# Graph Kernels, Logic, and Choice Axioms

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# Overview

- Kernels and solutions of digraphs
- Kernel existence and propositional logic
- Kernel existence and choice axioms
- Computational complexity of kernel existence
- Summary of results

# Overview

1. Kernels and solutions
2. Kernel existence and propositional logic
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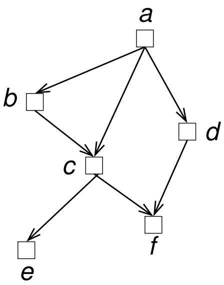
# Digraphs

A directed graph (digraph) $G = \langle V, \rightarrowtail \rangle$ consists of a set $V$ of vertices, and a set $\rightarrowtail \subseteq V \times V$ of directed edges. Notation for vertices $x$:

- $(x \rightarrowtail) := \{y \in V \mid x \rightarrowtail y\}$ set of successors of $x$
- $(\rightarrowtail x) := \{y \in V \mid y \rightarrowtail x\}$ set of predecessors of $x$
- extended to sets, e.g. $(\rightarrowtail X) := \bigcup_{x \in X} (\rightarrowtail x)$.




$$(a \rightarrowtail) = \{b, c, d\}$$

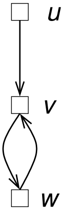
$$(\rightarrowtail \{d, f\}) = \{a, c, d\}$$

# Kernels

## Definition

A kernel of a digraph $G = \langle V, \rightarrowtail \rangle$ is a set $K \subseteq V$ such that:

1. $(K \rightarrowtail) \cap K = \emptyset$
   (no successor of a vertex in $K$ is in $K$);
2. $V \setminus K \subseteq (\rightarrowtail K)$
   (every vertex not in $K$ is the predecessor of a vertex in $K$).



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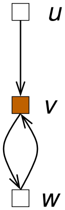



$K = \{v\}$ is a kernel $\qquad (K \rightarrowtail) = \{w\}$

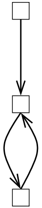
$V \setminus K = \{u, v\} \qquad (\rightarrowtail K) = \{u, v\}$

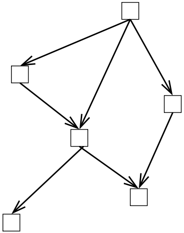
# Solutions

**Definition (von Neumann/Morgenstern, 1944)**

A solution of a digraph $G = \langle V, \rightarrowtail \rangle$ is an assignment $\alpha \in \{\mathbf{0}, \mathbf{1}\}^V$ of truth-values to the vertices such that:

$$\forall u \in V \left[\, \alpha(u) = \mathbf{1} \iff \forall v \in V \,(\, u \rightarrowtail v \;\Rightarrow\; \alpha(v) = \mathbf{0}\,)\,\right].$$





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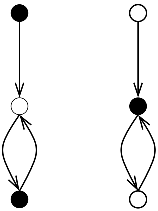

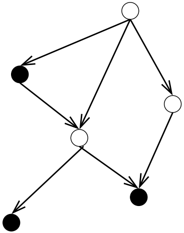

# Kernels versus solutions

For all assignments $\alpha \in \{\mathbf{0}, \mathbf{1}\}^V$, let $\alpha^{\mathbf{1}} := \{x \in V \mid \alpha(x) = \mathbf{1}\}$.

## Proposition

*For all assignments $\alpha$ on a digraph G:*

$$\alpha \textit{ is a solution of } G \iff \alpha^{\mathbf{1}} \textit{ is a kernel of } G.$$

## Proof.

1. $K \subseteq V$ is a kernel $\iff K = V \setminus (\rightarrowtail K)$;
2. $\alpha \in sol(G) \iff \alpha^{\mathbf{1}} = V \setminus (\rightarrowtail \alpha^{\mathbf{1}})$.

□

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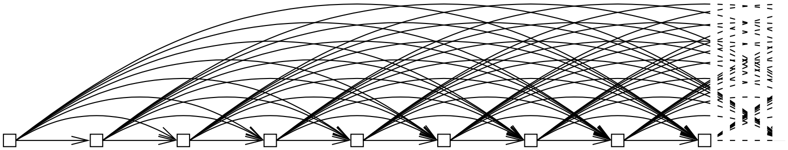
# Solvability: some results

- general digraphs
  - complete digraphs
  - fb digraphs without odd cycles (Richardson, 1953)
  - digraphs in which for all vertices $u$ and $v$, either all paths between them have even length, or all have odd length (W/Dyrkolbotn, 2009)
- dags (directed acyclic graphs)
  - finite
  - well-founded (von Neumann/Morgenstern, 1944)
  - fb (finitely branching)
  - trees (rooted or unrooted), forests

# Unsolvable dag

The infinitely-branching dag $\langle \mathbb{N}, < \rangle$ (Yablo dag) is unsolvable:



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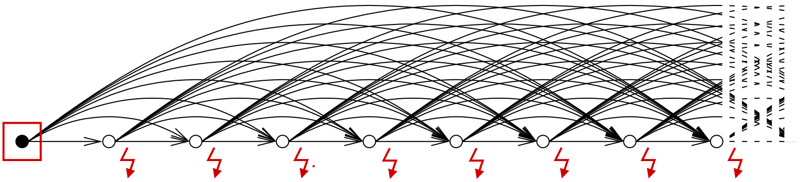

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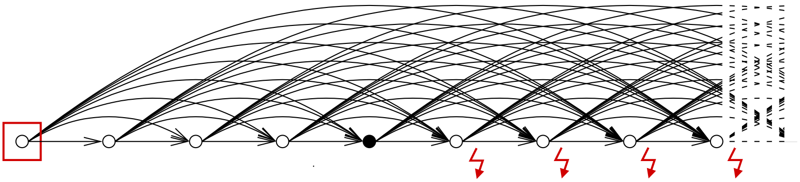
Case 2:



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# From digraphs to theories

Every digraph $G = \langle V, \rightarrowtail \rangle$ induces the (infinitary) propositional theory

$$\mathcal{T}(G) = \{x \leftrightarrow \bigwedge_{y \in (x \rightarrowtail)} \neg y \mid x \in V\}$$

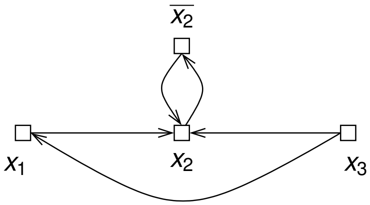
taking $(\bigwedge_{z \in \emptyset} z) := 1$. If $G$ is finitely-branching, then $\mathcal{T}(G)$ is finitary.

