Expressibility in the Lambda Calculus with letrec

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NWO-project *Realising Optimal Sharing*
Utrecht University

TeReSe-Meeting
RWTH Aachen, 28 March 2012

Which infinite λ-terms are expressible finitely in λ_{letrec}?

Example

$$\text{letrec}\, f = \lambda xy.f\,y\,x \text{ in } f \quad \twoheadrightarrow_{\triangledown} \quad \lambda xy.\lambda xy.(\lambda xy.(\ldots y\,x)\,y\,x)\,y\,x$$

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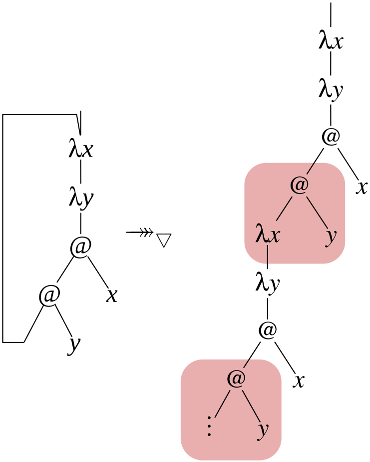

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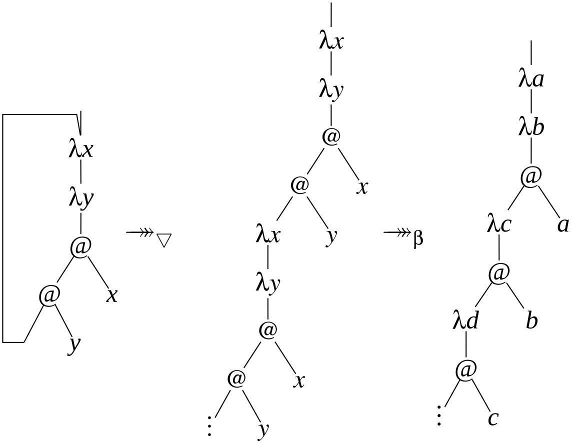

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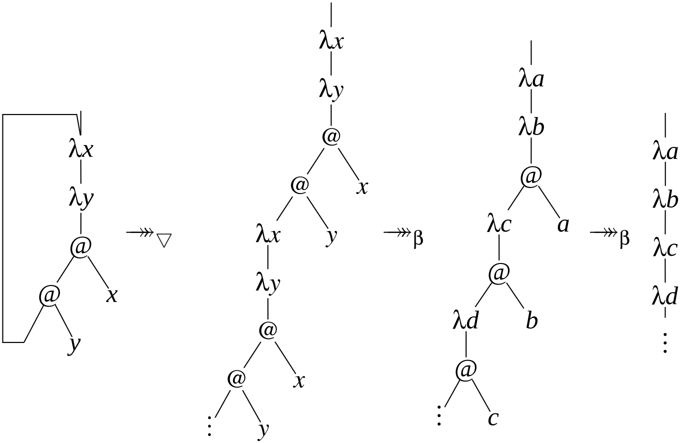

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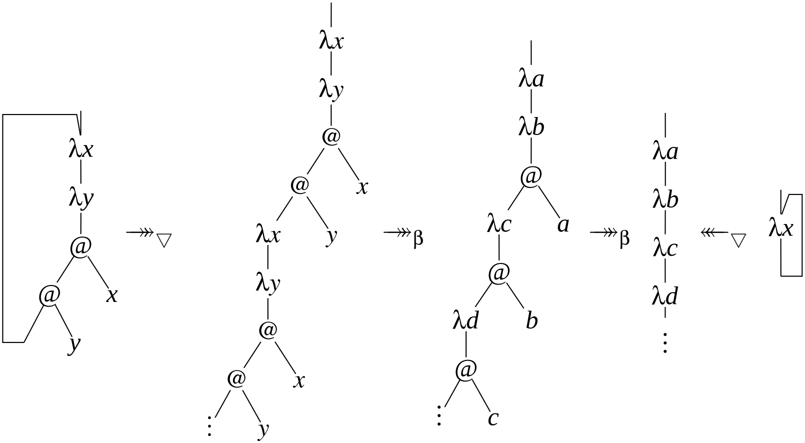

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What makes a λ-terms inexpressible in λ_{letrec}?

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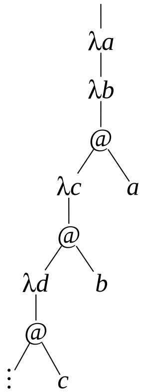

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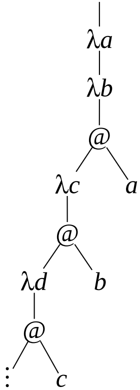

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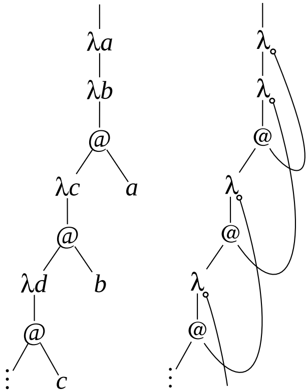

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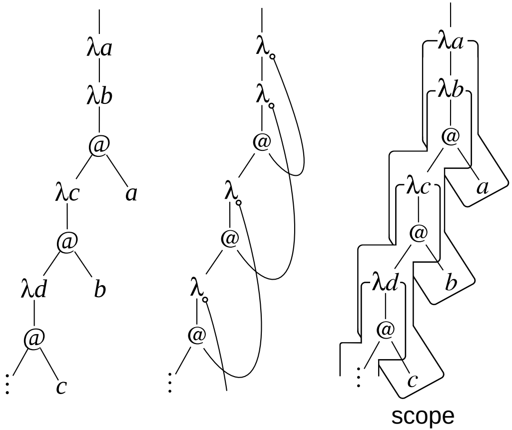

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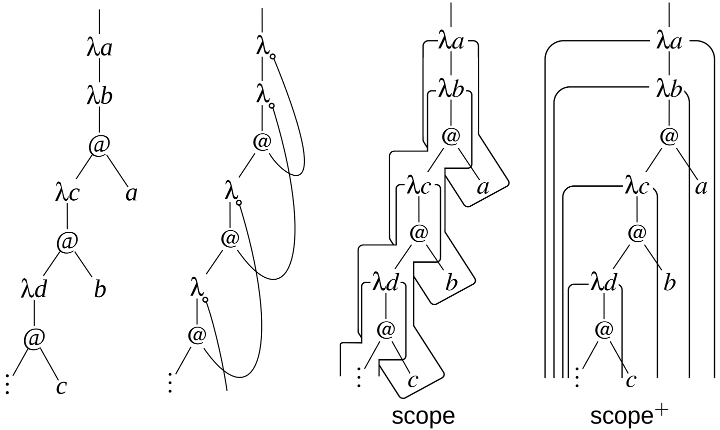

