Derivability and Admissibility of Inference Rules in Abstract Hilbert Systems

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Overview

About the following sections of the report (2003) with the same title:

2
- What *is* an inference rule?
- An ‘extensional’ abstract notion of inference rule. Some problems.
- An ‘intensional’ abstract notion of rules. (Motivated by: abstract reduction systems vs. abstract rewrite systems.)
- Abstract Hilbert Systems (AHS’s), and
- Abstract Hilbert Systems with rule/axiom names (n-AHS’s).
- Three consequence relations on these systems.

3
- Definition of “rule admissibility” in (n-)AHS’s.
- Definition of *three versions* of “rule derivability” in (n-)AHS’s.
- Some *basic facts* about these notions.

4 • Comparing abstract Hilbert systems w.r.t. consequence relations, rule derivability and admissibility: Introducing relations between abstract Hilbert systems.
• "Interrelation Prisms" between these relations.

5 • Three notions of "mimicking derivation".
• Four notions of "rule elimination" in (n)-AHS's and their relationships with rule derivability and admissibility.
• Some notions of "strong rule elimination" in n-AHS's, and their relationship with rule derivability and admissibility.

E (*Appendix E*) Relationship of (n)-AHS's with sequent-style "Hilbert systems for consequence" à la Avron.

Rule derivability and admissibility (informal def.'s)

Let $\mathcal{S}$ a formal system, R a rule 'on' of $\mathcal{S}$.

'Definition'. R is *derivable* in $\mathcal{S}$ if and only if every instance of R can be 'modelled', or 'mimicked', by an appropriate derivation in $\mathcal{S}$.

'Definition'. Frequently, two versions to define "rule admissibility": R is *admissible* in $\mathcal{S}$ if and only if . . .

(i) . . . by adding R to $\mathcal{S}$ not more theorems become derivable;
[Kleene, 1952; Lorenzen, 1955; Schütte, 1960]

(ii) . . . the theory of $\mathcal{S}$ (the collection of theorems of $\mathcal{S}$) is closed under applications of R (R is *correct* for $\mathcal{S}$).

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Both definitions presuppose the concept of inference rule.

What is an inference rule?

Rules in logic are defined in a variety of ways; here are some examples:

$$\frac{A \to B \quad A}{B}\,\mathsf{MP} \qquad \frac{\begin{array}{c}[A]^{u}\\ \mathcal{D}_1\\ B\end{array}}{A \to B}\to\mathsf{I},\ u \qquad \frac{A[t/x],\ \Gamma \Rightarrow \Delta}{\exists x A,\ \Gamma \Rightarrow \Delta}\,\mathsf{L}\exists$$

$$\frac{p_1 \xrightarrow{a} p_2}{p_1 + q \xrightarrow{a} p_2}\,\mathsf{L}+ \qquad \frac{\tau_1 = \tau[\tau_1/\alpha] \quad \tau_2 = \tau[\tau_2/\alpha]}{\tau_1 = \tau_2}\,\mathsf{UFP}$$

Mostly, rules are defined schematically (s.a.), using substitution on a meta-language of the formula language.

Desirable for studying general properties of rule derivability and admissibility: an *abstract notion of inference rule* that neglects language-specific details.

pure Hilbert Systems (informally)

- *Formulas*, *axioms*.

- *Rules* with applications $\dfrac{A_1 \;\; \ldots \;\; A_n}{B}\,R$ or $\dfrac{}{B}\,R$.

- In *derivations* assumptions are allowed to be made.

- Rules are *pure*: An application of a rule R in a derivations $\mathcal{D}$

$$\dfrac{\begin{array}{ccc}\mathcal{D}_1 & & \mathcal{D}_n \\ A_1 & \ldots & A_n\end{array}}{B}\,R$$

  does not depend on the presence, or absence, of assumptions in the subderivations $\mathcal{D}_1, \ldots, \mathcal{D}_n$.Example of a *impure* Hilbert-system rule: $\dfrac{\phi}{\Box\phi}\,\mathrm{UG}$

An 'extensional' abstract notion of rule

Definition ("Rule descriptions" in pure Hilbert-systems [Hindley, Seldin]). Let $n \in \omega$, Fo a nonempty set.

A *rule description* for an n-premise rule on Fo is a partial function

$$\Phi : \underbrace{Fo \times \ldots \times Fo}_{n} \rightharpoonup Fo \,;$$

it describes the rule R_Φ defined by:

$$\frac{A_1 \quad \ldots \quad A_n}{B} \text{ is application of } R_\Phi \quad \text{iff} \quad \Phi(A_1, \ldots, A_n) = B \,.$$

There are, however, some problems connected with rule descriptions.

Problems with rule descriptions (I)

Rules that allow more than one conclusion to be drawn from a given sequence of premises, e.g.:

$$\frac{A}{A \vee B}\,\vee\mathsf{I}_R \qquad\qquad \frac{\forall x A}{A[t/x]}\,\forall\mathsf{E}$$

Definition (“Rule descriptions”, generalized version).

A *rule description* for an n-premise rule on Fo is a function

$$\Phi : (Fo)^n \rightarrow \mathcal{P}(Fo) \; ;$$

it describes the rule R_Φ defined by:

$$\frac{A_1 \quad \ldots \quad A_n}{B} \text{ is application of } R_\Phi \quad \text{iff} \quad B \in \Phi(A_1, \ldots, A_n) \; .$$

Problems with rule descriptions (II)

Rules with '**behaviourally equivalent**' applications, i.e. applications with the same sequence of premises and the same conclusion:

$$\frac{A_1 \wedge A_2}{A_i} \wedge\mathsf{E} \ (i \in \{1, 2\})$$

has, for example, the two different applications

$$\frac{(x = 0) \wedge (x = 0)}{x = 0} \wedge\mathsf{E} \qquad\qquad \frac{(x = 0) \wedge (x = 0)}{x = 0} \wedge\mathsf{E}$$

Such *syntactic accidents* call for a different abstract framework.

(Problems with) Abstract Reduction Systems

Definition (Klop). An *abstract reduction system* is a structure $\langle A, \rightarrow \rangle$ consisting of a set A with a binary *reduction relation*.

Example. Consider the TRS $\mathcal{T}$

$$f(x) \rightarrow x \ .$$

There are two steps from $f(f(a))$,

$$\underline{f}(f(a)) \rightarrow f(a) \qquad \text{and} \qquad f(\underline{f}(a)) \rightarrow f(a),$$

both of which give rise to the same step

$$f(f(a)) \rightarrow_{\mathcal{T}} f(a)$$

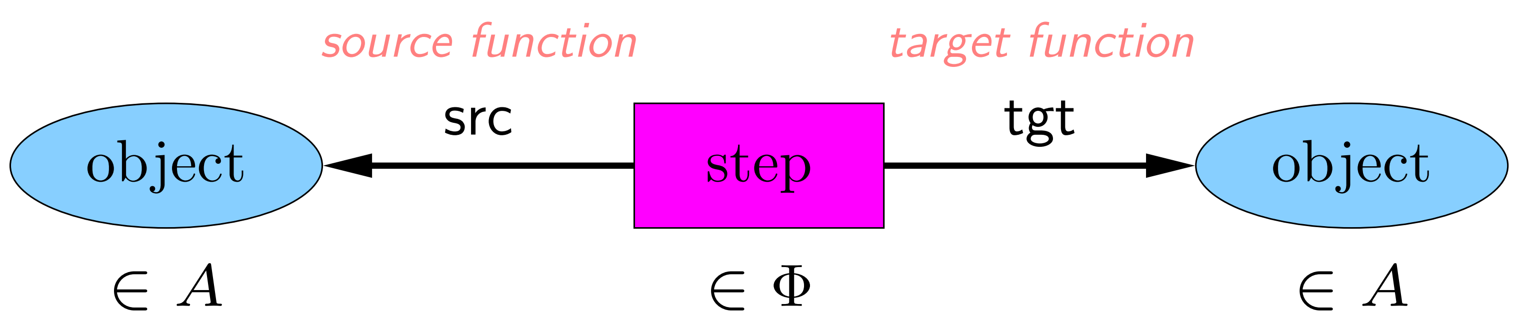
in the extensional description of $\mathcal{T}$ as abstract reduction system $(Ter, \rightarrow_{\mathcal{T}})$; this is called a *'syntactic accident'* (J.J. Lèvy).

Abstract Rewriting Systems

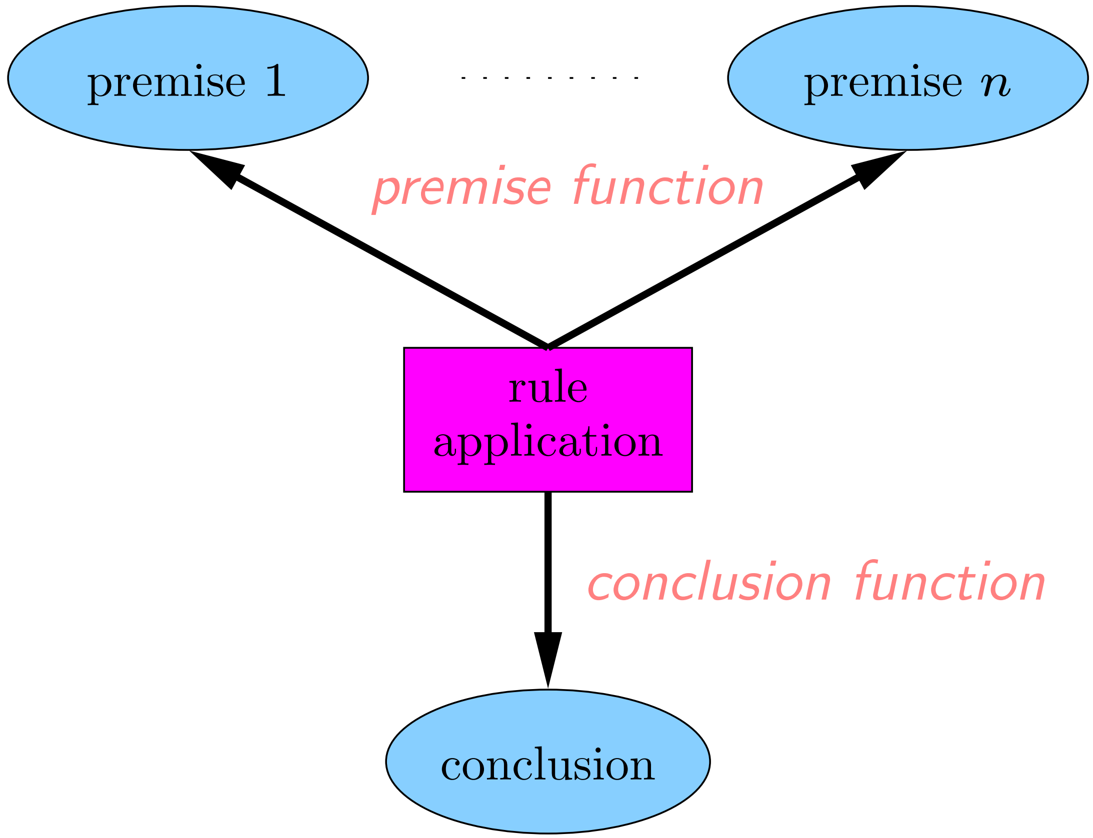
Definition (van Oostrom, de Vrijer). An *abstract rewriting system* is a quadrupel $\langle A, \Phi, \text{src}, \text{tgt} \rangle$ with

