On sets, functions and relations

Some solutions

Rosalie Iemhoff

December 5, 2008

1 Exercises Chapter 2

1. $\{n \in \mathbb{N} \mid \exists m \in \mathbb{N}(n = m^2)\}$.

3. For example $\{n \in \mathbb{N} \mid \exists m \in \mathbb{Z}(n = 3m)\}$, $\{n \in \mathbb{Z} \mid n \geq 0$ and $(n/3) \in \mathbb{Z}\}$, and $\{0, 3, 6, 9, 12, \dots\}$.

8. $\{0, 1\}$.

10.

$$\begin{array}{rcl} x \in C \backslash (A \cap B) & \Leftrightarrow & x \in C \text{ and } x \notin A \cap B \\ & \Leftrightarrow & x \in C \text{ and } (x \notin A \text{ or } x \notin B) \\ & \Leftrightarrow & (x \in C \text{ and } x \notin A) \text{ or } (x \in C \text{ and } x \notin B) \\ & \Leftrightarrow & x \in (C \backslash A) \cup (C \backslash B). \end{array}$$

14. $\emptyset \in P(X)$ because $\emptyset \subseteq X$, for any set X. Also $X \in P(X)$, as $X \subseteq X$.

15. Assume $X \subseteq Y$ and $Y \subseteq Z$. We show that $X \subseteq Z$. That is, that $\forall x(x \in X \to x \in Z)$. Therefore, assume $x \in X$. Then $x \in Y$ because $X \subseteq Y$. But then $x \in Z$ since $Y \subseteq Z$.

16. $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{1, 2, 3, 4\}$.

17. $2^6 = 64$ subsets. $\mathbb{N}$ clearly has infinitely many subsets.

2 Exercises Chapter 3

1. $\{\langle r,s\rangle \mid r,s\in\mathbb{R}, r=\sqrt{s}\}$ or $\{\langle r,s\rangle\in\mathbb{R}^2 \mid r^2=s\}$.

2. $\{\langle a,a\rangle,\langle a,c\rangle,\langle a,d\rangle,\langle b,a\rangle,\langle b,c\rangle,\langle b,d\rangle\}$.

3. The diagonal.

4. The pairs of reals which sum is a rational number. Let us call this relation R. R is symmetric: $\langle r,s\rangle\in R$ implies $(r+s)\in\mathbb{Q}$, which implies $(s+r)\in\mathbb{Q}$, which implies $\langle s,r\rangle\in R$. The relation is not linear: neither $\langle\pi,2\pi\rangle\in R$, nor $\pi=2\pi$, nor $\langle 2\pi,\pi\rangle\in R$.

5. The relation $R=\{\langle n,m\rangle\in\mathbb{Z}^2 \mid n^2=m\}$ is not dense: $\langle 2,4\rangle\in R$ since $2^2=4$, but there is no $k\in\mathbb{Z}$ such that $\langle 2,k\rangle\in R$ and $\langle k,4\rangle\in R$, as this would imply both $k=2^2$ and $k^2=4$, i.e. $k=4$ and $k=2$ or $k=-2$.

8. Let us start with the following observation. If we let R denote $A\times B$, then
$$aRb \;\Leftrightarrow\; a\in A \text{ and } b\in B.$$
Now we turn to the exercise. It contains a mistake. It should read: prove that $A\times B$ is serial if and only if B is not empty or A is empty. Recall that seriality of R means $\forall a\in A\exists b\in B aRb$. Thus, using the observation above, seriality in this case boils down to $\forall a\in A\exists b\in B$. Observe that $\forall a\in A\exists b\in B$ exactly holds when A is empty or B is not empty. This proves that $A\times B$ is serial if and only if B is not empty or A is empty.

   Next we show that $A\times B$ is symmetric if and only if $A=B$. Recall that R is symmetric if $\forall a\forall b(aRb\to bRa)$. By the observation above, in this case symmetry means $\forall a\forall b(a\in A\wedge b\in B\to b\in A\wedge a\in B)$. This holds exactly when $A=B$. Thus we have shown that $A\times B$ is symmetric if and only if $A=B$.

10. We treat some cases. Reflexivity is subset hereditary: if $R\subseteq A^2$ is a reflexive relation, then $\forall x\in ARxx$ holds. Clearly, if $B\subseteq A$ also $\forall x\in BRxx$ holds. Thus $R_{\uparrow B}$ is reflexive too.