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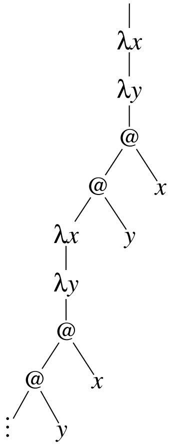

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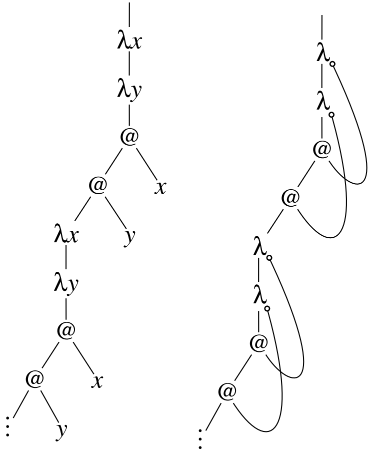

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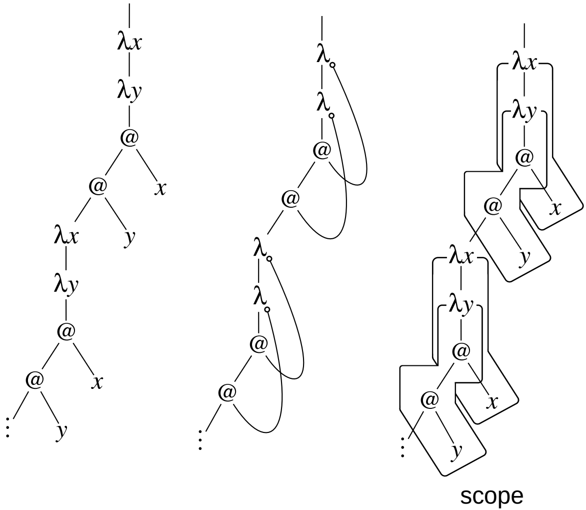

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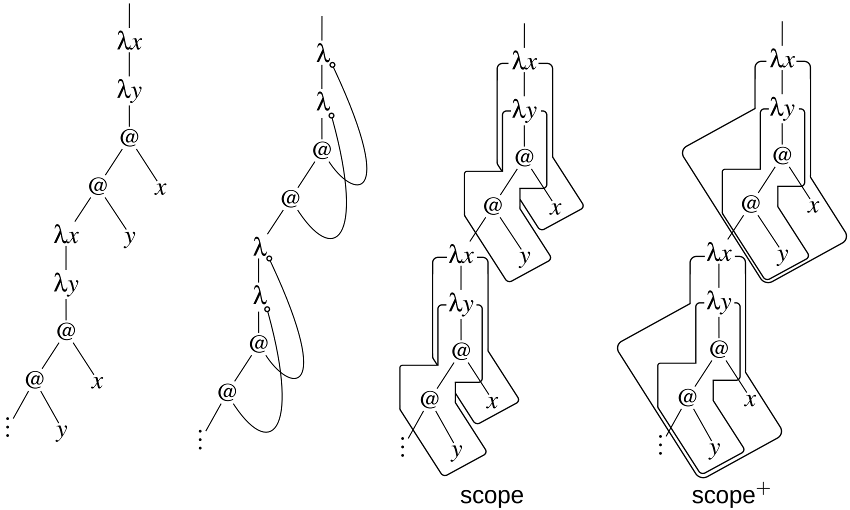

A CRS for observing infinite λ-terms: *Reg*$^{+}$

$$(\rho^{@_i}): \quad (\lambda x_1 \ldots x_n) M_0\, M_1 \to (\lambda x_1 \ldots x_n) M_i \qquad (i \in \{0,1\})$$

$$(\rho^{\lambda}): \quad (\lambda x_1 \ldots x_n) \lambda x_{n+1}.M_0 \to (\lambda x_1 \ldots x_{n+1}) M_0$$

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\end{aligned}
$$

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(\rho^{@_i}): &\quad \mathrm{pre}_n([x_1 \dots x_n]\,\mathrm{app}(Z_0(\vec{x}), Z_1(\vec{x}))) \to \mathrm{pre}_n([x_1 \dots x_n]\,Z_i(\vec{x})) &\quad (i \in \{0,1\})\\
(\rho^{\lambda}): &\quad \mathrm{pre}_n([x_1 \dots x_n]\,\mathrm{abs}([x_{n+1}]Z(\vec{x}))) \to \mathrm{pre}_n([x_1 \dots x_{n+1}]\,Z(\vec{x})) &\\
(\rho^{\mathsf{S}}): &\quad \mathrm{pre}_{n+1}([x_1 \dots x_{n+1}]\,Z(x_1, \dots, x_n)) \to \mathrm{pre}_n([x_1 \dots x_n]\,Z(x_1, \dots, x_n)) &
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$$(\rho^{@_i}):\quad (\lambda x_1 \ldots x_n) M_0\, M_1 \to (\lambda x_1 \ldots x_n) M_i \qquad (i \in \{0,1\})$$

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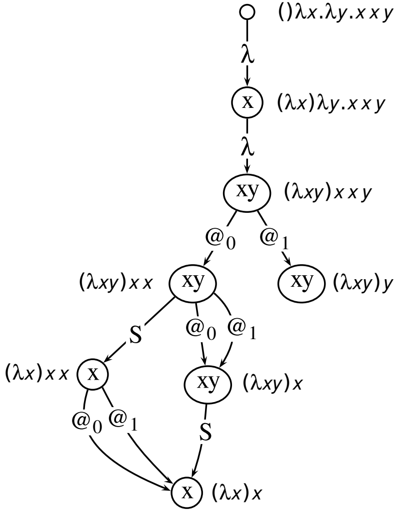
$$(\rho^{S}):\quad (\lambda x_1 \ldots x_{n+1}) M_0 \to (\lambda x_1 \ldots x_n) M_0 \qquad (\text{if } \lambda x_{n+1} \text{ is vacuous})$$

