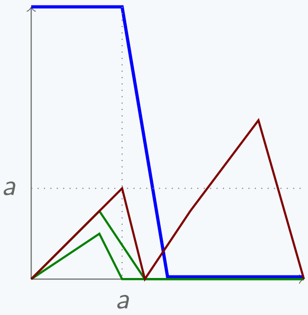

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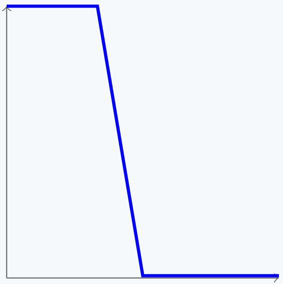

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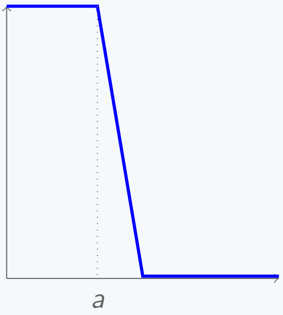

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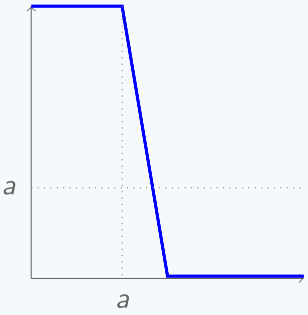

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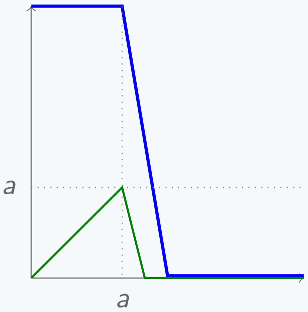

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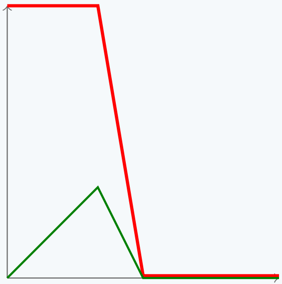

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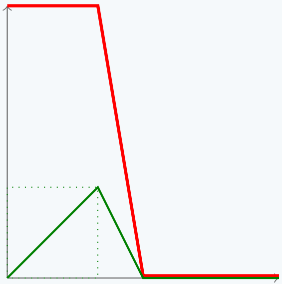

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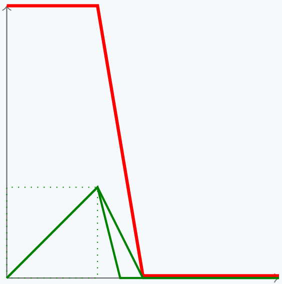

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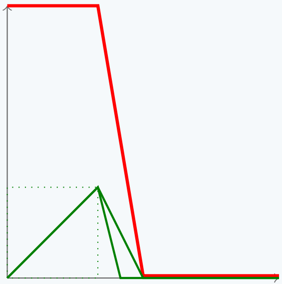

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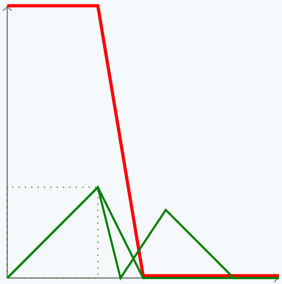

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