    Transitivity is subset hereditary: if R is transitive relation on A, then
$$\forall x,y,z\in A: Rxy\wedge Ryz\to Rxz.$$
    Clearly, then also $\forall x,y,z\in B: Rxy\wedge Ryz\to Rxz$, for any subsete B of A. Thus $R_{\uparrow B}$ is transitive too.

    Seriality is not subset hereditary: the relation $<$ on $\mathbb{N}$ is serial ($\forall x\in\mathbb{N}\exists y\in\mathbb{N}x<y$). But $<$ restricted to $\{0\}$, which means $<_{\uparrow\{0\}}$, is not, since there is no $y\in\{0\}$ such that $0<y$.

11. We have to show that $(\langle a,b\rangle=\langle c,d\rangle)\Leftrightarrow(a=c\wedge b=d)$.
    $\Rightarrow$: suppose $\langle a,b\rangle=\langle c,d\rangle$. Unwinding the definition of ordered pair this means that $\{\{a\},\{a,b\}\}=\{\{c\},\{c,d\}\}$. This implies $a=c$ and $b=d$.
    $\Leftarrow$: if $a=c$ and $b=d$, then of course $\langle a,b\rangle=\langle c,d\rangle$.

12. Because with these definitions one cannot distinguish which element of the ordered pair should come first.

13. $\{\langle a,b\rangle\}$ is, by definition, the set $\{\{\{a\},\{a,b\}\}\}$. Hence $\{a\} \notin \{\langle a,b\rangle\}$ and $\{b\} \notin \{\langle a,b\rangle\}$.

14. $\{\langle 1,2\rangle\} \subseteq \mathbb{N}$ means $\langle 1,2\rangle \in \mathbb{N}$, qoud non. Thus $\{\langle 1,2\rangle\} \not\subseteq \mathbb{N}$. $\{\langle 1,2\rangle\} \subseteq P(\mathbb{N})$ means $\langle 1,2\rangle \in P(\mathbb{N})$. But $\langle 1,2\rangle = \{\{1\},\{1,2\}\}$, which is not an element of $P(\mathbb{N})$ since it is not a subset of $\mathbb{N}$. Thus $\{\langle 1,2\rangle\} \not\subseteq P(\mathbb{N})$.

16. No, $1R2$ and $1R3$ but not $2R3$. To make it trasitive an arrow from 0 to 3 has to be added.

17. We treat the first relation. It is serial and well-founded, but not dense: a_1Rb_1, but no x such that a_1RxRb_1.

18. $\leq_{\mathbb{N}}$.

19. We have to show that $\leftrightarrow$ is reflexive, transitive and symmetric. Reflexivity is clear: $\varphi \leftrightarrow \varphi$ for all formulas φ. Transitivity is: if $\varphi \leftrightarrow \psi$ and $\psi \leftrightarrow \phi$, then $\varphi \leftrightarrow \phi$. But this is clearly true. Finally, symmetry follows easily too: if $\varphi \leftrightarrow \psi$, then $\psi \leftrightarrow \varphi$.

22. Two elements. For with one element, say 0, $P(\{0\}) = \{\emptyset,\{0\}\}$ which is totally ordered, and for $\emptyset$, $P(\emptyset) = \{\emptyset\}$, which is totally ordered too.

3 Exercises Chapter 4

2. The domain is $\mathbb{N}$ and the range is the set of natural numbers divisable by 7: $\{n \in \mathbb{N} \mid \exists m \in \mathbb{N}(n = 7m)\}$.

3. $\{\langle r,s\rangle \mid r,s \in \mathbb{R}_{\geq 0}, s = \sqrt{r}\}$. $f[\mathbb{R}_{\geq 4}] = \mathbb{R}_{\geq 2}$. $f^{-1}[\mathbb{R}_{\leq 4}] = \mathbb{R}_{\leq 16}$.

4. These are all the functions from $\{0\}$ to $\{0,1,2\}$: $\{\{\langle 0,0\rangle\},\{\langle 0,1\rangle\},\{\langle 0,2\rangle\}\}$.

6. $f \circ g(x) = \sqrt{x^3}$ and $g \circ f(x) = (\sqrt{x})^3$.

7. $f \circ g(\varphi) = (p \to \varphi) \to p$ and $g \circ f(\varphi) = p \to (\varphi \to p)$. For $\varphi = \top$:

$$f \circ g(\top) = ((\top \to \top) \to \top) \leftrightarrow \top \leftrightarrow (\top \to (\top \to \top)) = g \circ f(\top).$$

8. Given the bijection $f : A \to B$, g is defined as $\{\langle y,x\rangle \mid f(x) = y\}$, i.e. $g(y) = x$ iff $f(x) = y$. Since $f \subseteq A \times B$, it follows that $g \subseteq B \times A$, that is, g indeed is a function from B to A: $g : B \to A$.