- A a set of *objects*,
- Φ a set of *steps*,
- and $\text{src}, \text{tgt} : \Phi \to A$ the *source* and *target* functions.

Visualization of a step as a 'graph hyperedge':



An 'intensional' abstract notion of rule



An intensional abstract notion of rule

Let, for X a set, $Seqs_{\mathsf{f}}(X)$ be the set of *finite sequences* over X.

Definition. Let Fo be a set.

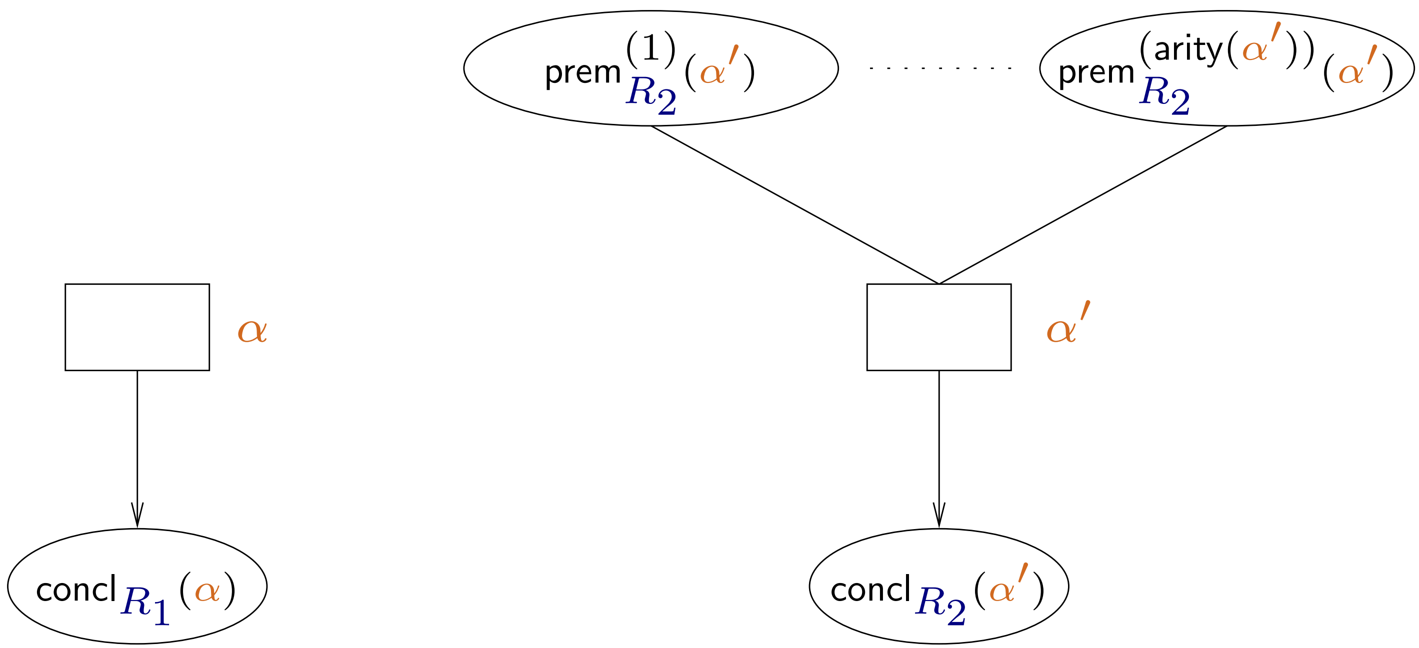
An *AHS-rule* R *on* Fo is a triple $\langle Apps, \text{prem}, \text{concl} \rangle$ where

- $Apps$ is the set of *applications of* R,
- $\text{prem} : Apps \rightarrow Seqs_{\mathsf{f}}(Fo)$ is the *premise* function of R,
- $\text{concl} : Apps \rightarrow Fo$ is the *conclusion* function of R.

By $\mathfrak{R}(Fo)$ we denote the *class of all AHS-rules* on Fo.

(Later an AHS-rule of Fo will only be called a *rule on* Fo.)

Visualization of applications of AHS-rules



Visualization as ‘graph hyperedges’ of

– a zero premise application α of an AHS-rule R_1, and

– of an application α' of an AHS-rule R_2.

Abstract Hilbert Systems

Definition. An *abstract Hilbert system* (an AHS) $\mathcal{H}$ is a triple $\langle Fo, Ax, \mathcal{R} \rangle$ where

- Fo, Ax and $\mathcal{R}$ the sets of *formulas*, *axioms*, and *rules* of $\mathcal{H}$,
- $Ax \subseteq Fo$,
- every $R \in \mathcal{R}$ is an AHS-rule on Fo.

We write $\mathfrak{H}$ for the class of all AHS's.

Derivations in an AHS

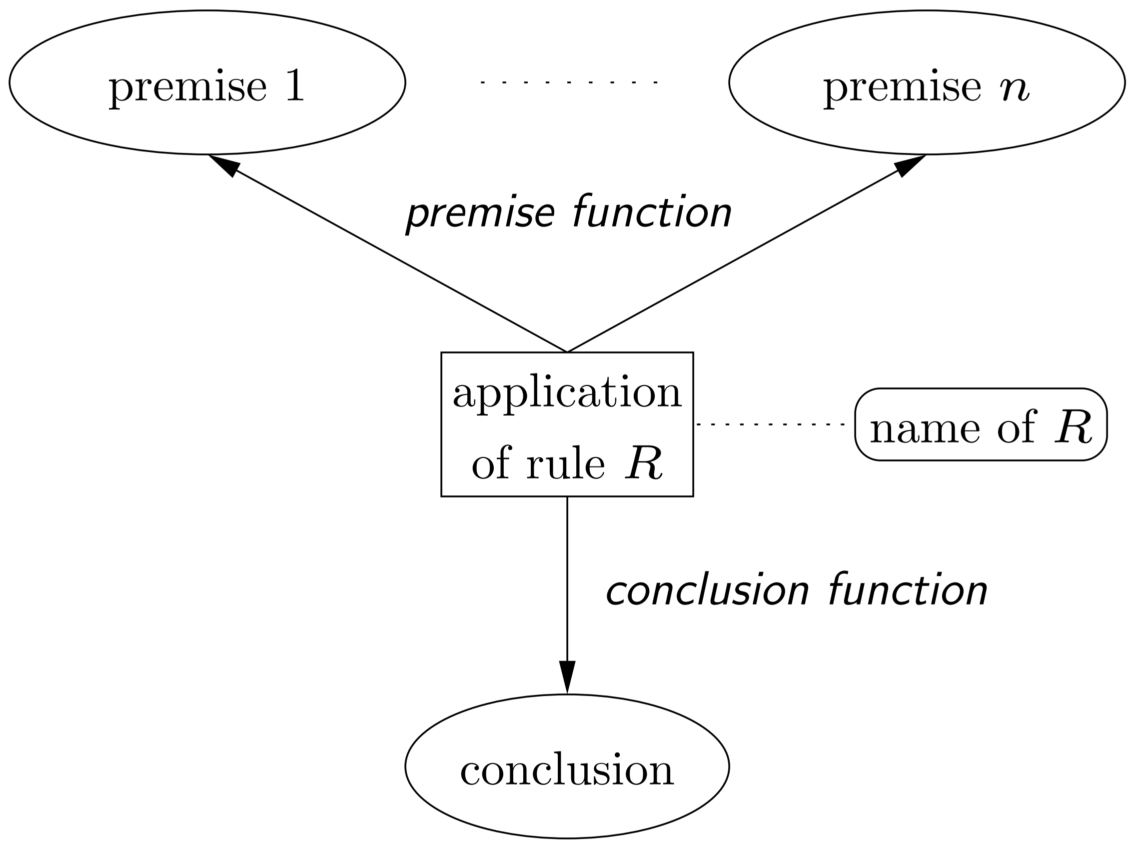
For a set X, we denote by $\mathcal{M}_{\mathsf{f}}(X)$ the set of finite multisets over X.

Notation. Let $\mathcal{H}$ be an AHS with formula set Fo.

By $Der(\mathcal{H})$ we denote the set of *derivations* in $\mathcal{H}$. And for a derivation $\mathcal{D}$ in $\mathcal{H}$, we denote by

- $\mathsf{assm}(\mathcal{D}) \in \mathcal{M}_{\mathsf{f}}(Fo)$ the *multiset of assumptions* of $\mathcal{D}$, and by
- $\mathsf{concl}(\mathcal{D}) \in Fo$ the *conclusion* of $\mathcal{D}$.