## Proposition

- *$\mathcal{T}(G)$ is consistent $\iff$ $G$ is solvable.*
- *Moreover: $sol(G) = mod(\mathcal{T}(G))$.*

# From theories to digraphs

Let $\mathsf{T}_1 = \{\, x_1 \leftrightarrow \neg\, x_2,\;\; x_3 \leftrightarrow \neg\, x_1 \wedge \neg\, x_2 \,\}$,




$\mathcal{G}(\mathsf{T}_1)$

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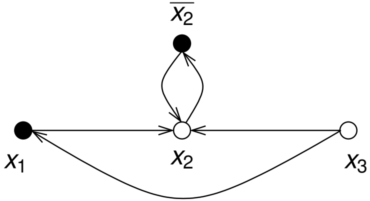


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solvable

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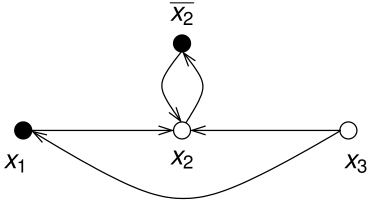
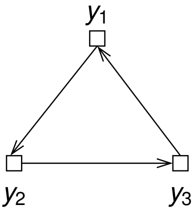
Let $\mathrm{T}_1 = \{\, x_1 \leftrightarrow \neg\, x_2,\;\; x_3 \leftrightarrow \neg\, x_1 \wedge \neg\, x_2 \,\}$,

$\mathrm{T}_2 = \{\, y_1 \leftrightarrow \neg\, y_2,\;\; y_2 \leftrightarrow \neg\, y_3,\;\; y_3 \leftrightarrow \neg\, y_1 \,\}$. Then:




$\mathcal{G}(\mathrm{T}_1)$

solvable

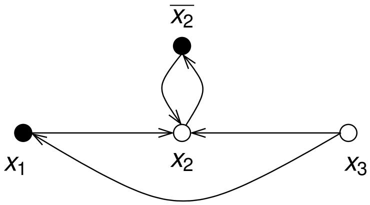



$\mathcal{G}(\mathrm{T}_2)$

# From theories to digraphs

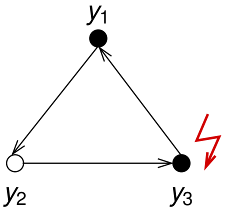
Let $T_1 = \{ x_1 \leftrightarrow \neg x_2,\ \ x_3 \leftrightarrow \neg x_1 \wedge \neg x_2 \}$,

$T_2 = \{ y_1 \leftrightarrow \neg y_2,\ \ y_2 \leftrightarrow \neg y_3,\ \ y_3 \leftrightarrow \neg y_1 \}$. Then:




$\mathcal{G}(T_1)$

solvable




$\mathcal{G}(T_2)$

unsolvable

# From theories to digraphs

Every finitary propositional theory (over var's $\mathbb{V}$) in normal form:

$$\mathrm{T} = \{\, x_i \leftrightarrow \bigwedge_{j\in J_i} \neg y_{ij} \,\big|\, i \in I \,\}$$

induces a digraph $\mathcal{G}(\mathrm{T}) = \langle V, \rightarrowtail \rangle$ with

$$V := \{z, \overline{z} \,\big|\, z \in \mathbb{V}, z \text{ not on the rhs of a formula in } \mathrm{T}\}$$

$$x \rightarrowtail y \quad :\Longleftrightarrow \quad \big(\, x \leftrightarrow \bigwedge_{j=1}^{n} \neg y_j \,\big) \in \mathrm{T} \;\;\&\;\; y \in \{y_1, \ldots, y_n\}$$

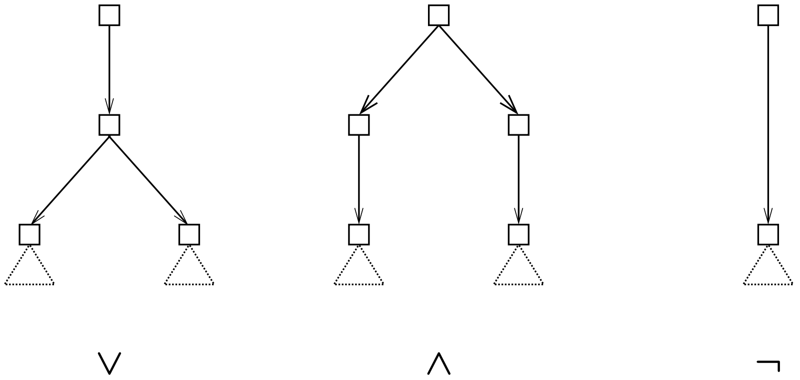
$$z \rightarrowtail \overline{z},\; \overline{z} \rightarrowtail z \;\; (z \text{ not on the rhs of a formula in } \mathrm{T})$$

**Proposition**

- *$\mathcal{G}(\mathrm{T})$ is solvable $\iff$ $\mathrm{T}$ is consistent.*
- *Moreover: $mod(\mathrm{T}) = sol(\mathcal{G}(\mathrm{T}))|_{\mathbb{V}(\mathrm{T})}$*

General theories can be brought into equiconsistent normal form by a simple procedure.

