On sets, functions and relations

Some solutions

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1 Exercises Chapter 4

2. The domain is $\mathbb{N}$ and the range is the set of natural numbers divisable by 7: $\{n \in \mathbb{N} \mid \exists m \in \mathbb{N}(n = 7m)\}$.

3. $\{\langle r, s\rangle \mid r, s \in \mathbb{R}_{\geq 0}, s = \sqrt{r}\}$. $f[\mathbb{R}_{\geq 4}] = \mathbb{R}_{\geq 2}$. $f^{-1}[\mathbb{R}_{\leq 4}] = \mathbb{R}_{\leq 16} \cap \mathbb{R}_{\geq 0}$.

4. These are all the functions from $\{0\}$ to $\{0, 1, 2\}$: $\{\{\langle 0, 0\rangle\}, \{\langle 0, 1\rangle\}, \{\langle 0, 2\rangle\}\}$.

6. $f \circ g(x) = \sqrt{x^3}$ and $g \circ f(x) = (\sqrt{x})^3$.

7. $f \circ g(\varphi) = (p \to \varphi) \to p$ and $g \circ f(\varphi) = p \to (\varphi \to p)$. For $\varphi = \top$:
$$f \circ g(\top) = ((\top \to \top) \to \top) \leftrightarrow \top \leftrightarrow (\top \to (\top \to \top)) = g \circ f(\top).$$

8. Given the bijection $f : A \to B$, g is defined as $\{\langle y, x\rangle \mid f(x) = y\}$, i.e. $g(y) = x$ iff $f(x) = y$. Since $f \subseteq A \times B$, it follows that $g \subseteq B \times A$, that is, g indeed is a function from B to A: $g : B \to A$.

   For $x \in A$, $g \circ f(x) = g(f(x))$. Suppose $f(x) = y$. Then by the definition of g, $g(y) = x$, and thus $g(f(x)) = g(y) = x$. Hence $g \circ f = id_A$. Also, for $y \in B$, $f \circ g(y) = f(g(y))$. Suppose $g(y) = x$. This implies that $f(x) = y$ by the definition of g. Thus $f(g(y)) = f(x) = y$. Hence $f \circ g = id_B$.

9. Suppose $|X| = m$ and $|Y| = n$. Thus $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$. For a function $f : X \to Y$ there are n possible choices for every $f(x_i)$. Thus in total there are $n \times n \times \cdots \times n$ (m times) different functions, i.e. n^m, which is $|Y|^{|X|}$.

10. It is surjective since for every $n \in \mathbb{Z}$, there is a $m \in \mathbb{Z}$, namely $m = n - 1$, such that $f(m) = f(n - 1) = n - 1 + 1 = n$. It is not surjective when considered as a function on the natural numbers: there is no $n \in \mathbb{N}$ such that $f(n) = 0$.

11. It is injective: $x \neq y$ implies $2^x \neq 2^y$. It is not surjective: e.g. there is no $x \in \mathbb{R}$ such that $2^x = 0$. $f[\{x \in \mathbb{R} \mid -2 \leq x \leq 2\}] = \{x \in \mathbb{R} \mid 1/4 \leq x \leq 4\}$ and $f^{-1}[\{x \in \mathbb{R} \mid 4 \leq x \leq 16\}] = \{x \in \mathbb{R} \mid 2 \leq x \leq 4\}$.

12. Consider two injective functions $f : A \to B$ and $g : C \to D$, where $C \subseteq B$. We show that $g \circ f$ is injective: $x \neq y$ implies $g \circ f(x) \neq g \circ f(y)$. Therefore, consider $x, y \in A$ such that $x \neq y$. Becasue f is injective it follows that $f(x) \neq f(y)$. Since g is injective it follows that then also $g(f(x)) \neq g(f(y))$. But $g(f(x)) = g \circ f(x)$ and $g(f(y)) = g \circ f(y)$, and thus $g \circ f(x) \neq g \circ f(y)$.

14. Consider $f : A \to \{a\}$ and assume $b \in A$ for some b (A is not empty). We have to show that for all $y \in \{a\}$ there exists a $x \in A$ such that $f(x) = y$, that is, that there is a $x \in A$ such that $f(x) = a$. But $f(b) = a$, and thus we can take b for x. Only when A contains one element the function is also an injection.

