On sets, functions and relations

Some solutions

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1 Exercises Chapter 2

1. $\{n \in \mathbb{N} \mid \exists m \in \mathbb{N}(n = m^2)\}$.

3. For example $\{n \in \mathbb{N} \mid \exists m \in \mathbb{Z}(n = 3m)\}$, $\{n \in \mathbb{Z} \mid n \geq 0 \text{ and } (n/3) \in \mathbb{Z}\}$, and $\{0, 3, 6, 9, 12, \ldots\}$.

8. $\{0, 1\}$.

9.

$$\begin{array}{rcl} x \in C \backslash (A \cap B) & \Leftrightarrow & x \in C \text{ and } x \notin A \cap B \\ & \Leftrightarrow & x \in C \text{ and } (x \notin A \text{ or } x \notin B) \\ & \Leftrightarrow & (x \in C \text{ and } x \notin A) \text{ or } (x \in C \text{ and } x \notin B) \\ & \Leftrightarrow & x \in (C \backslash A) \cup (C \backslash B). \end{array}$$

12. $\emptyset \in P(X)$ because $\emptyset \subseteq X$, for any set X. Also $X \in P(X)$, as $X \subseteq X$.

13. Assume $X \subseteq Y$ and $Y \subseteq Z$. We show that $X \subseteq Z$. That is, that $\forall x(x \in X \rightarrow x \in Z)$. Therefore, assume $x \in X$. Then $x \in Y$ because $X \subseteq Y$. But then $x \in Z$ since $Y \subseteq Z$.

14. $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$, $\{1,2,3,4\}$.

15. $2^6 = 64$ subsets. $\mathbb{N}$ clearly has infinitely many subsets.

2 Exercises Chapter 3

1. $\{\langle r,s\rangle \mid r,s\in\mathbb{R}, r=\sqrt{s}\}$ or $\{\langle r,s\rangle\in\mathbb{R}^2 \mid r^2=s\}$.
2. $\{\langle a,a\rangle,\langle a,c\rangle,\langle a,d\rangle,\langle b,a\rangle,\langle b,c\rangle,\langle b,d\rangle\}$.
3. The diagonal.
4. The negative fractions.
5. The pairs of reals which sum is a rational number. Let us call this relation R. R is symmetric: $\langle r,s\rangle\in R$ implies $(r+s)\in\mathbb{Q}$, which implies $(s+r)\in\mathbb{Q}$, which implies $\langle s,r\rangle\in R$. The relation is not linear: neither $\langle\pi,2\pi\rangle\in R$, nor $\pi=2\pi$, nor $\langle 2\pi,\pi\rangle\in R$.
6. The relation $R=\{\langle n,m\rangle\in\mathbb{Z}^2 \mid n^2=m\}$ is not dense: $\langle 2,4\rangle\in R$ since $2^2=4$, but there is no $k\in\mathbb{Z}$ such that $\langle 2,k\rangle\in R$ and $\langle k,4\rangle\in R$, as this would imply both $k=2^2$ and $k^2=4$, i.e. $k=4$ and $k=2$ or $k=-2$.

9. The cartesian product of two sets is a set of pairs, thus it has always arity 2.
10. Let us start with the following observation. If we let R denote $A\times B$, then
$$aRb \;\Leftrightarrow\; a\in A \text{ and } b\in B.$$
Now we turn to the exercise. It contains a mistake. It should read: prove that $A\times B$ is serial if and only if B is not empty or A is empty. Recall that seriality of R means $\forall a\in A\exists b\in BaRb$. Thus, using the observation above, seriality in this case boils down to $\forall a\in A\exists b\in B$. Observe that $\forall a\in A\exists b\in B$ exactly holds when A is empty or B is not empty. This proves that $A\times B$ is serial if and only if B is not empty or A is empty.

   Next we show that $A\times B$ is symmetric if and only if $A=B$ or A or B is empty (so also in this case the exercise in the notes contains a mistake). Recall that R is symmetric if $\forall a\forall b(aRb\to bRa)$. By the observation above, in this case symmetry means $\forall a\forall b(a\in A\wedge b\in B\to b\in A\wedge a\in B)$. This holds exactly when $A=B$ or A is empty or B is empty. Thus we have shown that $A\times B$ is symmetric if and only if $A=B$ or A is empty or B is empty.