An abstract notion of rule with (rule) names



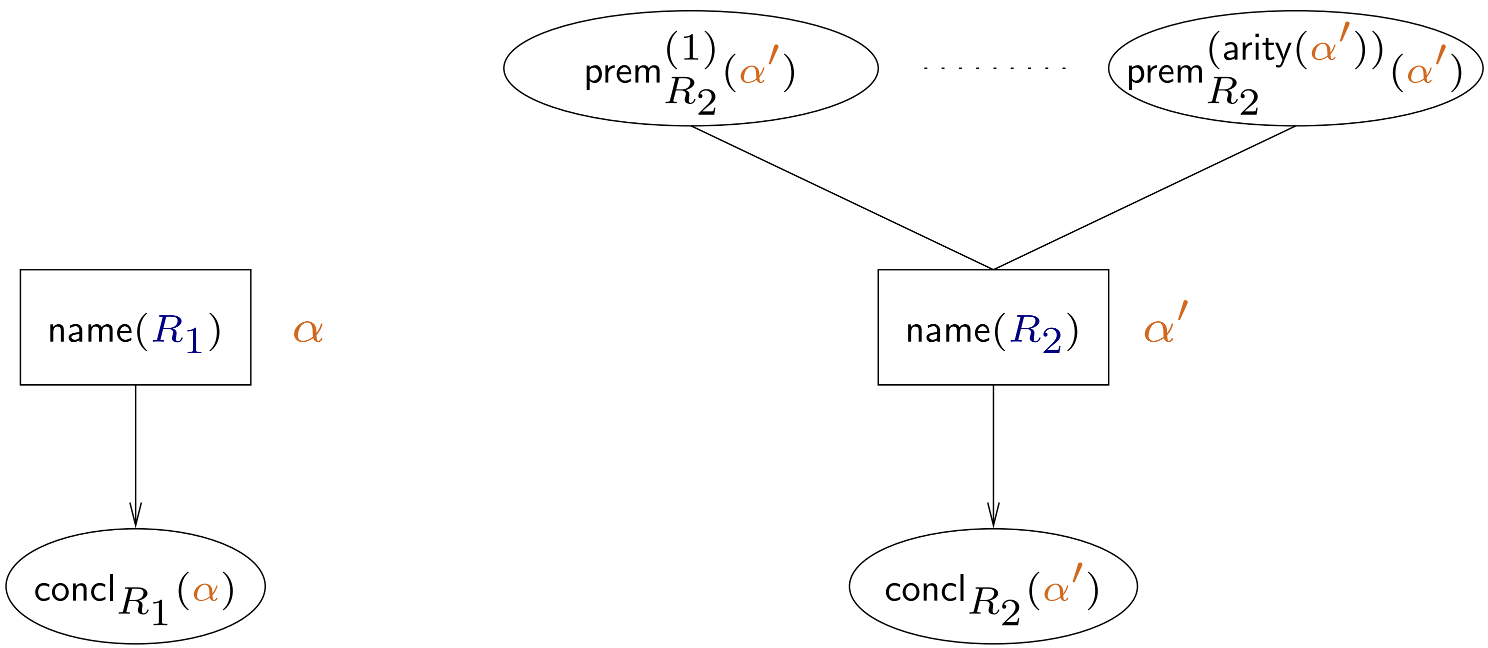
Abstract Hilbert systems with names

Definition. An *abstract Hilbert system with names (for axioms and rules)* (an n-AHS) $\mathcal{H}$ is a quadrupel $\langle Fo, Na, nAx, n\mathcal{R} \rangle$ where

- Fo, Na, nAx and $n\mathcal{R}$ are the *formulas*, *names*, *named axioms* and *named rules of* $\mathcal{H}$,
- $nAx \subseteq Fo \times Na$,
- $n\mathcal{R} \subseteq \mathfrak{R}(Fo) \times Na$, (we allow to write $R = \langle R, \text{name}(R) \rangle$, for arbitrary $R \in n\mathcal{R}$),
- – "axiom names" in nAx are different from "rule names" in $n\mathcal{R}$,
  – different rules are differently named in $n\mathcal{R}$.

We write $\mathfrak{Hn}$ for the class of all n-AHS's.

Visualization of applications of n-AHS-rules



Visualization as 'graph hyperedges' of

– a zero premise application α of a named rule R_1, and

– of an application α' of a named rule R_2 in an n-AHS $\mathcal{H}$.

Derivations in an n-AHS

Definition. (Derivations in abstr. Hilbert systems with names). Let $\mathcal{H} = \langle Fo, Na, nAx, n\mathcal{R} \rangle$ be an n-AHS.

A *derivation* $\mathcal{D}$ *in* $\mathcal{H}$ is the result (a prooftree) of carrying out a finite number of construction steps of the following three kinds:

(i) For every named axiom $\langle A, name \rangle \in nAx$, the prooftree $\mathcal{D}$ of the form

$$\frac{(name)}{A}$$

is a derivation in $\mathcal{H}$ with conclusion $\text{concl}(\mathcal{D}) = A$ and without assumptions, i.e. such that $\text{set}(\text{assm}(\mathcal{D})) = \emptyset$ holds.

(ii) For all formulas $A \in Fo$, the prooftree $\mathcal{D}$ consisting only of the formula

$$A$$

is a derivation in $\mathcal{H}$ with assumptions $\mathsf{assm}(\mathcal{D}) = \{A\}$ and with conclusion $\mathsf{concl}(\mathcal{D}) = A$.

(iii) Let $R = \langle R, \mathsf{name}(R)\rangle \in n\mathcal{R}$ a named rule of $\mathcal{H}$, and $\alpha \in Apps_R$ an appl. of R. We distinguish two cases concerning the arity of α:

Case 1. $\mathsf{arity}_R(\alpha) = 0$: Given that $\mathsf{concl}_R(\alpha) = A$, the prooftree

$$\frac{}{A}\,\mathsf{name}(R)$$

is a derivation $\mathcal{D}$ in $\mathcal{H}$ that has conclusion $\mathsf{concl}(\mathcal{D}) = A$ and no assumptions, i.e. $\mathsf{assm}(\mathcal{D}) = \emptyset$ holds.

Case 2. $\text{arity}_R(\alpha) = n \in \omega \backslash \{0\}$:

Given that $\text{prem}_R(\alpha) = \langle A_1, \ldots, A_n \rangle$ and that $\text{concl}_R(\alpha) = A$, and given further that $\mathcal{D}_1, \ldots, \mathcal{D}_n$ are derivations in $\mathcal{H}$ with respective conclusions $A_1, \ldots, A_n$, the prooftree of the form

$$\frac{\begin{array}{ccc} \mathcal{D}_1 & & \mathcal{D}_n \\ A_1 & \ldots & A_n \end{array}}{A}\,\text{name}(R)$$

is a derivation $\mathcal{D}$ in $\mathcal{H}$ with conclusion $\text{concl}(\mathcal{D}) = A$ and with assumptions and depth defined by

$$\text{assm}(\mathcal{D}) = \biguplus_{i=1}^{n} \text{assm}(\mathcal{D}_i)\ .$$

We denote by $Der(\mathcal{H})$ the set of all derivations in $\mathcal{H}$.

Three Consequence Relations on an AHS or n-AHS

Definition. For an AHS or n-AHS $\mathcal{H}$ we define:

$\Sigma \vdash_{\mathcal{H}} A \iff$ A is the conclusion of a derivation in $\mathcal{H}$ whose assumptions are contained in the set Σ;

$\Sigma \vdash_{\mathcal{H}}^{(\mathbf{s})} A \iff$ A is the conclusion of a derivation in $\mathcal{H}$ whose assumptions are contained in the set Σ and that uses every formula in Σ at least once;

$\Gamma \vdash_{\mathcal{H}}^{(\mathbf{m})} A \iff$ A is the conclusion of a derivation in $\mathcal{H}$ whose assumptions are contained in the multiset Γ and that uses every formula in Γ precisely once.

Three Consequence Relations on an AHS or n-AHS

Definition. Let $\mathcal{H}$ be an AHS or n-AHS with formula set Fo.
We define the consequence relations $\vdash_{\mathcal{H}}$, $\vdash_{\mathcal{H}}^{(\mathbf{s})}$ and $\vdash_{\mathcal{H}}^{(\mathbf{m})}$ by setting for all $A \in Fo$, finite *sets* Σ on Fo and *multisets* Γ on Fo:

$$\Sigma \vdash_{\mathcal{H}} A \iff (\exists\, \mathcal{D} \in Der(\mathcal{H}))\big[\,\mathsf{set}(\mathsf{assm}(\mathcal{D})) \subseteq \Sigma \;\;\&\;\; \mathsf{concl}(\mathcal{D}) = A\,\big]\,,$$

$$\Sigma \vdash_{\mathcal{H}}^{(\mathbf{s})} A \iff (\exists\, \mathcal{D} \in Der(\mathcal{H}))\big[\,\mathsf{set}(\mathsf{assm}(\mathcal{D})) = \Sigma \;\;\&\;\; \mathsf{concl}(\mathcal{D}) = A\,\big]\,,$$

$$\Gamma \vdash_{\mathcal{H}}^{(\mathbf{m})} A \iff (\exists\, \mathcal{D} \in Der(\mathcal{H}))\big[\,\mathsf{assm}(\mathcal{D}) = \Gamma \;\;\&\;\; \mathsf{concl}(\mathcal{D}) = A\,\big]\,,$$

whereby $\vdash_{\mathcal{H}}, \vdash_{\mathcal{H}}^{(\mathbf{s})} \subseteq \mathcal{P}_{\mathsf{f}}(Fo) \times Fo$ and $\vdash_{\mathcal{H}}^{(\mathbf{m})} \subseteq \mathcal{M}_{\mathsf{f}}(Fo) \times Fo$.

The neglected consequence relation

Definition. Let $\mathcal{H}$ be an AHS or n-AHS with formula set Fo.