# Simulating connectives



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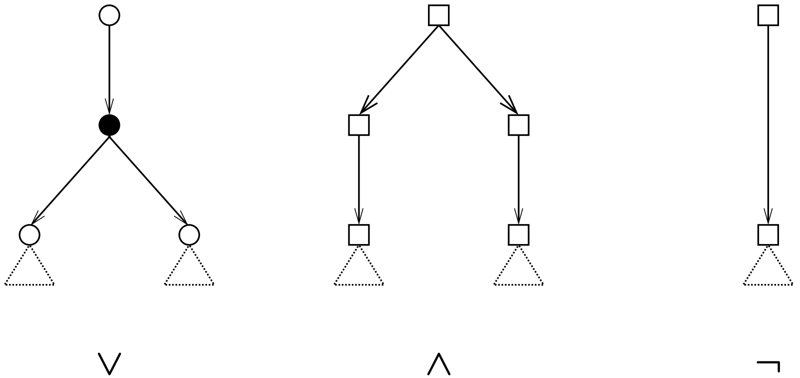

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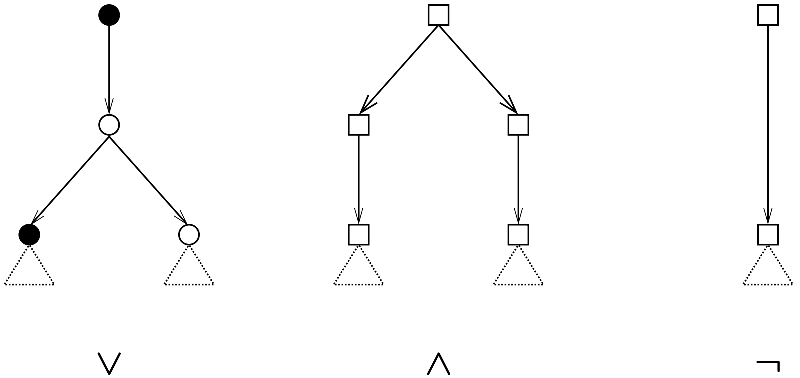

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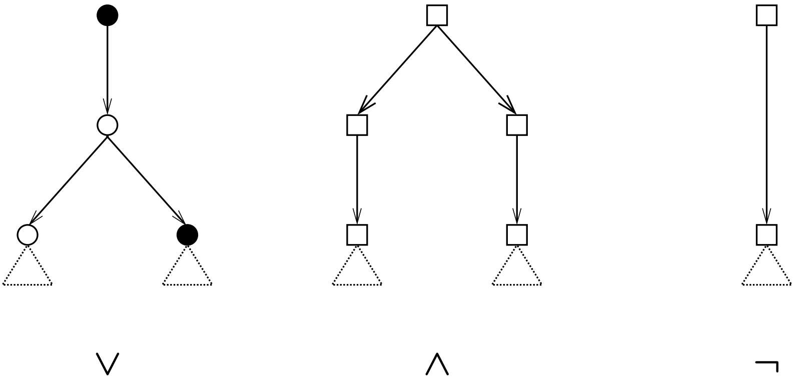

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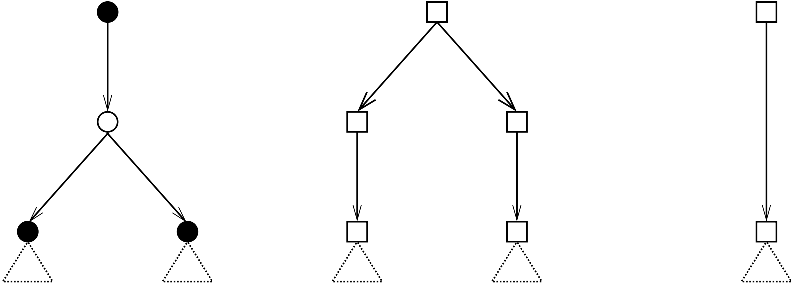

$\vee$ $\wedge$ $\neg$

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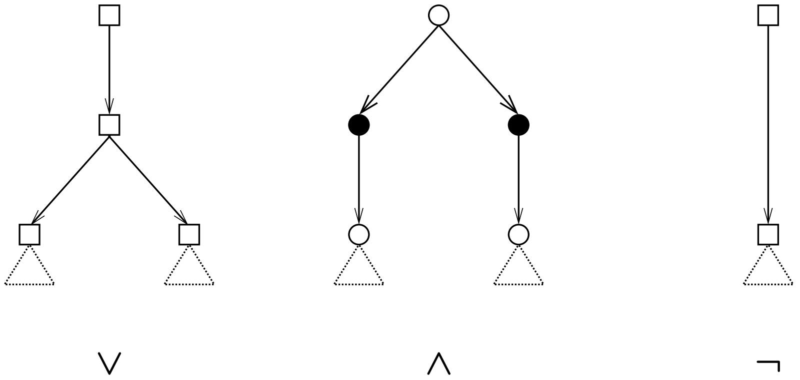

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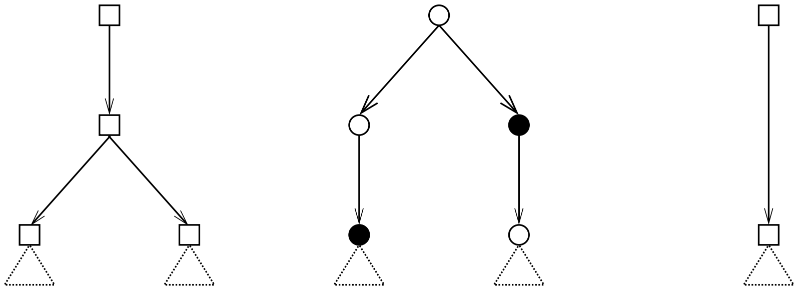