   For $x \in A$, $g \circ f(x) = g(f(x))$. Suppose $f(x) = y$. Then by the definition of g, $g(y) = x$, and thus $g(f(x)) = g(y) = x$. Hence $g \circ f = id_A$. Also, for $y \in B$, $f \circ g(y) = f(g(y))$. Suppose $g(y) = x$. This implies that $f(x) = y$ by the definition of g. Thus $f(g(y)) = f(x) = y$. Hence $f \circ g = id_B$.

9. Suppose $|X| = m$ and $|Y| = n$. Thus $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$. For a function $f : X \to Y$ there are n possible choices for every $f(x_i)$. Thus in total there are $n \times n \times \cdots \times n$ (m times) different functions, i.e. n^m, which is $|Y|^{|X|}$.

10. It is surjective since for every $n \in \mathbb{Z}$, there is a $m \in \mathbb{Z}$, namely $m = n - 1$, such that $f(m) = f(n-1) = n - 1 + 1 = n$. It is not surjective when considered as a function on the natural numbers: e.g. there is no $n \in \mathbb{N}$ such that $f(n) = 0$.

11. It is injective: $x \neq y$ implies $2^x \neq 2^y$. It is not surjective: e.g. there is no $x \in \mathbb{R}$ such that $2^x = 0$. $f[\{x \in \mathbb{R} \mid -2 \leq x \leq 2\}] = \{x \in \mathbb{R} \mid 1/4 \leq x \leq 4\}$ and $f^{-1}[\{x \in \mathbb{R} \mid 4 \leq x \leq 16\}] = \{x \in \mathbb{R} \mid 2 \leq x \leq 4\}$.

12. Consider two injective functions $f : A \to B$ and $g : C \to D$, where $C \subseteq B$. We show that $g \circ f$ is injective: $x \neq y$ implies $g \circ f(x) \neq g \circ f(y)$. Therefore, consider $x, y \in A$ such that $x \neq y$. Becasue f is injective it follows that $f(x) \neq f(y)$. Since g is injective it follows that then also $g(f(x)) \neq g(f(y))$. But $g(f(x)) = g \circ f(x)$ and $g(f(y)) = g \circ f(y)$, and thus $g \circ f(x) \neq g \circ f(y)$.

14. Consider $f : A \to \{a\}$ and assume $b \in A$ for some b (A is not empty). We have to show that for all $y \in \{a\}$ there exists a $x \in A$ such that $f(x) = y$, that is, that there is a $x \in A$ such that $f(x) = a$. But $f(b) = a$, and thus we can take b for x. Only when A contains one element the function is also an injection.

16. Given an injection $f : A \to B$ we have to show that the function $f : A \to f[A]$ is a bijection, thus both an injection and a surjection. That $f : A \to f[A]$ is an injection follows immediately from the injectivity of $f : A \to B$. That $f : A \to f[A]$ is a surjection follows from the fact that for all $y \in f[A]$ there exists a $x \in A$ such that $f(x) = y$, by the definition of $f[A]$.

17. f does not have a fixed point: $\neg\varphi$ is never equal (literally the same formua) as φ. Neither are there formulas that are equivalent to their negation. Thus there are no ψ such that $\psi \leftrightarrow f(\psi)$.

18. $f^n(x) = f \circ f \circ \cdots \circ f(x)$. Assume that x is a fixed point of f. Then, for every n,

$$f^n(x) = f^{n-1} \circ f(x) = f^{n-1}(f(x)) = f^{n-1}(x) =$$
$$f^{n-2} \circ f(x) = f^{n-2}(f(x)) = f^{n-2}(x) = \cdots = f(x) = x.$$

In the chapter on induction we will see how we can prove this theorem in a rigorous way by induction.

19.
$$f(x) = \begin{cases} 0 & \text{if } 0 < x < 1 \\ 1 & \text{if } x = 0 \text{ or } x = 1 \\ -1 & \text{otherwise} \end{cases}$$

20. Suppose $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$.

    If $m > n$ there are no injections from A to B, so assume $m \leq n$. For an injection f from A to B, there are n choices for $f(a_1)$, namely b_1 or b_2 or $\ldots$ b_n. There are $(n-1)$ choices for $f(a_2)$, namely b_1 or b_2 or $\ldots$ b_n except $f(a_1)$, as f has to be an injection. There are $(n-2)$ choices for $f(a_3)$, namely b_1 or b_2 or $\ldots$ b_n except $f(a_1)$ and $f(a_2)$, etc. Thus there are $n \cdot (n-1) \ldots (n-m+1)$ many injections from A to B.