We define the consequence relation $\vdash_{\mathcal{H}}^{\textbf{(mw)}}$ by letting for all $A \in Fo$ and *multisets* Γ on Fo

$$\Gamma \vdash_{\mathcal{H}}^{\textbf{(mw)}} A \iff$$

$$\iff (\exists \mathcal{D} \in Der(\mathcal{H}))\big[\, \mathsf{assm}(\mathcal{D}) \subseteq \Gamma \;\;\&\;\; \mathsf{concl}(\mathcal{D}) = A \,\big] \,,$$

whereby $\vdash_{\mathcal{H}}^{\textbf{(mw)}} \subseteq \mathcal{M}_{\mathsf{f}}(Fo) \times Fo$.

Rule Admissibility

Definition. Let $\mathcal{H}$ be an AHS or n-AHS with formula set Fo, and let $R = \langle Apps_R, \mathsf{prem}, \mathsf{concl} \rangle$ be a rule on Fo.

The rule R is *admissible in* $\mathcal{H}$ if and only if it holds that

$$(\forall \alpha \in Apps_R) \Big[(\forall A \in \mathsf{set}(\mathsf{prem}(\alpha))) [\vdash_{\mathcal{H}} A] \implies \implies \vdash_{\mathcal{H}} \mathsf{concl}(\alpha) \Big] ,$$

i.e. iff the *theory* of $\mathcal{H}$ (the set of theorems of $\mathcal{H}$) is closed under applications of R.

Three Versions of Rule Derivability

Definition. Let $\mathcal{H}$ be an AHS or an n-AHS. We consider a rule $R = \langle Apps_R, \mathsf{prem}, \mathsf{concl}\rangle$ on $Fo_{\mathcal{H}}$.

The rule R is *derivable in* $\mathcal{H}$ if and only if

$$(\forall \alpha \in Apps_R)\left[\,\mathsf{set}(\mathsf{prem}(\alpha)) \vdash_{\mathcal{H}} \mathsf{concl}(\alpha)\,\right]$$

holds, that is, for all applications α of R, there exists a "mimicking derivation" $\mathcal{D}$ in $\mathcal{H}$,i.e. a derivation $\mathcal{D}$ with conclusion $\mathsf{concl}(\alpha)$ and with its assumptions contained in $\mathsf{set}(\mathsf{assm}(\alpha))$.

Three Versions of Rule Derivability

Definition. Let $\mathcal{H}$ be an AHS or an n-AHS. We consider a rule $R = \langle Apps_R, \mathsf{prem}, \mathsf{concl}\rangle$ on $Fo_{\mathcal{H}}$.

The rule R is *derivable in* $\mathcal{H}$ if and only if

$$(\forall \alpha \in Apps_R)\ \big[\ \mathsf{set}(\mathsf{prem}(\alpha)) \vdash_{\mathcal{H}} \mathsf{concl}(\alpha)\ \big]$$

holds.

And we say that R is *s-derivable in* $\mathcal{H}$ or that R is *m-derivable in* $\mathcal{H}$ if and only if, respectively, the assertions (1) and (2) hold:

$$(\forall \alpha \in Apps_R)\ \big[\ \mathsf{set}(\mathsf{prem}(\alpha)) \vdash^{(\mathbf{s})}_{\mathcal{H}} \mathsf{concl}(\alpha)\ \big]\ , \tag{1}$$

$$(\forall \alpha \in Apps_R)\ \big[\ \mathsf{mset}(\mathsf{prem}(\alpha)) \vdash^{(\mathbf{m})}_{\mathcal{H}} \mathsf{concl}(\alpha)\ \big]\ . \tag{2}$$

Formula Derivability and Admissibility

Definition. Let $\mathcal{H}$ be an AHS on an n-AHS with formula set Fo.

We call a formula $A \in Fo$ *admissible*, *derivable*, *s-derivable* and *m-derivable* if and only if

$$\vdash_{\mathcal{H}} A$$

holds, i.e. iff A is a theorem of $\mathcal{H}$.

Admissible and (s-,m-)derivable rules: Examples (I)

Example. Let $\mathcal{H}$ be the AHS *without axioms* and with the three rules R_1, R_2 and $R_{AA.B}$ each of which has only one application:

$$\frac{C_1}{A}\,R_1 \qquad \frac{C_2}{A}\,R_2 \qquad \frac{A \quad A}{B}\,R_{AA.B} \ .$$

- $\dfrac{C_1 \quad C_2}{B}$ is derivable, s-derivable, m-derivable in $\mathcal{H}$:
$$\dfrac{\dfrac{C_1}{A}\,R_1 \qquad \dfrac{C_2}{A}\,R_2}{B}\,R_{AA.B} \ .$$

- $\dfrac{C_1}{B}$ is derivable, s-derivable (not m-derivable) in $\mathcal{H}$:
$$\dfrac{\dfrac{C_1}{A}\,R_1 \qquad \dfrac{C_1}{A}\,R_1}{B}\,R_{AA.B} \ .$$

Admissible and (s-,m-)derivable rules: Examples (I)

Example. (Continued) Let $\mathcal{H}$ be the AHS *without axioms* and with the three rules R_1, R_2 and $R_{AA.B}$ each of which has only one application:

$$\frac{C_1}{A}\,R_1 \qquad \frac{C_2}{A}\,R_2 \qquad \frac{A \quad A}{B}\,R_{AA.B} \quad .$$

- $\dfrac{C_1 \quad C_2}{A}$ is derivable (not s-derivable) (not m-derivable) in $\mathcal{H}$: $\dfrac{C_1}{A}\,R_1$.

- $\dfrac{B}{C}$ is admissible (not derivable) (not s-derivable) (not m-derivable) in $\mathcal{H}$: Due to $B \notin Th(\mathcal{H}) (= \emptyset)$.

Admissible and (s-,m-)derivable rules: Examples (II)

Example. Let $\mathcal{H}$ be the AHS with the single axiom

$$A$$

and with the two rules $R_{A.B}$ and $R_{A.C}$ each of which has only one application:

$$\frac{A}{B}\,R_{A.B} \qquad\qquad \frac{A}{C}\,R_{A.C}$$

- $\dfrac{D}{C}$ is admissible in $\mathcal{H}$. $\dfrac{D}{F}$ is admissible in $\mathcal{H}$.
- $\dfrac{A \quad D}{F}$ is admissible in $\mathcal{H}$: Since $D \notin Th(\mathcal{H})$.

Admissible and (s-,m-)derivable rules: Examples (II)

Example. (Continued) Let $\mathcal{H}$ be the AHS with the single axiom

$$A$$

and with the two rules $R_{A.B}$ and $R_{A.C}$ each of which has only one application:

$$\frac{A}{B}\,R_{A.B} \qquad\qquad \frac{A}{C}\,R_{A.C}$$

- $\dfrac{A \quad C}{D}$ is not admissible in $\mathcal{H}$: Since $A, C \in Th(\mathcal{H})$ and $D \notin Th(\mathcal{H})$.
- $\dfrac{A \quad C}{B}$ is derivable (not s-derivable) (not m-derivable) in $\mathcal{H}$: $\dfrac{A}{B}\,R_{A.B}$.

Rule Derivability and Admissibility: Basic Facts

Lemma. (Hindley, Seldin *[except (iv)]* **).**
Let $\mathcal{H}$ be an AHS and let R be a rule on the set of formulas of $\mathcal{H}$.

(i) R *is admissible in* $\mathcal{H}$ $\iff$ *the AHS* $\mathcal{H}+R$ *does not possess more theorems than* $\mathcal{H}$.

(ii) R *is derivable in* $\mathcal{H}$ $\Longrightarrow$ R *is also admissible in* $\mathcal{H}$. *(The inverse implication does not hold in general.)*

(iii) R *is derivable in* $\mathcal{H}$ $\Longrightarrow$ R *is derivable in every extension of* $\mathcal{H}$ *that is obtained by adding new formulas, axioms and/or rules.*

(iv) R *is* m*-derivable in* $\mathcal{H}$ $\Longrightarrow$ R *is* s*-derivable in* $\mathcal{H}$ $\Longrightarrow$ R *is derivable in* $\mathcal{H}$. *(The inverse implications aren't true in general).*

Rule Derivability and Admissibility: Basic Facts

Theorem. *Let $\mathcal{H}$ be an AHS with set Fo of formulas, and R a rule on Fo.*

Then the following three statements are equivalent:

(i) R *is derivable in* $\mathcal{H}$.

(ii) R *is admissible in the AHS* $\mathcal{H}+\Sigma$, *for every set* Σ *on* Fo.

(iii) R *is admissible in every extension of* $\mathcal{H}$ *that is obtained by adding new formulas, axioms and/or rules(in every* extension by enlargement *of* $\mathcal{H}$*).*

(Mutual) Inclusion Relations between Abstract Hilbert Systems

We will define *inclusion relations* $\preceq_{P,Q}$ between AHS's by stipulating, for all $\mathcal{H}_1, \mathcal{H}_2 \in \mathfrak{H}$,

$$\mathcal{H}_1 \preceq_{P,Q} \mathcal{H}_2 \iff \left\{ \begin{array}{l} \text{Every formula in } \mathcal{H}_1 \text{ is also a formula of } \mathcal{H}_2, \\ \text{and every object in } \mathcal{H}_1 \text{ having property } P \\ \text{appears in } \mathcal{H}_2 \text{ as an object with property } Q. \end{array} \right\}$$

for properties P and Q of 'objects' in AHS's (objects like theorems, rules, . . . , and properties like "is theorem" or "is derivable rule").