Example $(\lambda x.\lambda y.x\,x\,y)$

$$\begin{array}{l} ()\lambda x.\lambda y.x\,x\,y \to_\lambda \\ (\lambda x)\lambda y.x\,x\,y \to_\lambda \\ (\lambda xy)x\,x\,y \to_{@_1} \\ (\lambda xy)y \end{array} \qquad \begin{array}{l} ()\lambda x.\lambda y.x\,x\,y \to_\lambda \\ (\lambda x)\lambda y.x\,x\,y \to_\lambda \\ (\lambda xy)x\,x\,y \to_{@_0} \\ (\lambda xy)x\,x \to_{@_1} \\ (\lambda xy)x \to_S \\ (\lambda x)x \end{array} \qquad \begin{array}{l} ()\lambda x.\lambda y.x\,x\,y \to_\lambda \\ (\lambda x)\lambda y.x\,x\,y \to_\lambda \\ (\lambda xy)x\,x\,y \to_{@_0} \\ (\lambda xy)x\,x \to_S \\ (\lambda x)x\,x \to_{@_1} \\ (\lambda x)x \end{array}$$

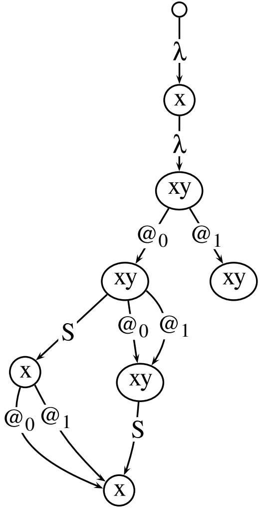
scope$^+$-delimiting strategies for *Reg*$^+$

Example $(\lambda x.\lambda y.x\,x\,y)$



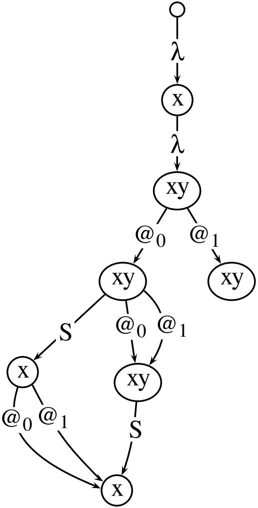
scope$^+$-delimiting strategies for *Reg*$^+$

Example $(\lambda x.\lambda y.x\,x\,y)$

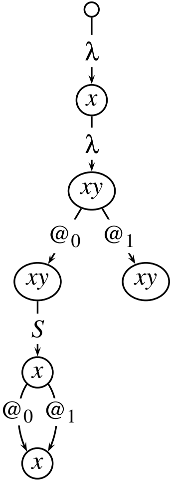


scope^{+}-delimiting strategies for ***Reg***$^{+}$

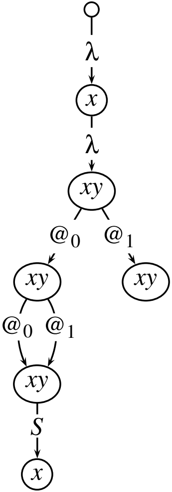
Example $(\lambda x.\lambda y.x\,x\,y)$



Reg^{+}



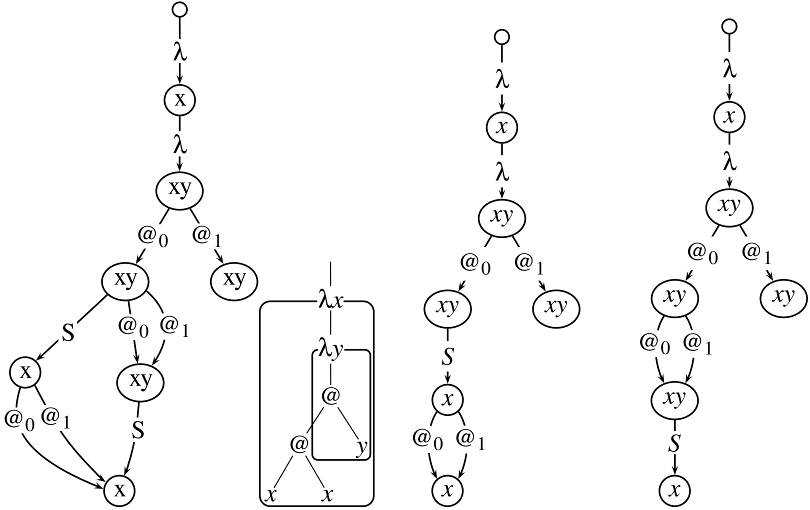
$\mathbb{S}_{\text{eag}}$



$\mathbb{S}_{\text{lazy}}$

scope$^+$-delimiting strategies for ***Reg***$^+$

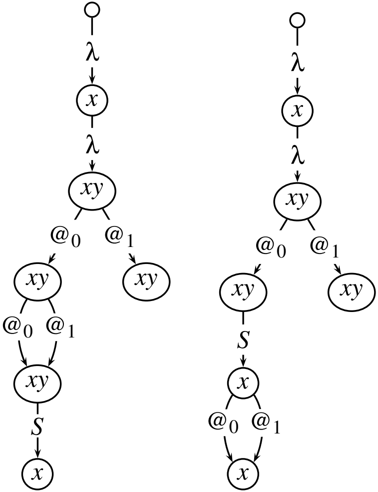
Example ($\lambda x.\lambda y.x\,x\,y$)



Reg$^+$ $\mathbb{S}_{\text{eag}}$ $\mathbb{S}_{\text{lazy}}$

Connection to the de-Bruijn notation

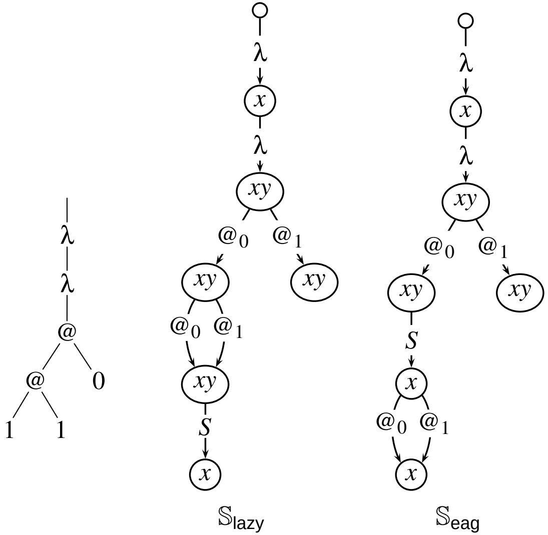
Example ($\lambda x.\lambda y.x\,x\,y$)