    If $m < n$ there are no surjections from A to B, therefore assume $m \geq n$. A surjection has to reach every element in B. There are $(n-1)^m$ many functions f from A to B such that $b_i \notin f[A]$ for at least one i. Since there are n^m many functions from A to B, there are $n^m - (n-1)^m$ surjections from A to B.

21. Let $R = \{\langle x, y \rangle \mid f(x) = f(y)\}$. We have to show that R is reflexive, transitive and symmetric. Reflexivity (xRx) follows from the fact that $f(x) = f(x)$, and thus xRx. For transitivity ($xRyRz \rightarrow xRz$), observe that if $xRyRz$, i.e. $f(x) = f(y)$ and $f(y) = f(z)$, then clearly $f(x) = f(z)$, and thus xRz as desired. For symmetry ($xRy \rightarrow yRx$), if $f(x) = f(y)$, of course also $f(y) = f(x)$, that is, yRx.

23. Consider a finite set A, and suppose that $f : A \rightarrow P(A)$ is a surjection. Then for every a there is a $a' \in A$ such that $f(a') = \{a\}$. But then all elements of A have been "used", and there is no $b \in A$ left for e.g. $f(b) = \emptyset$.

25. No. For suppose $f : \mathbb{N} \rightarrow \mathbb{Z}$ is an isomorphism between $(\mathbb{N}, \leq)$ and $(\mathbb{Z}, \leq)$. Suppose $f(0) = n$, for some $n \in \mathbb{Z}$. Because f is an isomorphism, every $m < n$ should be such that for the $i \in \mathbb{N}$ with $f(i) = m$, $i < 0$. But this cannot be, as in $\mathbb{N}$ there is no $i < 0$.

4 Exercises Chapter 5

1. Use the fact that $\mathbb{Z}$ is countable and the surjection given in the proof of the countability of the cartesian product of two countable sets.

5. $f : \mathbb{N} \rightarrow \mathbb{Z} \cup \{\langle 0, n \rangle \mid n \in \mathbb{Z}\}$ given by

$$f(n) = \begin{cases} m & \text{if } n = 4m \\ -m & \text{if } n = 4m+1 \\ \langle 0, m \rangle & \text{if } n = 4m+2 \\ \langle 0, -m \rangle & \text{if } n = 4m+3 \end{cases}$$

    is a surjection.

6. We know that $\mathbb{Q}$ is countable, so let $f : \mathbb{N} \to \mathbb{Q}$ be a surjection. Then $g : \mathbb{Q} \to \mathbb{Q} \cup \{\langle 0, q\rangle \mid q \in \mathbb{Q}\}$ given by

$$g(n) = \begin{cases} f(m) & \text{if } n = 2m \\ \langle 0, f(m)\rangle & \text{if } n = 2m+1 \end{cases}$$

is a surjection.

9. Clearly, $|\mathbb{R}_{\geq 0}| \leq |\mathbb{R}|$, and $\mathbb{R}$ is uncountable. Apply Theorem 12.

10. Use Theorem 12.

12. Suppose $|A| \leq |B|$ and that B is countable. Thus there exist an injection $f : A \to B$ and a surjection $g : \mathbb{N} \to B$. By one of the exercises of Chapter 4, $f : A \to f[A]$ is a bijection. Let f^{-1} be its inverse. Choose an $a \in A$. Now we define the function $h : \mathbb{N} \to f[A]$ as

$$h(n) = \begin{cases} g(n) & \text{if } g(n) \in f[A] \\ f(a) & \text{otherwise} \end{cases}$$

Because g is a surjection, so is h. Thus $f^{-1} \circ h$ is a surjection from $\mathbb{N}$ to A, which implies that A is countable.

Suppose $|A| \leq |B|$ and A is uncountable. If B would be countable, then by the previous observation, so would A be, qoud non. Thus B is uncountable.

13. Let S be the set of finite words from 0's and 1's. One can order S as follows: first the words of length 1, then the words of length 2, etc. Given two words of the same length, we enmuerate them lexicographically. Thus the begin of the enumeration is: 0, 1, 00, 01, 10, 11, 000, ... This enumeration implies a surjection $f : \mathbb{N} \to S$: $f(0) = 0$, $f(1) = 1$, $f(2) = 00$, $f(3) = 01$, $f(4) = 10$, $f(5) = 11$, $f(6) = 000$, ...