And, for every inclusion relation $\preceq_{P,Q}$, we will define the *induced mutual inclusion relation* $\preceq_{P,Q}$ by stipulating for all $\mathcal{H}_1, \mathcal{H}_2 \in \mathfrak{H}$:

$$\mathcal{H}_1 \sim_{P,Q} \mathcal{H}_2 \iff \mathcal{H}_1 \preceq_{P,Q} \mathcal{H}_2 \;\&\; \mathcal{H}_2 \preceq_{P,Q} \mathcal{H}_1 \,.$$

Relations between Abstract Hilbert Systems (I)

Definition. We define the inclusion relation $\preceq_{th}$ on the class $\mathfrak{H}$ by stipulating for all $\mathcal{H}_1, \mathcal{H}_2 \in \mathfrak{H}$:

$$\mathcal{H}_1 \preceq_{th} \mathcal{H}_2 \iff Fo_{\mathcal{H}_1} \subseteq Fo_{\mathcal{H}_2} \ \& \ Th(\mathcal{H}_1) \subseteq Th(\mathcal{H}_2) \ .$$

We define the inclusion relations $\preceq_{rth}$, $\preceq_{rth}^{(s)}$ and $\preceq_{rth}^{(m)}$ on $\mathfrak{H}$ by stipulating for all $\mathcal{H}_1, \mathcal{H}_2 \in \mathfrak{H}$:

$$\mathcal{H}_1 \preceq_{rth} \mathcal{H}_2 \iff Fo_{\mathcal{H}_1} \subseteq Fo_{\mathcal{H}_2} \ \& \ \vdash_{\mathcal{H}_1} \subseteq \vdash_{\mathcal{H}_2} \ ,$$

$$\mathcal{H}_1 \preceq_{rth}^{(s)} \mathcal{H}_2 \iff Fo_{\mathcal{H}_1} \subseteq Fo_{\mathcal{H}_2} \ \& \ \vdash_{\mathcal{H}_1}^{(s)} \subseteq \vdash_{\mathcal{H}_2}^{(s)} \ ,$$

$$\mathcal{H}_1 \preceq_{rth}^{(m)} \mathcal{H}_2 \iff Fo_{\mathcal{H}_1} \subseteq Fo_{\mathcal{H}_2} \ \& \ \vdash_{\mathcal{H}_1}^{(m)} \subseteq \vdash_{\mathcal{H}_2}^{(m)} \ .$$

Relations between Abstract Hilbert Systems (I)

Definition. We define the inclusion relation $\preceq_{th}$ on the class $\mathfrak{H}$ by stipulating for all $\mathcal{H}_1, \mathcal{H}_2 \in \mathfrak{H}$:

$$\mathcal{H}_1 \preceq_{th} \mathcal{H}_2 \iff Fo_{\mathcal{H}_1} \subseteq Fo_{\mathcal{H}_2} \,\&\, (\forall A \in Fo_{\mathcal{H}_1})[(\vdash_{\mathcal{H}_1} A) \Rightarrow (\vdash_{\mathcal{H}_2} A)]\,.$$

We define the inclusion relations $\preceq_{rth}$ and $\preceq_{rth}^{(\mathsf{m})}$ on $\mathfrak{H}$ by stipulating for all $\mathcal{H}_1, \mathcal{H}_2 \in \mathfrak{H}$

$$\mathcal{H}_1 \preceq_{rth} \mathcal{H}_2 \iff Fo_{\mathcal{H}_1} \subseteq Fo_{\mathcal{H}_2} \,\&$$
$$\&\ (\forall \Sigma \in \mathcal{P}(Fo_{\mathcal{H}_1}))(\forall A \in Fo_{\mathcal{H}_1})\big[(\Sigma \vdash_{\mathcal{H}_1} A) \Rightarrow (\Sigma \vdash_{\mathcal{H}_2} A)\big]\,,$$

$$\mathcal{H}_1 \preceq_{rth}^{(\mathsf{m})} \mathcal{H}_2 \iff Fo_{\mathcal{H}_1} \subseteq Fo_{\mathcal{H}_2} \,\&$$
$$\&\ (\forall \Gamma \in \mathcal{M}_{\mathsf{f}}(Fo_{\mathcal{H}_1}))(\forall A \in Fo_{\mathcal{H}_1})\big[(\Gamma \vdash_{\mathcal{H}_1}^{(\mathbf{m})} A) \Rightarrow (\Gamma \vdash_{\mathcal{H}_2}^{(\mathbf{m})} A)\big]\,.$$

These four inclusion relations *induce* respective mutual inclusion relations: For all $\mathcal{H}_1, \mathcal{H}_2 \in \mathfrak{H}$, we let

$$\mathcal{H}_1 \sim_{th} \mathcal{H}_2 \iff \mathcal{H}_1 \preceq_{th} \mathcal{H}_2 \;\&\; \mathcal{H}_2 \preceq_{th} \mathcal{H}_1$$

(if $\mathcal{H}_1 \sim_{th} \mathcal{H}_2$ holds, we say that $\mathcal{H}_1$ and $\mathcal{H}_2$ are *(theorem) equivalent*;and we use analogous stipulations for the mutual inclusion relations

$$\sim_{rth} \;, \quad \sim_{rth}^{(s)} \;\text{ and }\; \sim_{rth}^{(m)}$$

(if $\mathcal{H}_1 \sim_{rth} \mathcal{H}_2$ holds, we say that $\mathcal{H}_1$ and $\mathcal{H}_2$ are *equivalent with respect to relative theoremhood*).

Relations between Abstract Hilbert Systems (II)

Definition. We define the inclusion relation $\preceq_{adm}$ on the class $\mathfrak{H}$ by stipulating for all $\mathcal{H}_1, \mathcal{H}_2 \in \mathfrak{H}$:

$$\begin{aligned} \mathcal{H}_1 \preceq_{adm} \mathcal{H}_2 \iff & Fo_{\mathcal{H}_1} \subseteq Fo_{\mathcal{H}_2} \ \& \\ & \& \ (\forall A \in Fo_{\mathcal{H}_1})\big[\, A \text{ is adm. in } \mathcal{H}_1 \ \Rightarrow \ A \text{ is adm. in } \mathcal{H}_2 \,\big] \ \& \\ & \& \ (\forall R \text{ rule on } Fo_{\mathcal{H}_1}) \\ & \qquad \big[\, R \text{ is admissible in } \mathcal{H}_1 \ \Rightarrow \ R \text{ is admissible in } \mathcal{H}_2 \,\big] \,. \end{aligned}$$

The inclusion relations $\preceq_{der}$, $\preceq_{der}^{(s)}$ and $\preceq_{der}^{(m)}$ are defined analogously by using ‘derivable’, ‘s-derivable’ and ‘m-derivable’ instead of ‘admissible’.

The *induced* mutual incl. relations: $\sim_{adm}$, $\sim_{der}$, $\sim_{der}^{(s)}$ and $\sim_{der}^{(m)}$.

Relations between Abstract Hilbert Systems (III)

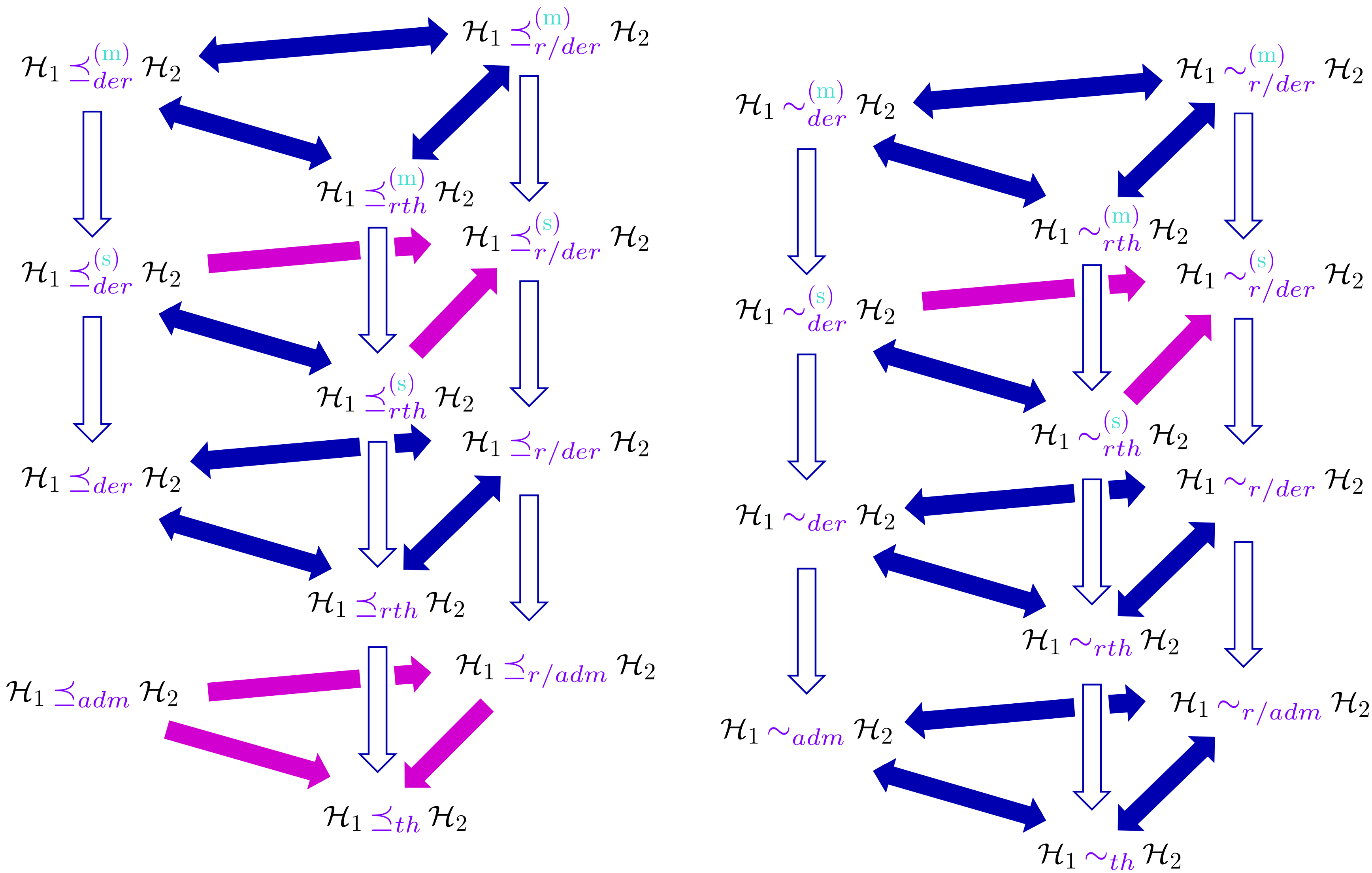
Definition. We define the inclusion relation $\preceq_{r/adm}$ on the class $\mathfrak{H}$ by stipulating for all $\mathcal{H}_1, \mathcal{H}_2 \in \mathfrak{H}$:

$$\mathcal{H}_1 \preceq_{r/adm} \mathcal{H}_2 \iff Fo_{\mathcal{H}_1} \subseteq Fo_{\mathcal{H}_2} \;\&$$
$$\& \;(\forall A \in Ax_{\mathcal{H}_1})\big[\,A \text{ is admissible in } \mathcal{H}_2\,\big] \;\&$$
$$\& \;(\forall R \in \mathcal{R}_{\mathcal{H}_1})\big[\,R \text{ is admissible in } \mathcal{H}_2\,\big]\,.$$

The inclusion relations $\preceq_{r/der}$, $\preceq^{(s)}_{r/der}$ and $\preceq^{(m)}_{r/der}$ are defined analogously by using 'derivable', 's-derivable' and 'm-derivable' instead of 'admissible'.