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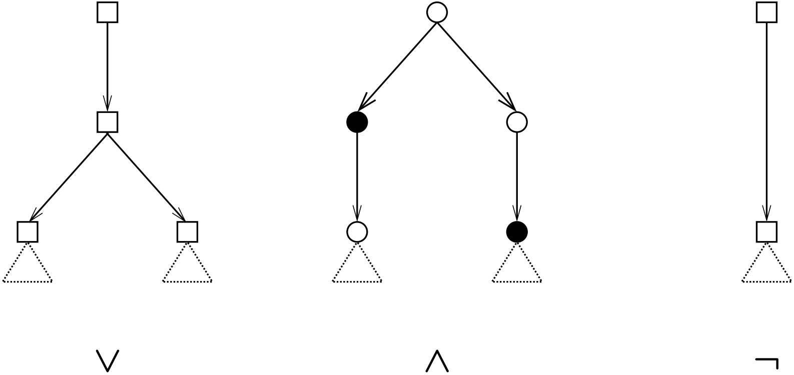

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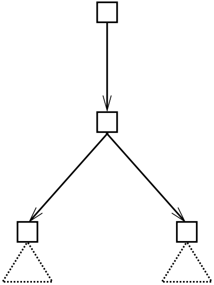

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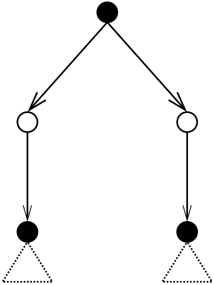

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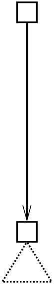

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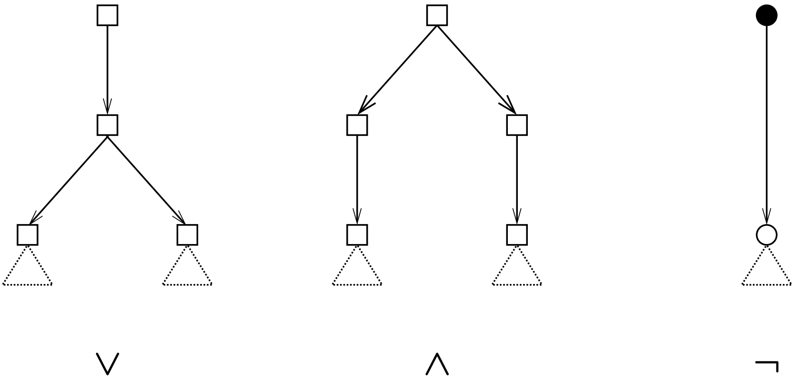

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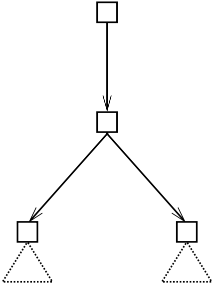

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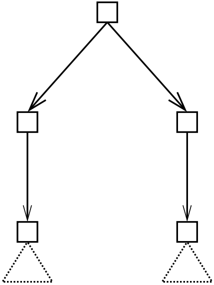

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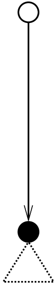

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# Overview

# Solvability and Choice Principles

## Proposition

*Solvability of fb dags follows from:*

- *in the general case:*
  - *compactness of propositional logic: every set of propositional formulas that is finitely satisfiable is satisfiable.*
- *for countable dags:*
  - *countable compactness,*
  - *Weak König's Lemma (WKL): Every infinite, ordered, and fb tree has an infinite path.*

- What about the converse implications?
- What choice principle corresponds precisely to solvability of a class of solvable digraphs?

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# Digraph Solvability over ZF

Our Results:

| digraph class $\mathcal{C}$ | additional principle needed for proving, and equivalent to, solvability of $\mathcal{C}$ over ZF |
|---|---|
| disjoint unions of solvable digraphs | AC |
| disjoint unions of solvable dags | |
| countable disjoint unions of solvable digraphs (solvable dags) | $AC_\omega$ |
| well-founded dags (e.g. finite dags); rooted trees; trees; forests of trees with roots or leafs | — |

# Digraph solvability and AC

**Theorem**

*Over ZF, the following are equivalent:*

*(AC): For every set $X$, there is a choice function on $X$.*

*(GS): Every disjoint union $\biguplus_{i \in I} G_i$ of solvable digraphs $G_i$ is solvable.*

*Idea*: Consider solutions of complete digraphs:

Every solution of a complete digraphs chooses one of the vertices.

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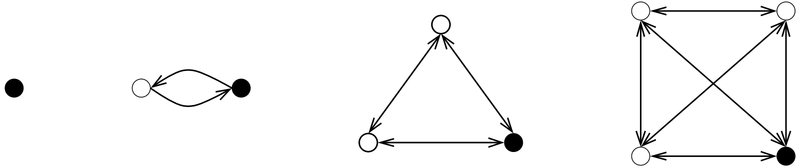

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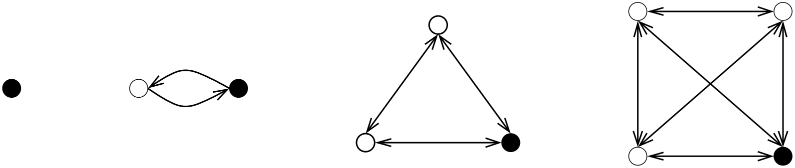

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*(DS): Every disjoint union $\biguplus_{i \in I} G_i$ of solvable dags $G_i$ is solvable.*

*Idea*: Consider a set $A = \{a, b\}$. Let $D(A)$ be the dag:

Solutions of $D(A)$ make a choice between $a$ and $b$.

