Co-Reflexivity = Symmetry + Anti-Symmetry

Clemens Grabmayer and Dimitri Hendriks
Department of Computer Science, VU University Amsterdam
room T-4.29, W&N building
c.a.grabmayer@vu.nl r.d.a.hendriks@vu.nl

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Definition. Let R be a binary relation on a set A, that is, $R \subseteq A \times A$.

- The *inverse* or *converse relation* of R is $R^{-1} := \{\langle y, x\rangle : \langle x, y\rangle \in R\}$.
- The *equality relation* $=$ on A is denoted by $= := \{\langle x, x\rangle : x \in A\}$.

Definition. Let A be a set, and let R be a binary relation, and $=$ be the equality relation, on A. The well-known properties of reflexivity, symmetry, and anti-symmetry, and the less well-known property of co-reflexivity, are defined for R by the following stipulations:

- R is called *reflexive* if $= \subseteq R$.
- R is called *co-reflexive* if $R \subseteq =$.
- R is called *symmetric* if $R^{-1} \subseteq R$.
- R is called *anti-symmetric* if $R \cap R^{-1} \subseteq =$.

Theorem (room T-4.29)**.** *Let R be a binary relation on a set A. Then it holds:*

$$R \text{ is co-reflexive} \iff R \text{ is symmetric and anti-symmetric.}$$

Proof. For "$\Rightarrow$", suppose that R is co-reflexive. Then $R \subseteq =$ holds. Now it follows that R is symmetric, because it is a subrelation of the equality relation $=$. Furthermore it follows that $R^{-1} \subseteq =^{-1} = =$. This entails that $(R \cap R^{-1}) \subseteq (= \cap =) = =$. Hence R is also anti-symmetric.

For "$\Leftarrow$", suppose that R is symmetric and anti-symmetric. Then it holds that $R^{-1} \subseteq R$ and $R \cap R^{-1} \subseteq =$. By the first statement (symmetry of R), we find $R = (R^{-1})^{-1} \subseteq R^{-1} \subseteq R$, and hence that also $R^{-1} = R$ holds. Then by using this and the second statement (anti-symmetry of R), we conclude that $R = R \cap R = R \cap R^{-1} \subseteq =$ holds, which shows that R is co-reflexive. □

Corollary. *Let A be a set. The following two statements hold:*

(i) The equality relation $=$ on A is the largest binary relation on A that is both symmetric and anti-symmetric.

(ii) The equality relation $=$ on A is the only binary relation on A that is reflexive, symmetric, and anti-symmetric.

Proof. For (i), let R be an arbitrary binary relation on A that is symmetric and anti-symmetric. By the theorem it follows that $R \subseteq =$. Since $=$ is symmetric and anti-symmetric, it follows that $=$ is indeed the largest binary relation on A with these properties.

For (ii), suppose that R is a binary relation on A that is reflexive, symmetric, and anti-symmetric. Again by the theorem it follows that $R \subseteq =$. Since by reflexivity of R also $R \supseteq =$ holds, it follows that $R = =$. □