These four relations on $\mathfrak{H}$ *induce* the four mutual inclusion relations $\sim_{r/adm}$, $\sim_{r/der}$, $\sim^{(s)}_{r/der}$ and $\sim^{(m)}_{r/der}$ on $\mathfrak{H}$, respectively.

Relationships between(mutual)inclusion relations



Relationships between (mutual) inclusion relations

Theorem. (Interrelation Prisms)

(i) *The implications and equivalences shown in the interrelations prisms hold, for all AHS's* $\mathcal{H}_1$ *and* $\mathcal{H}_2$, *between statements* $\mathcal{H}_1 \preceq \mathcal{H}_2$ *(where* $\preceq$ *is an introduced inclusion relation), and respectively, between statements of the form* $\mathcal{H}_1 \sim \mathcal{H}_2$ *(where* $\sim$ *is an introduced inclusion relation).*

(ii) *Not inverted arrows indicate that the implication in the opposite direction does not hold in general.*

(iii) *In the case of the int.rel. prism for the incl. relations, in general no implication holds in either direction between* $\mathcal{H}_1 \preceq_{r/adm} \mathcal{H}_2$ *and any of* $\mathcal{H}_1 \preceq_{r/der} \mathcal{H}_2$, $\mathcal{H}_1 \preceq^{(s)}_{r/der} \mathcal{H}_2$ *or* $\mathcal{H}_1 \preceq^{(m)}_{r/der} \mathcal{H}_2$.

A Consequence of the Interrelation Prisms (I)

Corollary. (Characterizations of rule admissibility, derivability and m-derivability)
Let $\mathcal{H}$ be an AHS and let R be a rule on the set of formulas of $\mathcal{H}$. Then the following hold:

$$R \text{ is } \textit{admissible} \text{ in } \mathcal{H} \iff \mathcal{H}{+}R \sim_{th} \mathcal{H} \;,$$

$$R \text{ is } \textit{derivable} \text{ in } \mathcal{H} \iff \mathcal{H}{+}R \sim_{rth} \mathcal{H} \;,$$

$$R \text{ is } \textit{s-derivable} \text{ in } \mathcal{H} \Longleftarrow \mathcal{H}{+}R \sim_{rth}^{(s)} \mathcal{H} \;,$$

$$R \text{ is } \textit{m-derivable} \text{ in } \mathcal{H} \iff \mathcal{H}{+}R \sim_{rth}^{(m)} \mathcal{H} \;.$$

A Consequence of the Interrelation Prisms (II)

Theorem. (Reformulation of a theorem by Schütte).
For all abstract Hilbert systems $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ it holds:

$$\mathcal{H}_1 \preceq_{r/der} \mathcal{H}_2 \;\;\&\;\; \mathcal{H}_2 \preceq_{r/adm} \mathcal{H}_3 \quad\Longrightarrow\quad \mathcal{H}_1 \preceq_{r/adm} \mathcal{H}_3 \,.$$

Wrong Proof. For all AHS's $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ it holds:

$$\mathcal{H}_1 \preceq_{r/der} \mathcal{H}_2 \;\;\&\;\; \mathcal{H}_2 \preceq_{r/adm} \mathcal{H}_3 \quad\Longrightarrow$$

$$\Longrightarrow \quad \mathcal{H}_1 \preceq_{r/adm} \mathcal{H}_2 \;\;\&\;\; \mathcal{H}_2 \preceq_{r/adm} \mathcal{H}_3 \qquad \text{(int.rels. prisma)}$$

$$\Longrightarrow \quad \mathcal{H}_1 \preceq_{r/adm} \mathcal{H}_2 \qquad \text{(if } \preceq_{r/adm} \textbf{ were} \text{ transitive)} \,.$$

However, the relation $\preceq_{r/adm}$ **is not transitive**.

A Consequence of the Interrelation Prisms (II)

Proof. For all AHS's $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ it holds:

$$\mathcal{H}_1 \preceq_{r/der} \mathcal{H}_2 \;\;\&\;\; \mathcal{H}_2 \preceq_{r/adm} \mathcal{H}_3 \implies$$

$$\implies \mathcal{H}_1{+}\mathcal{H}_2{+}\mathcal{H}_3 \sim_{r/der} \mathcal{H}_2{+}\mathcal{H}_3 \;\;\&\;\; \mathcal{H}_2{+}\mathcal{H}_3 \sim_{r/adm} \mathcal{H}_3$$

(due to defs.)

$$\implies \mathcal{H}_1{+}\mathcal{H}_2{+}\mathcal{H}_3 \sim_{r/adm} \mathcal{H}_2{+}\mathcal{H}_3 \;\;\&\;\; \mathcal{H}_2{+}\mathcal{H}_3 \sim_{r/adm} \mathcal{H}_3$$

(due int.rels. prisma)

$$\implies \mathcal{H}_1{+}\mathcal{H}_2{+}\mathcal{H}_3 \sim_{r/adm} \mathcal{H}_3$$

($\sim_{r/adm}$ is transitive)

$$\implies \mathcal{H}_1 \preceq_{r/adm} \mathcal{H}_2$$

(def. of $\preceq_{r/adm}$) .

□

Three notions of “mimicking derivation”

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be AHS’s or n-AHS’s, and let $\mathcal{D}_1 \in Der(\mathcal{H}_1)$ and $\mathcal{D}_2 \in Der(\mathcal{H}_2)$ be derivations.

We say that $\mathcal{D}_1$ *mimics* $\mathcal{D}_2$ (denoted by $\mathcal{D}_1 \precsim \mathcal{D}_2$) if and only if

$$\text{set}(\text{assm}(\mathcal{D}_1)) \subseteq \text{set}(\text{assm}(\mathcal{D}_2)) \quad \& \quad \text{concl}(\mathcal{D}_1) = \text{concl}(\mathcal{D}_2) \ ,$$

i.e. $\mathcal{D}_1$ and $\mathcal{D}_2$ have the same conclusion and all assumptions of $\mathcal{D}_1$ are contained in the set of assumptions of $\mathcal{D}_2$.

Three notions of “mimicking derivation”

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be AHS's or n-AHS's, and let $\mathcal{D}_1 \in Der(\mathcal{H}_1)$ and $\mathcal{D}_2 \in Der(\mathcal{H}_2)$ be derivations.

We say that $\mathcal{D}_1$ *mimics* $\mathcal{D}_2$ (denoted by $\mathcal{D}_1 \precsim \mathcal{D}_2$) if and only if

$$\mathsf{set}(\mathsf{assm}(\mathcal{D}_1)) \subseteq \mathsf{set}(\mathsf{assm}(\mathcal{D}_2)) \;\;\&\;\; \mathsf{concl}(\mathcal{D}_1) = \mathsf{concl}(\mathcal{D}_2) \,,$$

Furthermore, we stipulate that $\mathcal{D}_1$ *s-mimics* $\mathcal{D}_2$ (symb. $\mathcal{D}_1 \simeq^{(\mathsf{s})} \mathcal{D}_2$), and that $\mathcal{D}_1$ *m-mimics* $\mathcal{D}_2$ (symb. $\mathcal{D}_1 \simeq^{(\mathsf{m})} \mathcal{D}_2$) if and only if respectively (3) and (4) hold:

$$\mathsf{set}(\mathsf{assm}(\mathcal{D}_1)) = \mathsf{set}(\mathsf{assm}(\mathcal{D}_2)) \;\;\&\;\; \mathsf{concl}(\mathcal{D}_1) = \mathsf{concl}(\mathcal{D}_2) \,, \quad (3)$$

$$\mathsf{assm}(\mathcal{D}_1) = \mathsf{assm}(\mathcal{D}_2) \;\;\&\;\; \mathsf{concl}(\mathcal{D}_1) = \mathsf{concl}(\mathcal{D}_2) \,. \quad (4)$$

Examples. (The notions $\precsim$, $\simeq^{(s)}$ and $\simeq^{(m)}$ of mimicking deriv.).

(a)
$$\dfrac{\dfrac{C_1}{A}\,R_1}{B}\,R_{A.B} \quad \left\{\begin{array}{c} \precsim \\ \not\succsim \\ \not\simeq^{(s)} \\ \not\simeq^{(m)} \end{array}\right\} \quad \dfrac{\dfrac{C_1}{A}\,R_1 \qquad \dfrac{C_2}{A}\,R_2}{B}\,R_{AA.B}$$

(b)
$$\dfrac{\dfrac{C_1}{A}\,R_1}{B}\,R_{A.B} \quad \left\{\begin{array}{c} \precsim \\ \succsim \\ \simeq^{(s)} \\ \not\simeq^{(m)} \end{array}\right\} \quad \dfrac{\dfrac{C_1}{A}\,R_1 \qquad \dfrac{C_1}{A}\,R_1}{B}\,R_{AA.B}$$

(c)
$$\dfrac{\dfrac{C_1}{A}\,R_1 \qquad \dfrac{C_2}{A}\,R_2}{B}\,R_{AA.B} \quad \left\{\begin{array}{c} \precsim \\ \succsim \\ \simeq^{(s)} \\ \simeq^{(m)} \end{array}\right\} \quad \dfrac{\dfrac{C_2}{A}\,R_2 \qquad \dfrac{C_1}{A}\,R_1}{B}\,R_{AA.B}$$

Proposition. (The notions $\precsim$, $\simeq^{(\mathsf{s})}$ and $\simeq^{(\mathsf{m})}$ of mimicking deriv.).