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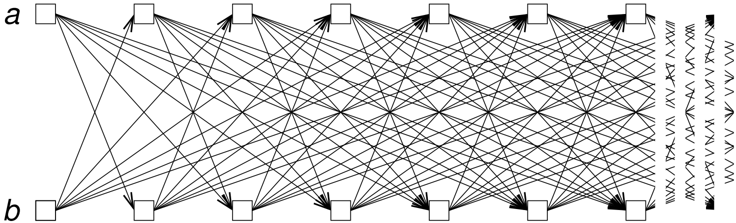


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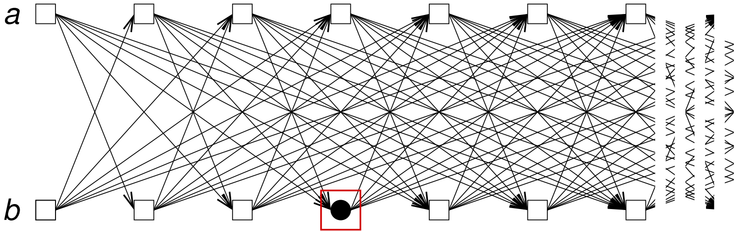


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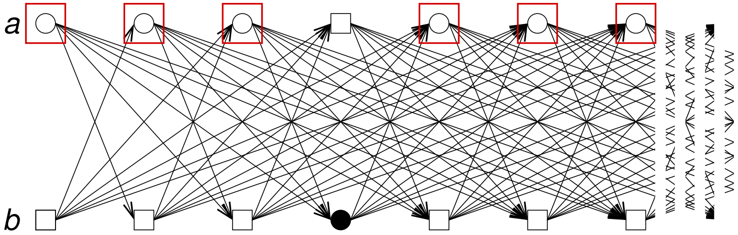


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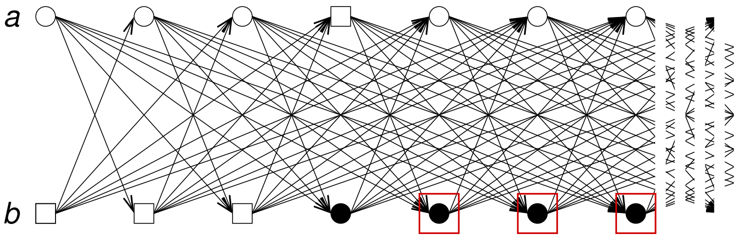


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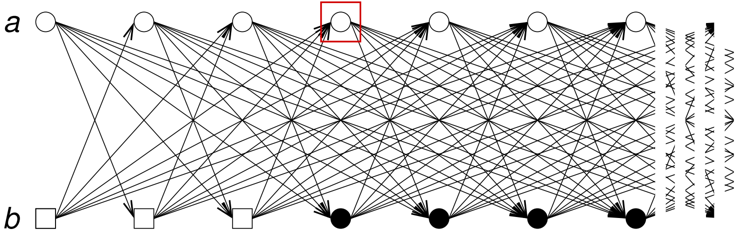


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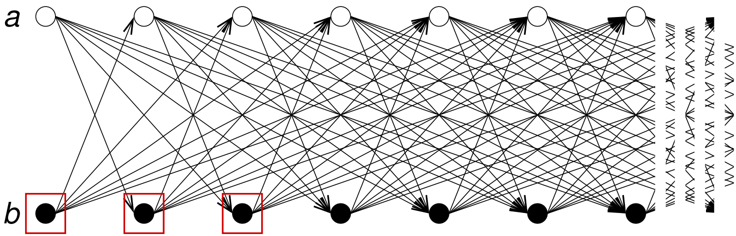


Solutions of $D(A)$ make a choice between $a$ and $b$.

# Dag solvability and AC

**Theorem**

*Over ZF, AC is also equivalent with:*

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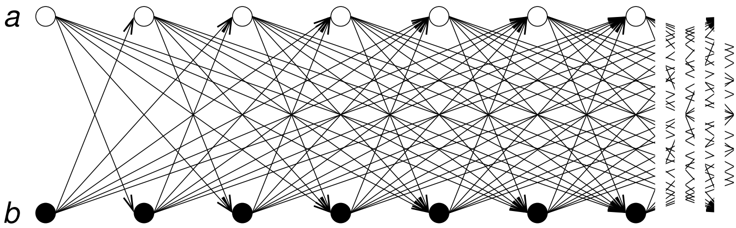


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# Digraph Solvability over $\mathrm{RCA}_0$

Our Results:

| digraph class $\mathcal{C}$ | additional principle needed for proving, and equivalent to, solvability of $\mathcal{C}$ over $\mathrm{RCA}_0$ |
|---|---|
| disjoint unions of solvable digraphs | AC |
| disjoint unions of solvable dags | |
| countable disjoint unions of solvable digraphs (solvable dags) | $\mathrm{AC}_\omega$ |
| countable fb dags | WKL |
| well-founded dags (e.g. finite dags); rooted trees; trees; forests of trees with roots or leafs | — |

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# Digraph Solvability over $\mathrm{RCA}_0$

**Theorem**

*Solvability of countable fb dags is, over $\mathrm{RCA}_0$, equivalent to:*

- *countable compactness: every countable set of propositional formulas that is finitely satisfiable is satisfiable.*

Since, over $\mathrm{RCA}_0$, countable compactness is equivalent to $\mathrm{WKL}$:

**Corollary**

*Solvability of countable fb dags is, over $\mathrm{RCA}_0$, equivalent to:*

- *WKL: Every infinite, ordered, and fb tree has an infinite path.*

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### Corollary

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# Overview

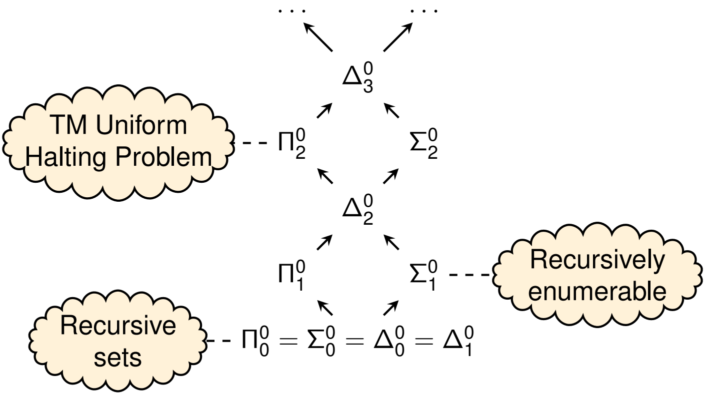
# Complexity of kernel/solution existence?

- is recursive: for classes of solvable digraphs (trivial).
- is NP-complete: for finite digraphs (Chvátal, 1973)
- is precisely what for classes including non-fb dags?