$\mathbb{S}_{\text{lazy}}$ $\mathbb{S}_{\text{eag}}$

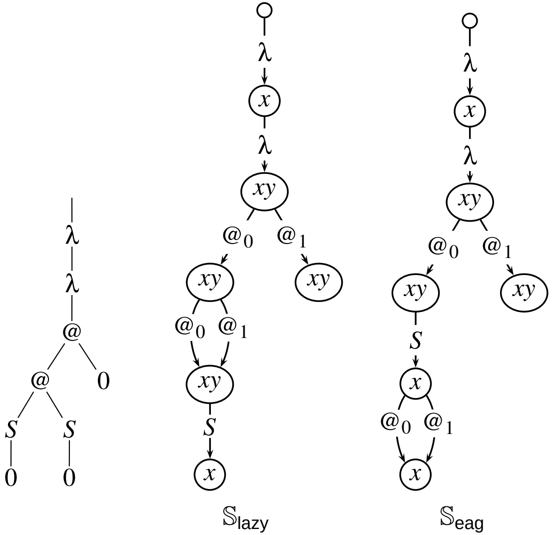
Connection to the de-Bruijn notation

Example $(\lambda x.\lambda y.x\,x\,y)$



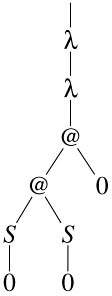
Connection to the de-Bruijn notation

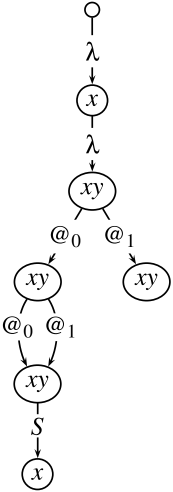
Example $(\lambda x.\lambda y.x\,x\,y)$



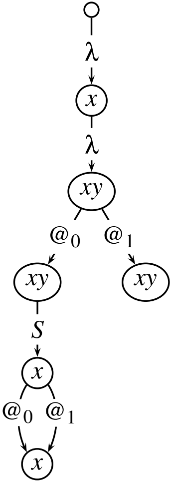
Connection to the de-Bruijn notation

Example $(\lambda x.\lambda y.x\,x\,y)$

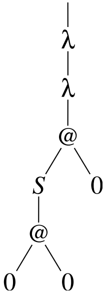




$\mathbb{S}_{\text{lazy}}$



$\mathbb{S}_{\text{eag}}$



Strong regularity

Definition ($\mathbb{S}$-regularity)

An infinite λ-term M is $\mathbb{S}$-*regular* the set of $\twoheadrightarrow_{\mathbb{S}}$-reducts of M is finite.

Strong regularity

Definition ($\mathbb{S}$-regularity)

An infinite λ-term M is *$\mathbb{S}$-regular* the set of $\twoheadrightarrow_{\mathbb{S}}$-reducts of M is finite.

Definition (strongly regular infinite λ-terms)

An infinite λ-term M is *strongly regular* if there exists a scope^{+}-delimiting strategy $\mathbb{S}^{+}$ for Reg^{+} such that M is $\mathbb{S}^{+}$-regular.

λ_{letrec}-expressibility

Definition

A λ_{letrec}-term L *expresses* an infinite λ-term M if $L \rightarrow^{\omega}_{\triangledown} M$.

Definition

M is *λ_{letrec}-expressible* if there is a λ_{letrec}-term which expresses M.

λ_{letrec}-expressibility

Definition

A λ_{letrec}-term L *expresses* an infinite λ-term M if $L \to^{\omega}_{\triangledown} M$.

Definition

M is *λ_{letrec}-expressible* if there is a λ_{letrec}-term which expresses M.

$$
\begin{array}{rrcl}
(\rho^{@}_{\triangledown}): & \text{letrec } B \text{ in } L_0\, L_1 & \to & (\text{letrec } B \text{ in } L_0)\,(\text{letrec } B \text{ in } L_1) \\
(\rho^{\lambda}_{\triangledown}): & \text{letrec } B \text{ in } \lambda x.L_0 & \to & \lambda x.\text{letrec } B \text{ in } L_0 \\
(\rho^{\text{letrec}}_{\triangledown}): & \text{letrec } B_0 \text{ in letrec } B_1 \text{ in } L & \to & \text{letrec } B_0, B_1 \text{ in } L \\
(\rho^{\text{rec}}_{\triangledown}): & \text{letrec } B \text{ in } f_i & \to & \text{letrec } B \text{ in } L_i \\
 & & & (\text{if } B \text{ is } f_1 = L_1 \ldots f_n = L_n) \\
(\rho^{\text{red}}_{\triangledown}): & \text{letrec } f_1 = L_1 \ldots f_n = L_n \text{ in } L & \to & \text{letrec } f_{j_1} = L_{j_1} \ldots f_{j_{n'}} = L_{j_{n'}} \text{ in } L \\
 & \multicolumn{3}{r}{(\text{if } f_{j_1}, \ldots, f_{j_{n'}} \text{ are the recursion variables reachable from } L)} \\
(\rho^{\text{nil}}_{\triangledown}): & \text{letrec in } L & \to & L
\end{array}
$$

Strong regularity and λ_{letrec}-expressibility

Theorem

An infinite λ-term is λ_{letrec}-expressible if and only if it is strongly regular.