(i) $\precsim$ *is reflexive and transitive.*

(ii) $\simeq^{(s)}$ *and* $\simeq^{(m)}$ *are equivalence relations.*

(iii) For all derivations $\mathcal{D}_1$ *and* $\mathcal{D}_2$

$$\mathcal{D}_1 \simeq^{(s)} \mathcal{D}_2 \iff \mathcal{D}_1 \precsim \mathcal{D}_2 \;\&\; \mathcal{D}_2 \precsim \mathcal{D}_1 \;.$$

holds, i.e. $\simeq^{(s)} \;=\; \precsim \cap \succsim$ *, where* $\succsim \;=\; (\precsim)^{-1}$*.*

(iv) $\simeq^{(m)} \;\subsetneqq\; \simeq^{(s)} \;\subsetneqq\; \precsim \;.$

Four notions of "rule elimination"

Definition. Let $\mathcal{H}$ be an AHS or n-AHS, and let R be a (named) rule of $\mathcal{H}$.

(i) We say that *R-elimination holds in $\mathcal{H}$* if and only

$$(\forall\,\mathcal{D}\in Der(\mathcal{H}))\,\Big[\,\mathsf{set}(\mathsf{assm}(\mathcal{D}))=\emptyset \implies \implies (\exists\,\mathcal{D}'\in Der(\mathcal{H}-R))\,[\,\mathcal{D}'\precsim\mathcal{D}\,]\,\Big]\,,$$

i.e. iff every derivation $\mathcal{D}$ in $\mathcal{H}$ *without assumptions* can be mimicked by a derivation $\mathcal{D}'$ in $\mathcal{H}-R$.

(ii) We say that R-*elimination holds in* $Der(\mathcal{H})$ *with respect to* $\precsim$ if and only if

$$(\forall\,\mathcal{D}\in Der(\mathcal{H}))\,(\exists\,\mathcal{D}'\in Der(\mathcal{H}-R))\,\left[\,\mathcal{D}'\precsim\mathcal{D}\,\right],$$

i.e. iff every derivation $\mathcal{D}$ of $\mathcal{H}$ can be mimicked by a derivation $\mathcal{D}'$ of $\mathcal{H}-R$.

We say that R-*elimination holds in* $Der(\mathcal{H})$ *with respect to* $\simeq^{(s)}$, and that R-*elimination holds in* $Der(\mathcal{H})$ *with respect to* $\simeq^{(m)}$ if and only if respectively (5) and (6) are the case:

$$(\forall\,\mathcal{D}\in Der(\mathcal{H}))\,(\exists\,\mathcal{D}'\in Der(\mathcal{H}-R))\,\left[\,\mathcal{D}'\simeq^{(s)}\mathcal{D}\,\right], \tag{5}$$

$$(\forall\,\mathcal{D}\in Der(\mathcal{H}))\,(\exists\,\mathcal{D}'\in Der(\mathcal{H}-R))\,\left[\,\mathcal{D}'\simeq^{(m)}\mathcal{D}\,\right] \tag{6}$$

How do these notions of rule elimination relate to rule derivability and admissibility?

Theorem. *Let $\mathcal{H}$ be an AHS or an n-AHS, and let R be a (named) rule of $\mathcal{H}$. Then the following statements hold:*

$$R\text{-elimination holds in } \mathcal{H} \iff R \text{ is admissible in } \mathcal{H}{-}R\,,$$

$$\left.\begin{array}{r}R\text{-elimination holds}\\ \text{in } Der(\mathcal{H}) \text{ w.r.t. } \precsim\end{array}\right\} \iff R \text{ is derivable in } \mathcal{H}{-}R\,,$$

$$\left.\begin{array}{r}R\text{-elimination holds}\\ \text{in } Der(\mathcal{H}) \text{ w.r.t. } \simeq^{(s)}\end{array}\right\} \Longrightarrow R \text{ is s-derivable in } \mathcal{H}{-}R\,,$$

$$\left.\begin{array}{r}R\text{-elimination holds}\\ \text{in } Der(\mathcal{H}) \text{ w.r.t. } \simeq^{(m)}\end{array}\right\} \iff R \text{ is m-derivable in } \mathcal{H}{-}R\,.$$

Effective rule elim. by “mimicking steps” in n-AHS’s

Let $\mathcal{H}$ be an n-AHS, and let R be a named rule of $\mathcal{H}$.

A *mimicking step for R-elimination* in $\mathcal{H}$ is a transition of the form

$$\phi \;:\; \dfrac{\begin{matrix}\mathcal{D}_1 & & \mathcal{D}_n \\ A_1 & \ldots & A_n\end{matrix}}{\begin{matrix}(A) \\ \mathcal{D}_0\end{matrix}}\,\textsf{name}(R) \quad \xrightarrow{(R)}_{\textsf{mim}} \quad \begin{matrix}\mathcal{D}_{i_1} & & \mathcal{D}_{i_k} \\ (A_{i_1}) & \ldots & (A_{i_k}) \\ & \mathcal{D}_\alpha & \\ & (A) & \\ & \mathcal{D}_0 & \end{matrix}$$

where the derivation $\mathcal{D}_\alpha \in Der(\mathcal{H}{-}R)$ mimics the application α of R displayed in the left derivation.

Observation: If R is derivable in $\mathcal{H}{-}R$, then each R-application in an $\mathcal{H}$-derivation can be eliminated by a mimicking step.

ARS's of rule elimination by mimicking steps

Let again $\mathcal{H}$ be an n-AHS and R a named rule of $\mathcal{H}$.

The described kind of steps give rise to the *ARS* $\to^{(R)}_{mim}(\mathcal{H})$ *of R-elimination on $Der(\mathcal{H})$ by mimicking steps*

$$\to^{(R)}_{\mathsf{mim}}(\mathcal{H}) = \langle Der(\mathcal{H}), \Phi^{(R)}_{\mathsf{mim}}(\mathcal{H}), \mathsf{src}, \mathsf{tgt} \rangle ,$$

where $\Phi^{(R)}_{\mathsf{mim}}(\mathcal{H})$ the set of mimicking steps for R-elimination on $Der(\mathcal{H})$, and src and tgt the source and target functions on $\Phi^{(R)}_{\mathsf{mim}}(\mathcal{H})$.

Effective rule elim. by s- and m-mimicking steps

Let $\mathcal{H}$ be an n-AHS and R a named rule of $\mathcal{H}$. We define similarly:

- *s-mimicking steps for R-elimination in $\mathcal{H}$* replace R-applications in $\mathcal{H}$-derivations by s-mimicking derivations.
- *m-mimicking steps for R-elimination in $\mathcal{H}$* replace R-applications in $\mathcal{H}$-derivations by m-mimicking derivations.

Analogously as before, these notions give rise to

$$\to^{(R)}_{\text{s-mim}}(\mathcal{H}) \quad \text{and} \quad \to^{(R)}_{\text{m-mim}}(\mathcal{H}) \ ,$$

the *ARS of R-elimination on $Der(\mathcal{H})$ by s-mimicking steps*, and the *ARS of R-elimination on $Der(\mathcal{H})$ by m-mimicking steps*.

Weak normalization of rule elimination by mimicking steps

For an ARS $\rightarrow$ we denote by $\mathcal{NF}(\rightarrow)$ the set of its *normal forms*.

Lemma. *Let* $\mathcal{H}$ *be an* n*-AHS. Let* R *be a named rule of* $\mathcal{H}$ *that is derivable in* $\mathcal{H}-R$.

(i) $\mathcal{NF}(\rightarrow^{(R)}_{mim}(\mathcal{H})) = Der(\mathcal{H}-R)$,
i.e. a derivation of $\mathcal{H}$ *is a normal form of* $\rightarrow^{(R)}_{mim}(\mathcal{H})$ *if and only if it does not contain applications of* R.

(ii) $\rightarrow^{(R)}_{mim}(\mathcal{H})$ *is weakly normalizing.*

Analogous statements hold for $\rightarrow^{(R)}_{\text{s-mim}}(\mathcal{H})$ and $\rightarrow^{(R)}_{\text{m-mim}}(\mathcal{H})$.

Correctness of rule elim. by (s-,m-)mimicking steps

Theorem. *Let $\mathcal{H}$ be an n-AHS and R be a named rule of $\mathcal{H}$. Then it holds:*

(i) R-elim. by mimicking steps in $Der(\mathcal{H})$ is correct w.r.t. $\precsim$:

$$(\forall \mathcal{D}, \mathcal{D}' \in Der(\mathcal{H}))$$

$$(\exists\, \phi)\left[\, \phi : \mathcal{D} \xrightarrow[mim]{*}{}^{(R)} \mathcal{D}' \;\;\&\;\; \mathcal{D}' \in Der(\mathcal{H}-R) \,\right] \;\Longrightarrow\; \mathcal{D}' \precsim \mathcal{D}\,.$$

(ii) R-elimination in $Der(\mathcal{H})$ by s-mim. steps is correct w.r.t. $\precsim$; but it is not in general also correct w.r.t. $\simeq^{(s)}$.

(iii) R-elimination in $Der(\mathcal{H})$ by m-mim. steps is correct w.r.t. $\simeq^{(m)}$.

Termination of rule elimination by mimicking steps

Lemma. *Let $\mathcal{H}$ be an n-AHS, and let R be a named rule of $\mathcal{H}$.*

(i) If R is derivable in $\mathcal{H}{-}R$, then the ARS $\to^{(R)}_{mim}(\mathcal{H})$ is strongly normalizing.

(ii) If R is s-derivable in $\mathcal{H}{-}R$, then the ARS $\to^{(R)}_{s\text{-}mim}(\mathcal{H})$ is strongly normalizing.