DAG-SOLVABILITY PROBLEM DSP
Instance: $G = \langle \mathbb{N}, \rightarrowtail \rangle$ a recursive dag
Answer: Is $G$ solvable?
Recognition problem: $\{\ulcorner G \urcorner : G \text{ is a recursive dag that is solvable}\}$

Where does DSP figure in the arithmetical hierarchy?

# The arithmetical hierarchy

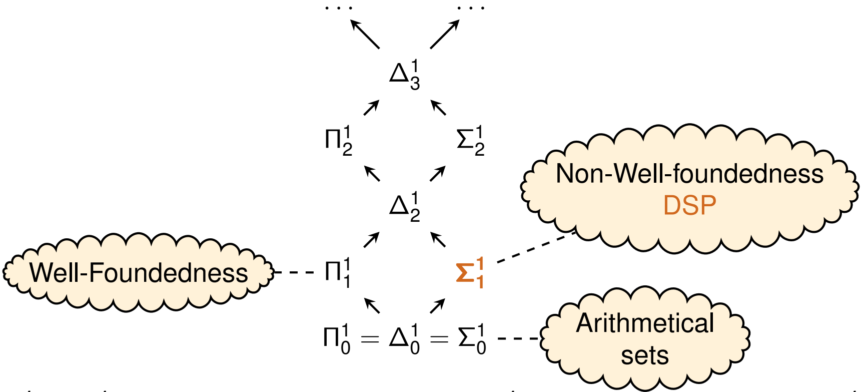



$\mathbf{\Pi_0^0} := \mathbf{\Sigma_0^0} :=$ 1st-order arithmetic formulas with bounded quantifiers

$\mathbf{\Sigma_{n+1}^0} := \{\exists x_1 \ldots \exists x_k \Psi \mid \Psi \in \mathbf{\Pi_n^0}\}$
$\mathbf{\Pi_{n+1}^0} := \{\forall x_1 \ldots \forall x_k \Psi \mid \Psi \in \mathbf{\Sigma_n^0}\}$

$\Sigma_n^0(\Pi_n^0) :=$ interpretations of formulas in $\mathbf{\Sigma_n^0}(\mathbf{\Pi_n^0})$ over $\mathbb{N}$ $\quad \Delta_n^0 := \Sigma_n^0 \cap \Pi_n^0$

# The analytical hierarchy




$\mathbf{\Pi^1_0} := \mathbf{\Sigma^1_0} :=$ 2nd-order arithm. formulas without set quantifiers

$\mathbf{\Sigma^1_{n+1}} := \{\exists X_1 \ldots \exists X_k \Psi \mid \Psi \in \mathbf{\Pi^1_n}\}$

$\mathbf{\Pi^1_{n+1}} := \{\forall X_1 \ldots \forall X_k \Psi \mid \Psi \in \mathbf{\Sigma^1_n}\}$

$\Sigma^1_n(\Pi^1_n) :=$ interpretations of formulas in $\mathbf{\Sigma^1_n}(\mathbf{\Pi^1_n})$ over $\mathbb{N}$ $\qquad \Delta^1_n := \Sigma^1_n \cap \Pi^1_n$

## Theorem

DSP *is* $\Sigma^1_1$*-complete.*

## Proof.

- Contained in $\Sigma^1_1$:

  solvability is expressible by the $\boldsymbol{\Sigma^1_1}$-formula:

  $$\exists K \, \forall n \big[\, n \in K \;\leftrightarrow\; \forall n' (\, \mathit{EdgeBetweenIn}(n, n', m) \;\to\; n' \notin K \,) \big]$$

- $\Sigma^1_1$-complete:

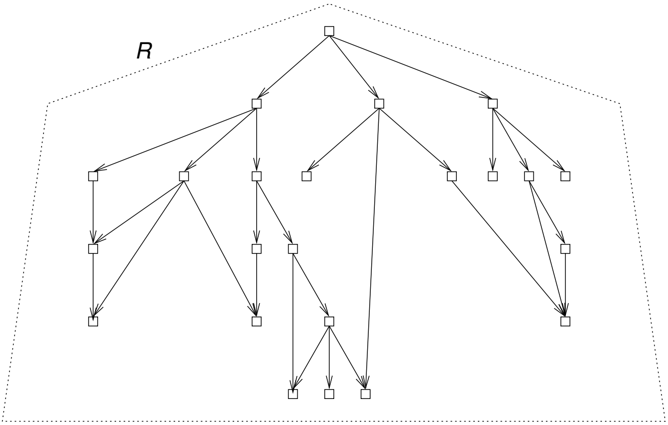
  Reducing the non-well-foundedness problem NWFP for binary recursive relations ($\Sigma^1_1$-complete!), to DSP via a recursive many-one reduction $D(\cdot)$:
  For every recursive binary rel. $R$ build a recursive dag $D(R)$ s.th.:

  $$D(R) \text{ is solvable} \iff R \text{ is not well-founded}$$

□

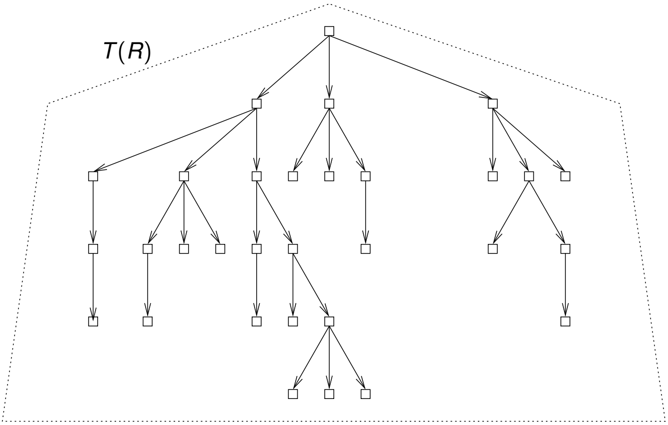
# Reducing NWFP to DSP (Case 1)

Case 1: $R$ well-founded.