Strong regularity

Example ($M = \lambda xy.M\,y\,x$)

$$\frac{\mathbb{S}^{+}_{\text{lazy}}}{()M}$$

Strong regularity

Example ($M = \lambda xy.M\,y\,x$)

$$\frac{\mathbb{S}^{+}_{\text{lazy}}}{\begin{array}{ll}()M & = \\ ()\lambda xy.M\,y\,x & \end{array}}$$

Strong regularity

Example ($M = \lambda xy.M\,y\,x$)

$\mathbb{S}^{+}_{\text{lazy}}$	
$()M$	$=$
$()\lambda xy.M\,y\,x$	$\to_{\lambda}$
$(\lambda x)\lambda y.M\,y\,x$	

Strong regularity

Example ($M = \lambda xy.M\,y\,x$)

$\mathbb{S}^{+}_{\text{lazy}}$	
$()M$	$=$
$()\lambda xy.M\,y\,x$	$\to_{\lambda}$
$(\lambda x)\lambda y.M\,y\,x$	$\to_{\lambda}$
$(\lambda xy)M\,y\,x$	

Strong regularity

Example ($M = \lambda xy.M\,y\,x$)

$\mathbb{S}^{+}_{\text{lazy}}$	
$()M$	$=$
$()\lambda xy.M\,y\,x$	$\to_{\lambda}$
$(\lambda x)\lambda y.M\,y\,x$	$\to_{\lambda}$
$(\lambda xy)M\,y\,x$	$\to_{@_0}$
$(\lambda xy)M\,y$	

Strong regularity

Example ($M = \lambda xy.M\,y\,x$)

$\mathbb{S}^{+}_{\text{lazy}}$	
$()M$	$=$
$()\lambda xy.M\,y\,x$	$\to_{\lambda}$
$(\lambda x)\lambda y.M\,y\,x$	$\to_{\lambda}$
$(\lambda xy)M\,y\,x$	$\to_{@_0}$
$(\lambda xy)M\,y$	$\to_{@_0}$
$(\lambda xy)M$	

Strong regularity

Example ($M = \lambda xy.M\,y\,x$)

$\mathbb{S}^{+}_{\text{lazy}}$	
$()M$	$=$
$()\lambda xy.M\,y\,x$	$\rightarrow_{\lambda}$
$(\lambda x)\lambda y.M\,y\,x$	$\rightarrow_{\lambda}$
$(\lambda xy)M\,y\,x$	$\rightarrow_{@_0}$
$(\lambda xy)M\,y$	$\rightarrow_{@_0}$
$(\lambda xy)M$	$=$
$(\lambda xy)\lambda x'y'.M\,y'\,x'$	

Strong regularity

Example ($M = \lambda xy.M\,y\,x$)

$$
\begin{array}{ll}
\mathbb{S}^{+}_{\text{lazy}} & \\
\hline
()M & = \\
()\lambda xy.M\,y\,x & \to_{\lambda} \\
(\lambda x)\lambda y.M\,y\,x & \to_{\lambda} \\
(\lambda xy)M\,y\,x & \to_{@_0} \\
(\lambda xy)M\,y & \to_{@_0} \\
(\lambda xy)M & = \\
(\lambda xy)\lambda x'y'.M\,y'\,x' & \to_{\lambda} \\
(\lambda xyx')\lambda y'.M\,y'\,x' &
\end{array}
$$

Strong regularity

Example ($M = \lambda xy.M\,y\,x$)

$\mathbb{S}^{+}_{\text{lazy}}$	
$()M$	$=$
$()\lambda xy.M\,y\,x$	$\to_\lambda$
$(\lambda x)\lambda y.M\,y\,x$	$\to_\lambda$
$(\lambda xy)M\,y\,x$	$\to_{@_0}$
$(\lambda xy)M\,y$	$\to_{@_0}$
$(\lambda xy)M$	$=$
$(\lambda xy)\lambda x'y'.M\,y'\,x'$	$\to_\lambda$
$(\lambda xyx')\lambda y'.M\,y'\,x'$	$\to_\lambda$
$(\lambda xyx'y')M\,y'\,x'$	

Strong regularity

Example ($M = \lambda xy.M\,y\,x$)

$\mathbb{S}^{+}_{\text{lazy}}$	
$()M$	$=$
$()\lambda xy.M\,y\,x$	$\to_{\lambda}$
$(\lambda x)\lambda y.M\,y\,x$	$\to_{\lambda}$
$(\lambda xy)M\,y\,x$	$\to_{@_0}$
$(\lambda xy)M\,y$	$\to_{@_0}$
$(\lambda xy)M$	$=$
$(\lambda xy)\lambda x'y'.M\,y'\,x'$	$\to_{\lambda}$
$(\lambda xyx')\lambda y'.M\,y'\,x'$	$\to_{\lambda}$
$(\lambda xyx'y')M\,y'\,x'$	$\to_{@_0}$
$(\lambda xyx'y')M\,y'$	

Strong regularity

Example ($M = \lambda xy.M\,y\,x$)

$\mathbb{S}^{+}_{\text{lazy}}$	
$()M$	$=$
$()\lambda xy.M\,y\,x$	$\to_\lambda$
$(\lambda x)\lambda y.M\,y\,x$	$\to_\lambda$
$(\lambda xy)M\,y\,x$	$\to_{@_0}$
$(\lambda xy)M\,y$	$\to_{@_0}$
$(\lambda xy)M$	$=$
$(\lambda xy)\lambda x'y'.M\,y'\,x'$	$\to_\lambda$
$(\lambda xyx')\lambda y'.M\,y'\,x'$	$\to_\lambda$
$(\lambda xyx'y')M\,y'\,x'$	$\to_{@_0}$
$(\lambda xyx'y')M\,y'$	$\to_{@_0}$
$(\lambda xyx'y')M$	$\ldots$

Strong regularity

Example ($M = \lambda xy.M\,y\,x$)

$\mathbb{S}^{+}_{\text{lazy}}$	
$()M$	$=$
$()\lambda xy.M\,y\,x$	$\to_{\lambda}$
$(\lambda x)\lambda y.M\,y\,x$	$\to_{\lambda}$
$(\lambda xy)M\,y\,x$	$\to_{@_0}$
$(\lambda xy)M\,y$	$\to_{@_0}$
$(\lambda xy)M$	$=$
$(\lambda xy)\lambda x'y'.M\,y'\,x'$	$\to_{\lambda}$
$(\lambda xyx')\lambda y'.M\,y'\,x'$	$\to_{\lambda}$
$(\lambda xyx'y')M\,y'\,x'$	$\to_{@_0}$
$(\lambda xyx'y')M\,y'$	$\to_{@_0}$
$(\lambda xyx'y')M$	$\dots$