(iii) If R is m-derivable in $\mathcal{H}{-}R$, then the ARS $\to^{(R)}_{m\text{-}mim}(\mathcal{H})$ is strongly normalizing.

Proof: Reducing the termination problem of these ARS's to a multiset-ordening. ($\sim$: Colonies of amoebae have a finite life-span).

Strong rule elimination by (s-, m-) mimicking steps

Definition. Let $\mathcal{H}$ be an n-AHS and let R be a named rule of $\mathcal{H}$.

Strong R-elimination by mimicking steps holds in $Der(\mathcal{H})$ iff

$$\mathrm{SN}(\rightarrow^{(R)}_{\mathrm{mim}}(\mathcal{H}))\text{, i.e. } \rightarrow^{(R)}_{\mathrm{mim}}(\mathcal{H}) \text{ is strongly normalizing,}$$
$$\text{and } \mathcal{NF}(\rightarrow^{(R)}_{\mathrm{mim}}(\mathcal{H})) = Der(\mathcal{H}-R)\,.$$

And similarly, we say that *strong R-elimination by s-mimicking steps holds in* $Der(\mathcal{H})$, and that *strong R-elimination by m-mimicking steps holds in* $Der(\mathcal{H})$ iff respectively (7) and (8) holds:

$$\mathrm{SN}(\rightarrow^{(R)}_{\mathrm{s\text{-}mim}}(\mathcal{H}))\text{, and } \mathcal{NF}(\rightarrow^{(R)}_{\mathrm{s\text{-}mim}}(\mathcal{H})) = Der(\mathcal{H}-R)\,, \quad (7)$$

$$\mathrm{SN}(\rightarrow^{(R)}_{\mathrm{m\text{-}mim}}(\mathcal{H}))\text{, and } \mathcal{NF}(\rightarrow^{(R)}_{\mathrm{m\text{-}mim}}(\mathcal{H})) = Der(\mathcal{H}-R)\,. \quad (8)$$

How do these notions of strong rule elimination relate to rule derivability and admissibility?

Theorem. *Let* $\mathcal{H}$ *be an* n*-AHS and let* R *be a named rule of* $\mathcal{H}$*.*

Then the following three logical equivalences hold:

$$\begin{aligned}
&\textit{Strong } R\textit{-elimination by mimicking steps holds in } Der(\mathcal{H}) \\
&\quad\Longleftrightarrow \quad R \textit{ is derivable in } \mathcal{H}{-}R\,, \\
&\textit{strong } R\textit{-elimination by s-mimicking steps holds in } Der(\mathcal{H}) \\
&\quad\Longleftrightarrow \quad R \textit{ is s-derivable in } \mathcal{H}{-}R\,, \\
&\textit{strong } R\textit{-elimination by m-mimicking steps holds in } Der(\mathcal{H}) \\
&\quad\Longleftrightarrow \quad R \textit{ is m-derivable in } \mathcal{H}{-}R\,.
\end{aligned}$$

How do the notions of strong rule elimination relate to the notions of rule elimination?

Corollary. *Let $\mathcal{H}$ be an n-AHS and let R be a named rule of $\mathcal{H}$.*

Then the following three statements hold:

$$\begin{aligned}
&\textit{Strong } R\textit{-elimination by mimicking steps holds in } Der(\mathcal{H}) \\
&\quad\Longleftrightarrow \quad R\textit{-elimination holds in } Der(\mathcal{H}) \textit{ w.r.t. } \precsim \;, \\
&\textit{strong } R\textit{-elimination by s-mimicking steps holds in } Der(\mathcal{H}) \\
&\quad\Longleftarrow \quad R\textit{-elimination holds in } Der(\mathcal{H}) \textit{ w.r.t. } \simeq^{(s)} \;, \\
&\textit{strong } R\textit{-elimination by m-mimicking steps holds in } Der(\mathcal{H}) \\
&\quad\Longleftrightarrow \quad R\textit{-elimination holds in } Der(\mathcal{H}) \textit{ w.r.t. } \simeq^{(m)} \;.
\end{aligned}$$

Sequent-style Hilbert systems à la Avron

Definition. A *Hilbert system for consequence* (a HSC) $\mathcal{HC}$ in the language L is an axiomatic system such that:

1. The *formulas* of $\mathcal{HC}$ are sequents in L, i.e. expressions $\Gamma \Rightarrow \Delta$ with Γ, Δ *multisets* of wff in L.
2. – The *axioms* of $\mathcal{HC}$ include $A \Rightarrow A$ for all A.
   – All other axioms of $\mathcal{HC}$ are of the form $\Rightarrow A$.
3. Every *rule* R of $\mathcal{HC}$ is an n*-premise rule* for some $n \in \omega$.
4. With the possible exception of the *structural rules* weakening and contraction and of the *cut rule*, all rules of $\mathcal{HC}$ fulfill the *left-hand side property*.

Left-hand side property of HSC-rules

The *set* of formulas that appear on the left-hand side of the conclusion of a rule is the union of the *sets* of formulas that appear on the left-hand side of the premises.

– An n-*premise rule* (where $n \in \omega \backslash \{0\}$) in a HSC has the *left-hand side property* if and only if for all its applications of the form

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \ldots \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta} \quad \text{holds:} \quad \mathsf{set}(\Gamma) = \textstyle\bigcup_{i=1}^{n} \mathsf{set}(\Gamma_i) \; .$$

– A *zero-premise rule* of $\mathcal{HC}$ fulfills the *left-hand side property* if and only if all of its applications are of the form

$$\frac{}{\Rightarrow \Delta} \quad .$$

Pure Rules in HSC's

Definition. Let $\mathcal{HC}$ be a HSC with language L, and R a rule of $\mathcal{HC}$.

The rule R is called *pure* if and only if the following holds: Whenever, for some $n \in \omega \setminus \{0\}$,

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$$

is an application of R, then

$$\Gamma = \Gamma_1 \dots \Gamma_n$$

holds, and for all multisets $\Gamma'_1, \dots, \Gamma'_n$ of formulas in L, also

$$\frac{\Gamma'_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma'_n \Rightarrow \Delta_n}{\Gamma'_1 \dots \Gamma'_n \Rightarrow \Delta}$$

is an application of R (hence zero-premise rules are pure trivially).

Structural Rules and Cut for HSC's

Weakening and contraction rules:

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\,\mathsf{Weak}_l \qquad \left(\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\,\mathsf{Weak}_r \right)$$

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\,\mathsf{Contr}_l \qquad \left(\frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\,\mathsf{Contr}_r \right)$$

Cut rule:

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma\Gamma' \Rightarrow \Delta\Delta'}\,\mathsf{Cut}$$

Cut-elimination in pure, single-conclusioned HSC's

Proposition. *Cut-elimination holds in every pure, single-conclusioned Hilbert system for consequence* $\mathcal{HC}$, *that is, for all sequents* $\Gamma \Rightarrow A$ *in* $\mathcal{HC}$ *it holds:*

$$\vdash_{\mathcal{HC}} \Gamma \Rightarrow A \quad \Longleftrightarrow \quad \vdash_{\mathcal{HC}-\mathsf{Cut}} \Gamma \Rightarrow A \ .$$

Moreover: Every derivation $\mathcal{D}$ *in* $\mathcal{HC}$ *can effectively be transformed into a cut-free derivation* $\mathcal{D}'$ *in* $\mathcal{HC}$ *with the same conclusion.*

Correspondence between AHS's and HSC's

Theorem. *For every AHS $\mathcal{H}$ there exists a pure, single-conclusioned HSC $\mathcal{HC}(\mathcal{H})$* without structural rules *such that for*[1] *all $A \in Fo_{\mathcal{H}}$ and $\Gamma \in \mathcal{M}_f(Fo_{\mathcal{H}})$ and $\Sigma \in \mathcal{P}_f(Fo_{\mathcal{H}})$ the following assertions hold:*

$$\Gamma \vdash_{\mathcal{H}}^{(\mathrm{m})} A \quad \Longleftrightarrow \quad \vdash_{\mathcal{HC}(\mathcal{H})} \Gamma \Rightarrow A \,,$$

$$\Gamma \vdash_{\mathcal{H}}^{(\mathrm{mw})} A \quad \Longleftrightarrow \quad \vdash_{\mathcal{HC}(\mathcal{H})+Weak} \Gamma \Rightarrow A \,,$$

$$\Sigma \vdash_{\mathcal{H}}^{(\mathrm{s})} A \quad \Longleftrightarrow \quad \vdash_{\mathcal{HC}(\mathcal{H})+Contr} \mathsf{mset}(\Sigma) \Rightarrow A \,,$$

$$\Sigma \vdash_{\mathcal{H}} A \quad \Longleftrightarrow \quad \vdash_{\mathcal{HC}(\mathcal{H})+Weak+Contr} \mathsf{mset}(\Sigma) \Rightarrow A \,.$$

[1] $Fo_{\mathcal{H}}$ is the set of formulas of $\mathcal{H}$.

Summary

We have introduced / we have found:

2
- Abstract Hilbert Systems (AHS's), and
- Abstract Hilbert Systems with rule/axiom names (n-AHS's).
- Three consequence relations on these systems.

3
- Definition of rule admissibility in (n-)AHS's.
- Definition of *three versions* of rule derivability in (n-)AHS's (derivability, s- and m-derivability).
- Some basic facts about these notions. A theorem that characterizes derivability of a rule R in an AHS $\mathcal{H}$ by admissibility of R in extensions of $\mathcal{H}$.

4
- (Mutual) inclusion relations $[2 \times 12$ relations].
- Two Interrelation Prisms between these relations.
- As a corollary: alternative characterizations of rule admissibility and rule (m-)derivability.

5
- Three notions of mimicking derivation between derivations in an AHS or n-AHS.
- Four notions of rule elimination in AHS's and n-AHS's. Correspondences with rule admissibility and (s-,m-)derivability.
- Three notions of strong rule elimination in n-AHS's, and their correspondences with the three notions of rule derivability.

E (*Appendix E*) A close relationship of (n)-AHS's with sequent-style *Hilbert systems for consequence* à la Avron.

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