# Reducing NWFP to DSP (Case 1)

Case 1: $R$ well-founded. Tree unfolding $T(R)$ well-founded.



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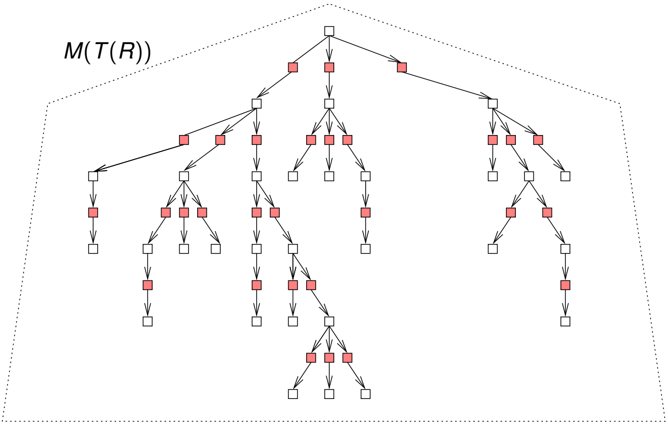
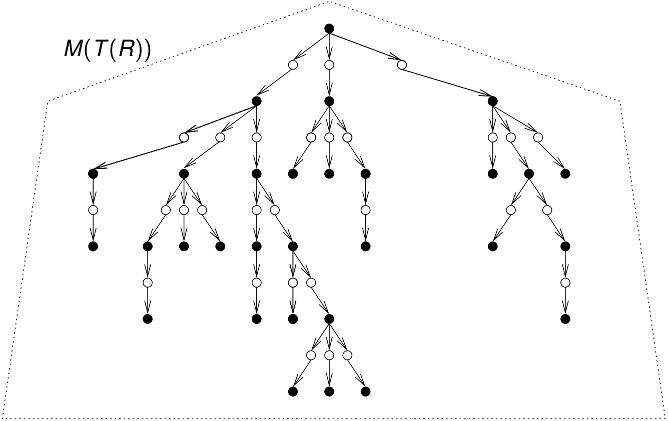


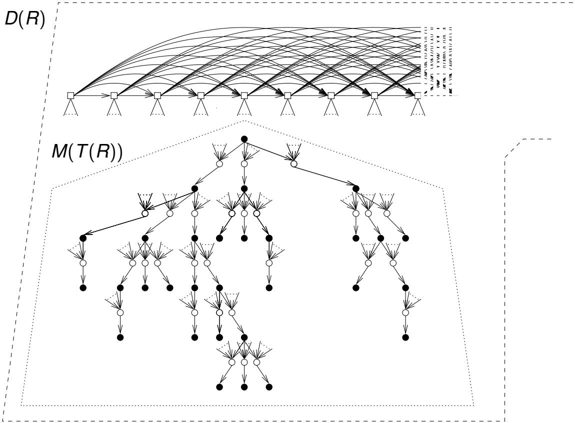
# Reducing NWFP to DSP (Case 1)

Case 1: $R$ well-founded. Modification $M(T(R))$ of $T(R)$ solvable.



# Reducing NWFP to DSP (Case 1)

Case 1: $R$ well-founded. Dag $D(R)$ associated with $R$:



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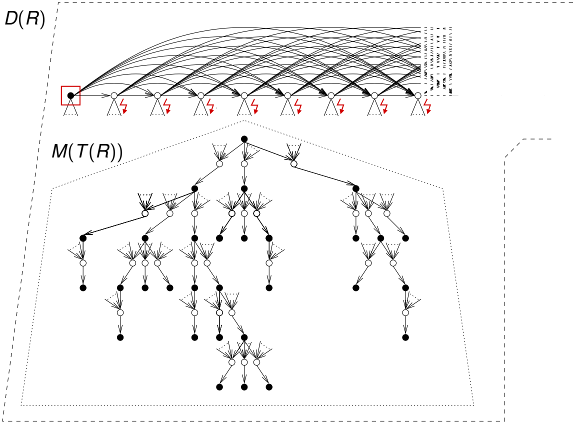

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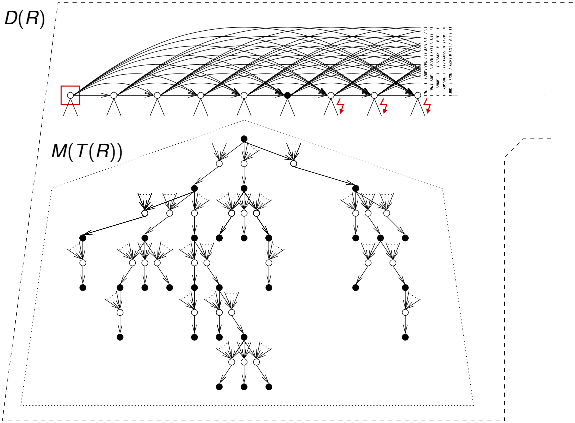
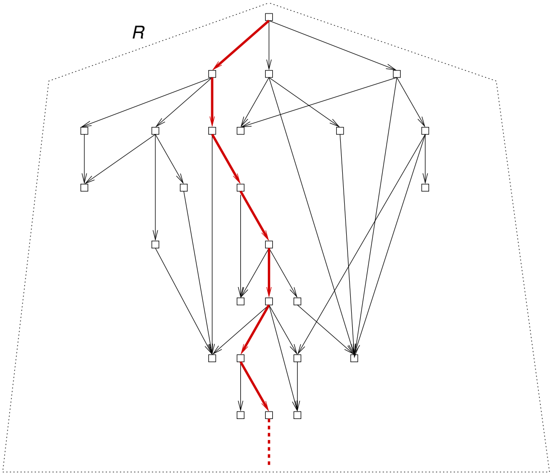


# Reducing NWFP to DSP (Case 2)

Case 2: not well-founded binary relation $R$.