$\mathbb{S}^{+}_{\text{eag}}$
$()M$

Strong regularity

Example ($M = \lambda xy.M\,y\,x$)

$$
\begin{array}{ll}
\multicolumn{2}{l}{\mathbb{S}^{+}_{\text{lazy}}} \\
\hline
()M & = \\
()\lambda xy.M\,y\,x & \to_{\lambda} \\
(\lambda x)\lambda y.M\,y\,x & \to_{\lambda} \\
(\lambda xy)M\,y\,x & \to_{@_0} \\
(\lambda xy)M\,y & \to_{@_0} \\
(\lambda xy)M & = \\
(\lambda xy)\lambda x'y'.M\,y'\,x' & \to_{\lambda} \\
(\lambda xyx')\lambda y'.M\,y'\,x' & \to_{\lambda} \\
(\lambda xyx'y')M\,y'\,x' & \to_{@_0} \\
(\lambda xyx'y')M\,y' & \to_{@_0} \\
(\lambda xyx'y')M & \ldots
\end{array}
$$

$$
\begin{array}{ll}
\multicolumn{2}{l}{\mathbb{S}^{+}_{\text{eag}}} \\
\hline
()M & = \\
()\lambda xy.M\,y\,x &
\end{array}
$$

Strong regularity

Example ($M = \lambda xy.M\,y\,x$)

$$
\begin{array}{ll}
\mathbb{S}^{+}_{\text{lazy}} & \\
\hline
()M & = \\
()\lambda xy.M\,y\,x & \to_{\lambda} \\
(\lambda x)\lambda y.M\,y\,x & \to_{\lambda} \\
(\lambda xy)M\,y\,x & \to_{@_0} \\
(\lambda xy)M\,y & \to_{@_0} \\
(\lambda xy)M & = \\
(\lambda xy)\lambda x'y'.M\,y'\,x' & \to_{\lambda} \\
(\lambda xyx')\lambda y'.M\,y'\,x' & \to_{\lambda} \\
(\lambda xyx'y')M\,y'\,x' & \to_{@_0} \\
(\lambda xyx'y')M\,y' & \to_{@_0} \\
(\lambda xyx'y')M & \ldots
\end{array}
$$

$$
\begin{array}{ll}
\mathbb{S}^{+}_{\text{eag}} & \\
\hline
()M & = \\
()\lambda xy.M\,y\,x & \to_{\lambda} \\
(\lambda x)\lambda y.M\,y\,x &
\end{array}
$$

Strong regularity

Example ($M = \lambda xy.M\,y\,x$)

$\mathbb{S}^{+}_{\text{lazy}}$	
$()M$	$=$
$()\lambda xy.M\,y\,x$	$\to_{\lambda}$
$(\lambda x)\lambda y.M\,y\,x$	$\to_{\lambda}$
$(\lambda xy)M\,y\,x$	$\to_{@_0}$
$(\lambda xy)M\,y$	$\to_{@_0}$
$(\lambda xy)M$	$=$
$(\lambda xy)\lambda x'y'.M\,y'\,x'$	$\to_{\lambda}$
$(\lambda xyx')\lambda y'.M\,y'\,x'$	$\to_{\lambda}$
$(\lambda xyx'y')M\,y'\,x'$	$\to_{@_0}$
$(\lambda xyx'y')M\,y'$	$\to_{@_0}$
$(\lambda xyx'y')M$	$\dots$

$\mathbb{S}^{+}_{\text{eag}}$	
$()M$	$=$
$()\lambda xy.M\,y\,x$	$\to_{\lambda}$
$(\lambda x)\lambda y.M\,y\,x$	$\to_{\lambda}$
$(\lambda xy)M\,y\,x$	

Strong regularity

Example ($M = \lambda xy.M\,y\,x$)

$\mathbb{S}^{+}_{\text{lazy}}$	
$()M$	$=$
$()\lambda xy.M\,y\,x$	$\to_{\lambda}$
$(\lambda x)\lambda y.M\,y\,x$	$\to_{\lambda}$
$(\lambda xy)M\,y\,x$	$\to_{@_0}$
$(\lambda xy)M\,y$	$\to_{@_0}$
$(\lambda xy)M$	$=$
$(\lambda xy)\lambda x'y'.M\,y'\,x'$	$\to_{\lambda}$
$(\lambda xyx')\lambda y'.M\,y'\,x'$	$\to_{\lambda}$
$(\lambda xyx'y')M\,y'\,x'$	$\to_{@_0}$
$(\lambda xyx'y')M\,y'$	$\to_{@_0}$
$(\lambda xyx'y')M$	$\ldots$

$\mathbb{S}^{+}_{\text{eag}}$	
$()M$	$=$
$()\lambda xy.M\,y\,x$	$\to_{\lambda}$
$(\lambda x)\lambda y.M\,y\,x$	$\to_{\lambda}$
$(\lambda xy)M\,y\,x$	$\to_{@_0}$
$(\lambda xy)M\,y$	

Strong regularity

Example ($M = \lambda xy.M\,y\,x$)

$\mathbb{S}^{+}_{\text{lazy}}$	
$()M$	$=$
$()\lambda xy.M\,y\,x$	$\to_{\lambda}$
$(\lambda x)\lambda y.M\,y\,x$	$\to_{\lambda}$
$(\lambda xy)M\,y\,x$	$\to_{@_0}$
$(\lambda xy)M\,y$	$\to_{@_0}$
$(\lambda xy)M$	$=$
$(\lambda xy)\lambda x'y'.M\,y'\,x'$	$\to_{\lambda}$
$(\lambda xyx')\lambda y'.M\,y'\,x'$	$\to_{\lambda}$
$(\lambda xyx'y')M\,y'\,x'$	$\to_{@_0}$
$(\lambda xyx'y')M\,y'$	$\to_{@_0}$
$(\lambda xyx'y')M$	$\dots$

$\mathbb{S}^{+}_{\text{eag}}$	
$()M$	$=$
$()\lambda xy.M\,y\,x$	$\to_{\lambda}$
$(\lambda x)\lambda y.M\,y\,x$	$\to_{\lambda}$
$(\lambda xy)M\,y\,x$	$\to_{@_0}$
$(\lambda xy)M\,y$	$\to_{@_0}$
$(\lambda xy)M$	

Strong regularity

Example ($M = \lambda xy.M\,y\,x$)

$\mathbb{S}^{+}_{\text{lazy}}$	
$()M$	$=$
$()\lambda xy.M\,y\,x$	$\to_{\lambda}$
$(\lambda x)\lambda y.M\,y\,x$	$\to_{\lambda}$
$(\lambda xy)M\,y\,x$	$\to_{@_0}$
$(\lambda xy)M\,y$	$\to_{@_0}$
$(\lambda xy)M$	$=$
$(\lambda xy)\lambda x'y'.M\,y'\,x'$	$\to_{\lambda}$
$(\lambda xyx')\lambda y'.M\,y'\,x'$	$\to_{\lambda}$
$(\lambda xyx'y')M\,y'\,x'$	$\to_{@_0}$
$(\lambda xyx'y')M\,y'$	$\to_{@_0}$
$(\lambda xyx'y')M$	$\ldots$