12. We treat some cases. Given the relation $R\subseteq A^2$ and set $B\subseteq A$. Reflexivity is subset-hereditary. Suppose R is reflexive. Then for every $b\in B$, bRb holds, because R is reflexive on A and $b\in A$. Thus $R_{\uparrow B}$ is reflexive.

   Transitivity is subset-hereditary. Suppose R is transitive, and let $a,b,c\in B$ and $aR_{\uparrow B}bR_{\uparrow B}c$. Then, since $R_{\uparrow B}$ is just the restriction of R to B also $aRbRc$. But R is transitive, whence aRc. But $a,c\in B$, thus $sR_{\uparrow B}c$, and this proves that $R_{\uparrow B}$ is transitive.

Density is not subset-hereditary. We show this by giving a counter example. The relation $<$ on $\mathbb{Q}$ is dense. $\mathbb{N}$ is a subset of $\mathbb{Q}$. But $<$ on $\mathbb{N}$ is not densen, that is, $<_{\upharpoonright \mathbb{N}}$ is not dense.

Seriality is not subset-hereditary. We show this by giving a counter example. The relation $<$ on $\mathbb{N}$ is serial. $\{0\}$ is a subset of $\mathbb{N}$. But there is no $x \in \{0\}$ such that $0 < x$. Thus $<_{\upharpoonright \{0\}}$ is not serial.

13. We have to show that $(\langle a, b\rangle = \langle c, d\rangle) \Leftrightarrow (a = c \wedge b = d)$.
$\Rightarrow$: suppose $\langle a, b\rangle = \langle c, d\rangle$. Unwinding the definition of ordered pair this means that $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. This implies $a = c$ and $b = d$.
$\Leftarrow$: if $a = c$ and $b = d$, then of course $\langle a, b\rangle = \langle c, d\rangle$.

14. Because with these definitions one cannot distinguish which element of the ordered pair should come first.

15. $\{\langle a, b\rangle\}$ is, by definition, the set $\{\{\{a\}, \{a, b\}\}\}$. Hence $\{a\} \notin \{\langle a, b\rangle\}$ and $\{b\} \notin \{\langle a, b\rangle\}$.

16. $\{\langle 1, 2\rangle\} \subseteq \mathbb{N}$ means $\langle 1, 2\rangle \in \mathbb{N}$, qoud non. Thus $\{\langle 1, 2\rangle\} \nsubseteq \mathbb{N}$. $\{\langle 1, 2\rangle\} \subseteq P(\mathbb{N})$ means $\langle 1, 2\rangle \in P(\mathbb{N})$. But $\langle 1, 2\rangle = \{\{1\}, \{1, 2\}\}$, which is not an element of $P(\mathbb{N})$ since it is not a subset of $\mathbb{N}$. Thus $\{\langle 1, 2\rangle\} \nsubseteq P(\mathbb{N})$.

18. No, $1R2$ and $1R3$ but not $2R3$. To make it trasitive an arrow from 0 to 3 has to be added.

18. Serial and well-founded, but not dense: $a_1 R b_1$, but no x such that $a_1 R x R b_1$.

20. $\leq_{\mathbb{N}}$.

21. We have to show that $\leftrightarrow$ is reflexive, transitive and symmetric. Reflexivity is clear: $\varphi \leftrightarrow \varphi$ for all formulas φ. Transitivity is: if $\varphi \leftrightarrow \psi$ and $\psi \leftrightarrow \phi$, then $\varphi \leftrightarrow \phi$. But this is clearly true. Finally, symmetry follows easily too: if $\varphi \leftrightarrow \psi$, then $\psi \leftrightarrow \varphi$.

24. Two elements. For with one element, say 0, $P(\{0\}) = \{\emptyset, \{0\}\}$ which is totally ordered, and for $\emptyset$, $P(\emptyset) = \{\emptyset\}$, which is totally ordered too.