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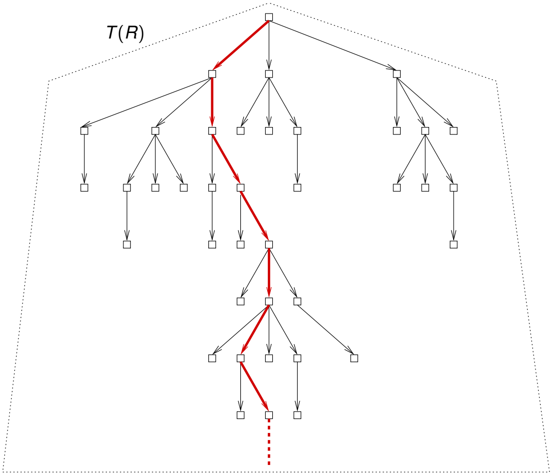

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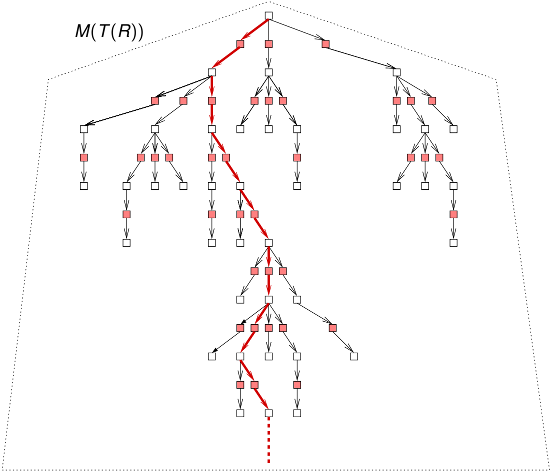

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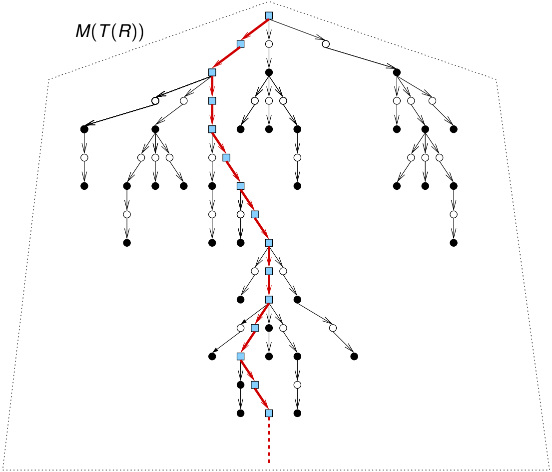

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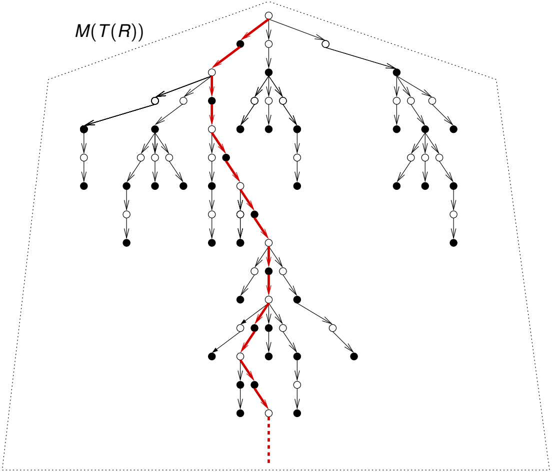
Case 2: $R$ not well-founded. Modification $M(T(R))$ of $T(R)$ solvable.



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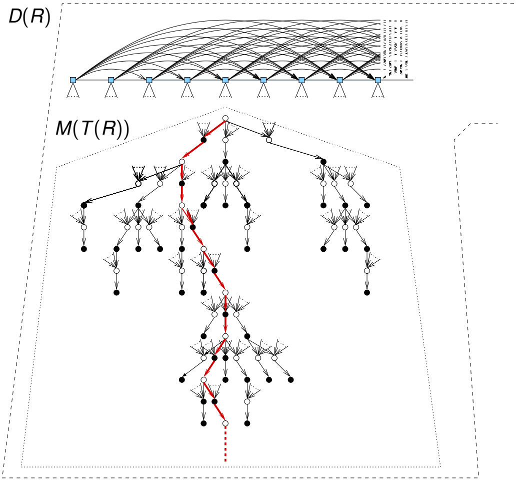

# Reducing NWFP to DSP (Case 2)

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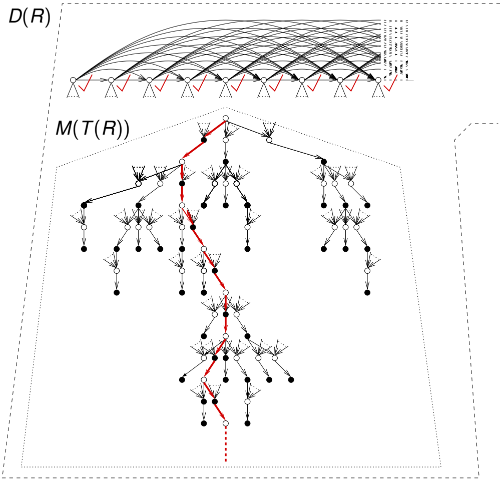

# Related result

- There exist recursive binary trees without recursive solutions.

# Overview

# Open questions

- Which choice principle corresponds, over $\mathrm{ZF}$:
  - to fb-dag solvability?
  - to forest solvability (forests possibly including unrooted trees)?

# Summary of results

- kernels and logic
  - back-and-forth correspondences between solvable digraphs and consistent propositional theories
- kernels and choice axioms
  - statements on digraph-/dag-solvability equivalent to $\mathrm{AC}$, and $\mathrm{AC}_\omega$, over $\mathrm{ZF}$
  - comparable statements over $\mathrm{RCA}_0$
  - **main result**: over $\mathrm{RCA}_0$, solvability of countable, fb dags is equivalent to countable compactness, and to $\mathrm{WKL}$
  - solvability of trees (rooted/unrooted) in ZF.
- computational complexity of kernel existence
  - $\Sigma^1_1$-completeness of dag-solvability (and of digraph-solvability)