$\mathbb{S}^{+}_{\text{eag}}$	
$()M$	$=$
$()\lambda xy.M\,y\,x$	$\to_{\lambda}$
$(\lambda x)\lambda y.M\,y\,x$	$\to_{\lambda}$
$(\lambda xy)M\,y\,x$	$\to_{@_0}$
$(\lambda xy)M\,y$	$\to_{@_0}$
$(\lambda xy)M$	$\to_{\mathsf{S}}$
$(\lambda x)M$	

Strong regularity

Example ($M = \lambda xy.M\,y\,x$)

$\mathbb{S}^{+}_{\text{lazy}}$	
$()M$	$=$
$()\lambda xy.M\,y\,x$	$\to_{\lambda}$
$(\lambda x)\lambda y.M\,y\,x$	$\to_{\lambda}$
$(\lambda xy)M\,y\,x$	$\to_{@_0}$
$(\lambda xy)M\,y$	$\to_{@_0}$
$(\lambda xy)M$	$=$
$(\lambda xy)\lambda x'y'.M\,y'\,x'$	$\to_{\lambda}$
$(\lambda xyx')\lambda y'.M\,y'\,x'$	$\to_{\lambda}$
$(\lambda xyx'y')M\,y'\,x'$	$\to_{@_0}$
$(\lambda xyx'y')M\,y'$	$\to_{@_0}$
$(\lambda xyx'y')M$	$\dots$

$\mathbb{S}^{+}_{\text{eag}}$	
$()M$	$=$
$()\lambda xy.M\,y\,x$	$\to_{\lambda}$
$(\lambda x)\lambda y.M\,y\,x$	$\to_{\lambda}$
$(\lambda xy)M\,y\,x$	$\to_{@_0}$
$(\lambda xy)M\,y$	$\to_{@_0}$
$(\lambda xy)M$	$\to_{\mathsf{S}}$
$(\lambda x)M$	$\to_{\mathsf{S}}$
$()M$	$\dots$

Strong regularity and $\mathbb{S}^{+}_{\text{eag}}$

Proposition

M is strongly regular if and only if M is $\mathbb{S}^{+}_{\text{eag}}$-regular.

Proof.

$\Leftarrow$: by definition.

$\Rightarrow$: follows from $\twoheadrightarrow_{\mathbb{S}^{+}_{\text{eag}}} \subseteq \twoheadrightarrow_{\mathbb{S}^{+}} \cdot \twoheadrightarrow_{\text{S}}$

Strong regularity and $\mathbb{S}^{+}_{\text{eag}}$

Proposition

M is strongly regular if and only if M is $\mathbb{S}^{+}_{\text{eag}}$-regular.

Proof.

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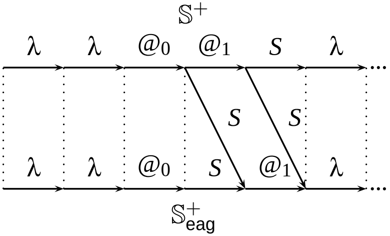
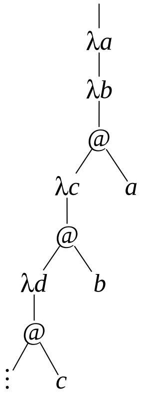
$\Rightarrow$: follows from $\twoheadrightarrow_{\mathbb{S}^{+}_{\text{eag}}} \subseteq \twoheadrightarrow_{\mathbb{S}^{+}} \cdot \twoheadrightarrow_{\mathsf{S}}$



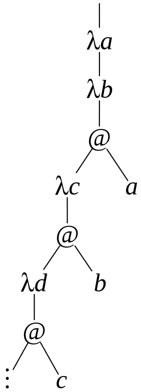
$\square$

Strong regularity and $\mathbb{S}^{+}_{\text{eag}}$



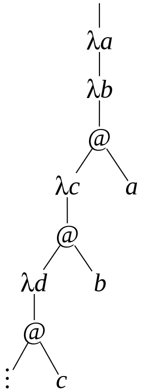
$$()\lambda a.\lambda b.\ldots \quad \to_{\lambda} \quad (\lambda a)\lambda b.\lambda c.\ldots a$$

Strong regularity and $\mathbb{S}^{+}_{\text{eag}}$



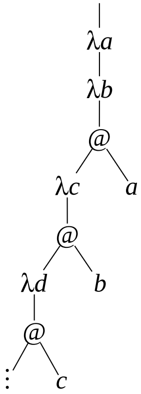
$$
\begin{aligned}
()\lambda a.\lambda b.\ldots \quad &\to_{\lambda} \quad (\lambda a)\lambda b.\lambda c.\ldots.a \\
&\to_{\lambda} \quad (\lambda ab)\lambda c.\ldots.a
\end{aligned}
$$

Strong regularity and $\mathbb{S}^{+}_{\text{eag}}$



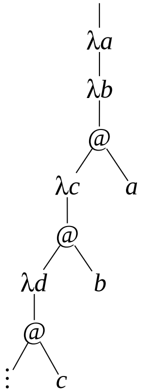
$$
\begin{aligned}
()\lambda a.\lambda b.\ldots \quad & \to_{\lambda} \quad (\lambda a)\lambda b.\lambda c.\ldots.a \\
& \to_{\lambda} \quad (\lambda ab)\lambda c.\ldots.a \\
& \to_{@_0} \quad (\lambda ab)\lambda c.\lambda d.\ldots.b
\end{aligned}
$$

Strong regularity and $\mathbb{S}^{+}_{\text{eag}}$



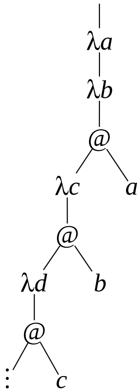
$$
\begin{aligned}
()\lambda a.\lambda b.\ldots \;&\to_{\lambda}\; (\lambda a)\lambda b.\lambda c.\ldots.a \\
&\to_{\lambda}\; (\lambda ab)\lambda c.\ldots.a \\
&\to_{@_0}\; (\lambda ab)\lambda c.\lambda d.\ldots.b \\
&\to_{\lambda}\; (\lambda abc)\lambda d.\ldots.b
\end{aligned}
$$