16. Given an injection $f : A \to B$ we have to show that the function $f : A \to f[A]$ is a bijection, thus both an injection and a surjection. That $f : A \to f[A]$ is an injection follows immediately from the injectivity of $f : A \to B$. That $f : A \to f[A]$ is a surjection follows from the fact that for all $y \in f[A]$ there exists a $x \in A$ such that $f(x) = y$, by the definition of $f[A]$.

18. f does not have a fixed point: $\neg\varphi$ is never equal (literally the same formua) as φ. Neither are there formulas that are equivalent to their negation. Thus there are no ψ such that $\psi \leftrightarrow f(\psi)$.

19. $f^n(x) = f \circ f \circ \cdots \circ f(x)$. Assume that x is a fixed point of f. Then, for every n,

$$f^n(x) = f^{n-1} \circ f(x) = f^{n-1}(f(x)) = f^{n-1}(x) =$$
$$f^{n-2} \circ f(x) = f^{n-2}(f(x)) = f^{n-2}(x) = \cdots = f(x) = x.$$

In the chapter on induction we will see how we can prove this theorem in a rigorous way by induction.

20. $f : \mathbb{Q} \to \mathbb{Q}$ with

$$f(x) = \begin{cases} 0 & \text{if } 0 < x < 1 \\ 1 & \text{if } x = 0 \text{ or } x = 1 \\ -1 & \text{otherwise} \end{cases}$$

21. Suppose $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$.

If $m > n$ there are no injections from A to B, so assume $m \leq n$. For an injection f from A to B, there are n choices for $f(a_1)$, namely b_1 or b_2 or $\ldots$ b_n. There are $(n-1)$ choices for $f(a_2)$, namely b_1 or b_2 or $\ldots$ b_n except $f(a_1)$, as f has to be an injection. There are $(n-2)$ choices for $f(a_3)$, namely b_1 or b_2 or $\ldots$ b_n except $f(a_1)$ and $f(a_2)$, etc. Thus there are $n \cdot (n-1) \ldots (n-m+1) = n!/(n-m)!$ many injections from A to B.

22. Let $R = \{\langle x, y\rangle \mid f(x) = f(y)\}$. We have to show that R is reflexive, transitive and symmetric. Reflexivity (xRx) follows from the fact that $f(x) = f(x)$, and thus xRx. For transitivity ($xRyRz \rightarrow xRz$), observe that if $xRyRz$, i.e. $f(x) = f(y)$ and $f(y) = f(z)$, then clearly $f(x) = f(z)$, and thus xRz as desired. For symmetry ($xRy \rightarrow yRx$), if $f(x) = f(y)$, of course also $f(y) = f(x)$, that is, yRx.

23. Recall that we denote the elements in A/R by $[a]$, where $a \in A$. Define $g : A/R \rightarrow B$ as $g([a]) = f(a)$. First, we have to see that this indeed defines a function, that is, that g does not depend on the chosen representation $[a]$. But indeed, if $[a] = [b]$ this means that aRb, i.e. $f(a) = f(b)$, and thus $g([a]) = g([b])$.

    g is an injection: if $[a] \neq [b]$ this means that $f(a) \neq f(b)$, and thus $g([a]) \neq g([b])$.

24. Consider a finite set A, and suppose that $f : A \rightarrow P(A)$ is a surjection. Then for every a there is a $a' \in A$ such that $f(a') = \{a\}$. But then all elements of A have been "used", and there is no $b \in A$ left for e.g. $f(b) = \emptyset$.

26. No. For suppose $f : \mathbb{N} \rightarrow \mathbb{Z}$ is an isomorphism between $(\mathbb{N}, \leq)$ and $(\mathbb{Z}, \leq)$. Suppose $f(0) = n$, for some $n \in \mathbb{Z}$. Because f is an isomorphism, every $m < n$ should be such that for the $i \in \mathbb{N}$ with $f(i) = m$, $i < 0$. But this cannot be, as in $\mathbb{N}$ there is no $i < 0$.

2 Exercises Chapter 5

1. Use the fact that $\mathbb{Z}$ is countable and the surjection given in the proof of the countability of the cartesian product of two countable sets.