5 Exercises Chapter 6

1. The set $\mathcal{F}$ of formulas in which the only connectives are negations and implications can be inductively defined as follows:

   (a) all propositional variables and $\bot$ and $\top$ are in $\mathcal{F}$,

   (b) if φ and ψ belong to $\mathcal{F}$, then so do $\neg\varphi$, $(\neg\varphi)$, $\varphi \to \psi$ and $(\varphi \to \psi)$,

   (c) no other expressions than the ones obtained via (a) and (b) are in $\mathcal{F}$.

2. The set $\mathcal{W}$ of binairy words in which the number of 0's is even can be defined as follows.

   (a) 1 and 00 belong to $\mathcal{W}$,

   (b) if the words w and v belong to $\mathcal{W}$, then so do $1w$, $w1$, $00w$, $w00$, $0w0$, $0w0v$, $w00v$, and $w0v0$,

(c) no other expressions than the ones obtained via (a) and (b) are in $\mathcal{W}$.

4. *The case* $n = 0$: $\Sigma_{k=0}^{0} k^2 = 0^2 = 0 = 0(0+1)(2 \cdot 0 + 1)/6$.
   The induction step: suppose $\Sigma_{k=0}^{n} k^2 = n(n+1)(2n+1)/6$. We have to show that

   $$\Sigma_{k=0}^{n+1} k^2 = (n+1)((n+1)+1)(2(n+1)+1)/6 = (n+1)(n+2)(2n+3)/6.$$

   Now $\Sigma_{k=0}^{n+1} k^2 = (n+1)^2 + \Sigma_{k=0}^{n} k^2$. By the induction hypothesis, $\Sigma_{k=0}^{n} k^2 = n(n+1)(2n+1)/6$, we have $(n+1)^2 + \Sigma_{k=0}^{n} k^2 = (n+1)^2 + n(n+1)(2n+1)/6$. Thus

   $$\Sigma_{k=0}^{n+1} k^2 = (n+1)^2 + n(n+1)(2n+1)/6 = n^2 + 2n + 1 + n(n+1)(2n+1)/6.$$

   Since

   $$(n+1)^2 + n(n+1)(2n+1)/6 = (6(n+1)^2 + n(n+1)(2n+1))/6 = \\ ((n+1)(n(2n+1) + 6(n+1))/6 = (n+1)(n+2)(2n+3)/6,$$

   we have showed what we wanted to show.

6. *The case* $n = 0$: $\Sigma_{k=0}^{0} 3^k = 3^0 = 1 = (3^1 - 1)/2$.
   The induction step: suppose $\Sigma_{k=0}^{n} 3^k = (3^{n+1} - 1)/2$. We have to show that $\Sigma_{k=0}^{n+1} 3^k = (3^{n+2} - 1)/2$. This is shown by the following equalities, using the induction hypothesis, for the second equality:

   $$\Sigma_{k=0}^{n+1} 3^k = 3^{n+1} + \Sigma_{k=0}^{n} 3^k = 3^{n+1} + (3^{n+1} - 1)/2 = \\ (2 \cdot 3^{n+1} + 3^{n+1} - 1)/2 = (3 \cdot 3^{n+1} - 1)/2 = (3^{n+2} - 1)/2.$$

9. *The base case*: If φ is a propositional formula, then clearly it contains no other connectives than $\neg$ and $\wedge$ since it contains no connectives at all.
   The induction step: Suppose that φ and ψ are equivalent to formulas φ' and ψ' in which only $\wedge$ and $\neg$ occur (the induction hypothesis). Then $\varphi \wedge \psi$ is equivalent to $\varphi' \wedge \psi'$, which contains only $\wedge$ and $\neg$, so that finishes the case for conjunction. Since $\neg\varphi$ is equivalent to $\neg\varphi'$, also the negation case is done. For disjunction, observe that $\varphi \vee \psi$ is equivalent to $\neg(\neg\varphi \wedge \neg\psi)$, and thus also to $\neg(\neg\varphi' \wedge \neg\psi')$, which shows that $\varphi \vee \psi$ is equivalent to a formula which contains only $\wedge$ and $\neg$. For implication, note that $\varphi \rightarrow \psi$ is equivalent to $\neg\varphi \vee \psi$, and thus to $\neg(\varphi \wedge \neg\psi)$, and thus to $\neg(\varphi' \wedge \neg\psi')$. This proves that $\varphi \rightarrow \psi$ is equivalent to a formula which contains only $\wedge$ and $\neg$.