3. We follow the same pattern as the proof of the countability of the cartesian product of two countable sets. Let f be a surjection from $\mathbb{N}$ to $\mathbb{Z}$, which exists because $\mathbb{Z}$ is countable. Let $z_1, z_2, \ldots$ be the enumeration of $\mathbb{Z}$ implied by f, i.e. $z_n = f(n)$. Now define a surjection from $\mathbb{N}$ to $\mathbb{Z}^n$, given by the following enumeration:

$$\langle z_1, \ldots, z_1\rangle, \langle z_2, z_1, \ldots, z_1\rangle, \langle z_1, z_2, \ldots, z_1\rangle, \ldots, \langle z_1, \ldots, z_2\rangle,$$
$$\langle z_2, z_2, z_1, \ldots, z_1\rangle, \langle z_2, z_1, z_2, \ldots, z_1\rangle, \ldots \langle z_2, z_1, \ldots, z_1, z_2\rangle,$$
$$\langle z_1, z_2, z_2, \ldots, z_1\rangle, \ldots$$

   Thus we first enumerate all the sequences the sum of which indices in n, then all the sequences which indices have sum $n + 1$, etc.

4. $f : \mathbb{N} \rightarrow \mathbb{Z} \cup \{\langle 0, n\rangle \mid n \in \mathbb{Z}\}$ given by

$$f(n) = \begin{cases} m & \text{if } n = 4m \\ -m & \text{if } n = 4m + 1 \\ \langle 0, m\rangle & \text{if } n = 4m + 2 \\ \langle 0, -m\rangle & \text{if } n = 4m + 3 \end{cases}$$

is a surjection.

5. We know that $\mathbb{Q}$ is countable, so let $f : \mathbb{N} \to \mathbb{Q}$ be a surjection. Then $g : \mathbb{Q} \to \mathbb{Q} \cup \{\langle 0, q\rangle \mid q \in \mathbb{Q}\}$ given by
$$g(n) = \begin{cases} f(m) & \text{if } n = 2m \\ \langle 0, f(m)\rangle & \text{if } n = 2m+1 \end{cases}$$
is a surjection.

6. Clearly, $|\mathbb{R}_{\geq 0}| \leq |\mathbb{R}|$, and $\mathbb{R}$ is uncountable. Apply Theorem 12.

7. Use Theorem 12.

8. Use Theorem 12.

9. Suppose $|A| \leq |B|$ and that B is countable. Thus there exist an injection $f : A \to B$ and a surjection $g : \mathbb{N} \to B$. By one of the exercises of Chapter 4, $f : A \to f[A]$ is a bijection. Let f^{-1} be its inverse. Choose an $a \in A$. Now we define the function $h : \mathbb{N} \to f[A]$ as
$$h(n) = \begin{cases} g(n) & \text{if } g(n) \in f[A] \\ f(a) & \text{otherwise} \end{cases}$$
Because g is a surjection, so is h. Thus $f^{-1} \circ h$ is a surjection from $\mathbb{N}$ to A, which implies that A is countable.

   Suppose $|A| \leq |B|$ and A is uncountable. If B would be countable, then by the previous observation, so would A be, qoud non. Thus B is uncountable.

10. Let S be the set of finite words from 0's and 1's. One can order S as follows: first the words of length 1, then the words of length 2, etc. Given two words of the same length, we enmuerate them lexicographically. Thus the begin of the enumeration is: 0, 1, 00, 01, 10, 11, 000, ... This enumeration implies a surjection $f : \mathbb{N} \to S$: $f(0) = 0$, $f(1) = 1$, $f(2) = 00$, $f(3) = 01$, $f(4) = 10$, $f(5) = 11$, $f(6) = 000$, ...

11. We can enumerate this set in the following way. First the formulas of length ≤ 1 (at most 1 symbol in it) containing as atoms only p_1. Then the formulas of length ≤ 2 containing as atoms only p_1 and p_2. Then the formulas of length ≤ 3 containing as atoms only p_1, p_2 and p_3, etc. Formulas of the same length are ordered lexicographically, with alphabet: $\top, \bot, p_1, p_2, \ldots, \neg, \wedge, \vee, \to, (,)$. Thus the enumeration starts as
$$\begin{aligned} &\top, \bot, p_1, p_2, \\ &\neg\top, \neg\bot, \neg p_1, \neg p_2, p_3, \neg p_3, \\ &\neg\neg\top, \neg\neg\bot, \neg\neg p_1, \neg\neg p_2, \neg\neg p_3, \top\wedge\top, \top\wedge\bot, \bot\wedge\top, \bot\wedge\bot, \top\wedge p_1, \ldots \end{aligned}$$
This enumeration naturally leads to a surjection from $\mathbb{N}$ to the set of formulas.

3 Exercises Chapter 6

1. The set $\mathcal{F}$ of formulas in which the only connectives are negations and implications can be inductively defined as follows:

   (a) all propositional variables and $\bot$ and $\top$ are in $\mathcal{F}$,

   (b) if φ and ψ belong to $\mathcal{F}$, then so do $\neg\varphi$ and $(\varphi \to \psi)$,

   (c) no other expressions than the ones obtained via (a) and (b) are in $\mathcal{F}$.

2. The set $\mathcal{W}$ of binairy words in which the number of 0's is even can be defined as follows.

   (a) 1 and 00 belong to $\mathcal{W}$,

   (b) if the words w and v belong to $\mathcal{W}$, then so do $1w$, $w1$, $00w$, $w00$, $0w0$,

   (c) no other expressions than the ones obtained via (a) and (b) are in $\mathcal{W}$.

4. *The case* $n = 0$: $\Sigma_{k=0}^{0} k^2 = 0^2 = 0 = 0(0+1)(2 \cdot 0 + 1)/6$.
   The induction step: suppose $\Sigma_{k=0}^{n} k^2 = n(n+1)(2n+1)/6$. We have to show that

   $$\Sigma_{k=0}^{n+1} k^2 = (n+1)((n+1)+1)(2(n+1)+1)/6 = (n+1)(n+2)(2n+3)/6.$$

   Now $\Sigma_{k=0}^{n+1} k^2 = (n+1)^2 + \Sigma_{k=0}^{n} k^2$. By the induction hypothesis, $\Sigma_{k=0}^{n} k^2 = n(n+1)(2n+1)/6$, we have $(n+1)^2 + \Sigma_{k=0}^{n} k^2 = (n+1)^2 + n(n+1)(2n+1)/6$. Thus

   $$\Sigma_{k=0}^{n+1} k^2 = (n+1)^2 + n(n+1)(2n+1)/6 = n^2 + 2n + 1 + n(n+1)(2n+1)/6.$$

   Since

   $$(n+1)^2 + n(n+1)(2n+1)/6 = \big(6(n+1)^2 + n(n+1)(2n+1)\big)/6 = \big((n+1)(n(2n+1) + 6(n+1))\big)/6 = (n+1)(n+2)(2n+3)/6,$$

   and we are done.

6. *The case* $n = 0$: $\Sigma_{k=0}^{0} 3^k = 3^0 = 1 = (3^1 - 1)/2$.
   The induction step: suppose $\Sigma_{k=0}^{n} 3^k = (3^{n+1} - 1)/2$. We have to show that $\Sigma_{k=0}^{n+1} 3^k = (3^{n+2} - 1)/2$. This is shown by the following equalities, using the induction hypothesis for the second equality:

   $$\Sigma_{k=0}^{n+1} 3^k = 3^{n+1} + \Sigma_{k=0}^{n} 3^k = 3^{n+1} + (3^{n+1} - 1)/2 = (2 \cdot 3^{n+1} + 3^{n+1} - 1)/2 = (3 \cdot 3^{n+1} - 1)/2 = (3^{n+2} - 1)/2.$$

9. *The base case*: If φ is a propositional formula or $\top$ or $\bot$, then clearly it contains no other connectives than $\neg$ and $\wedge$ since it contains no connectives at all.
   The induction step: Suppose that φ and ψ are equivalent to formulas φ' and ψ' in which only $\wedge$ and $\neg$ occur (the induction hypothesis). Then $(\varphi \wedge \psi)$ is equivalent to $(\varphi' \wedge \psi')$, which contains only $\wedge$ and $\neg$, so that finishes the case for conjunction. Since $\neg\varphi$ is equivalent to $\neg\varphi'$, also the negation case is done. For disjunction, observe that $(\varphi \vee \psi)$ is equivalent to $\neg(\neg\varphi \wedge \neg\psi)$, and thus also to $\neg(\neg\varphi' \wedge \neg\psi')$, which shows that $\varphi \vee \psi$ is equivalent to a formula which contains only $\wedge$ and $\neg$. For implication, note that $(\varphi \to \psi)$ is equivalent to $(\neg\varphi \vee \psi)$, and thus to $\neg(\varphi \wedge \neg\psi)$, and thus to $\neg(\varphi' \wedge \neg\psi')$. This proves that $(\varphi \to \psi)$ is equivalent to a formula which contains only $\wedge$ and $\neg$.