Strong regularity and $\mathbb{S}^{+}_{\mathrm{eag}}$



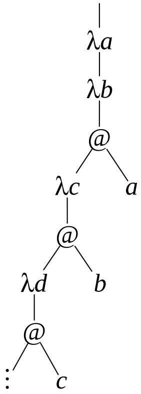
$$
\begin{aligned}
()\lambda a.\lambda b.\ldots \;&\rightarrow_{\lambda}\; (\lambda a)\lambda b.\lambda c.\ldots.a \\
&\rightarrow_{\lambda}\; (\lambda ab)\lambda c.\ldots.a \\
&\rightarrow_{@_0}\; (\lambda ab)\lambda c.\lambda d.\ldots.b \\
&\rightarrow_{\lambda}\; (\lambda abc)\lambda d.\ldots.b \\
&\rightarrow_{@_0}\; (\lambda abc)\lambda d.\lambda e.\ldots.c
\end{aligned}
$$

Strong regularity and $\mathbb{S}^{+}_{\text{eag}}$



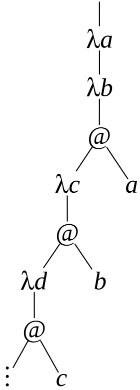
$$
\begin{aligned}
()\lambda a.\lambda b.\ldots \quad &\to_{\lambda} \quad (\lambda a)\lambda b.\lambda c.\ldots.a\\
&\to_{\lambda} \quad (\lambda ab)\lambda c.\ldots.a\\
&\to_{@_0} \quad (\lambda ab)\lambda c.\lambda d.\ldots.b\\
&\to_{\lambda} \quad (\lambda abc)\lambda d.\ldots.b\\
&\to_{@_0} \quad (\lambda abc)\lambda d.\lambda e.\ldots.c\\
&\to_{\lambda} \quad (\lambda abcd)\lambda e.\ldots.c
\end{aligned}
$$

Strong regularity and $\mathbb{S}^{+}_{\mathrm{eag}}$



$$
\begin{aligned}
()\lambda a.\lambda b.\ldots \;&\to_{\lambda}\; (\lambda a)\lambda b.\lambda c.\ldots a\\
&\to_{\lambda}\; (\lambda ab)\lambda c.\ldots a\\
&\to_{@_0}\; (\lambda ab)\lambda c.\lambda d.\ldots b\\
&\to_{\lambda}\; (\lambda abc)\lambda d.\ldots b\\
&\to_{@_0}\; (\lambda abc)\lambda d.\lambda e.\ldots c\\
&\to_{\lambda}\; (\lambda abcd)\lambda e.\ldots c\\
&\to_{@_0}\; (\lambda abcd)\lambda e.\lambda f.\ldots d
\end{aligned}
$$

Strong regularity and $\mathbb{S}^{+}_{\mathrm{eag}}$



$$
\begin{array}{lll}
()\lambda a.\lambda b.\dots & \to_{\lambda} & (\lambda a)\lambda b.\lambda c.\dots a \\
& \to_{\lambda} & (\lambda ab)\lambda c.\dots a \\
& \to_{@_0} & (\lambda ab)\lambda c.\lambda d.\dots b \\
& \to_{\lambda} & (\lambda abc)\lambda d.\dots b \\
& \to_{@_0} & (\lambda abc)\lambda d.\lambda e.\dots c \\
& \to_{\lambda} & (\lambda abcd)\lambda e.\dots c \\
& \to_{@_0} & (\lambda abcd)\lambda e.\lambda f.\dots d \\
& \to_{\lambda} & (\lambda abcde)\lambda f.\dots d \\
& \dots &
\end{array}
$$

Proof system $\mathbf{Reg}^+$ for strong regularity

$$\frac{}{(\lambda\vec{x}y)y}\ 0 \qquad \frac{(\lambda\vec{x}y)M_0}{(\lambda\vec{x})\lambda y.M_0}\ \lambda \qquad \frac{(\lambda\vec{x})M_0 \qquad (\lambda\vec{x})M_1}{(\lambda\vec{x})M_0\,M_1}\ @$$

$$\frac{(\lambda x_1 \ldots x_{n-1})M}{(\lambda x_1 \ldots x_n)M}\ \mathsf{S}$$ (if the binding λx_n is vacuous)

$$\frac{\begin{array}{c}[(\lambda\vec{x})M]^u \\ \mathcal{D}_0 \\ (\lambda\vec{x})M\end{array}}{(\lambda\vec{x})M}\ \mathrm{FIX}, u \quad (\text{if } |\mathcal{D}_0| \geq 1)$$

Proof system $\mathbf{Reg}^+$ for strong regularity

$$\frac{}{(\lambda\vec{x}y)y}\;0 \qquad \frac{(\lambda\vec{x}y)M_0}{(\lambda\vec{x})\lambda y.M_0}\;\lambda \qquad \frac{(\lambda\vec{x})M_0 \qquad (\lambda\vec{x})M_1}{(\lambda\vec{x})M_0\,M_1}\;@$$

$$\frac{(\lambda x_1 \ldots x_{n-1})M}{(\lambda x_1 \ldots x_n)M}\;\mathsf{S} \quad \text{(if the binding } \lambda x_n \text{ is vacuous)} \qquad \frac{\begin{matrix}[(\lambda\vec{x})M]^u \\ \mathcal{D}_0 \\ (\lambda\vec{x})M\end{matrix}}{(\lambda\vec{x})M}\;\mathrm{FIX}, u \quad (\text{if } |\mathcal{D}_0| \geq 1)$$

$$(\rho^{@_i}):\quad (\lambda x_1 \ldots x_n)M_0\,M_1 \to (\lambda x_1 \ldots x_n)M_i \qquad (i \in \{0,1\})$$

$$(\rho^{\lambda}):\quad (\lambda x_1 \ldots x_n)\lambda x_{n+1}.M_0 \to (\lambda x_1 \ldots x_{n+1})M_0$$

$$(\rho^{\mathsf{S}}):\quad (\lambda x_1 \ldots x_{n+1})M_0 \to (\lambda x_1 \ldots x_n)M_0 \qquad (\text{if } \lambda x_{n+1} \text{ is vacuous})$$

Proof system $\mathbf{Reg}^{+}$ for strong regularity

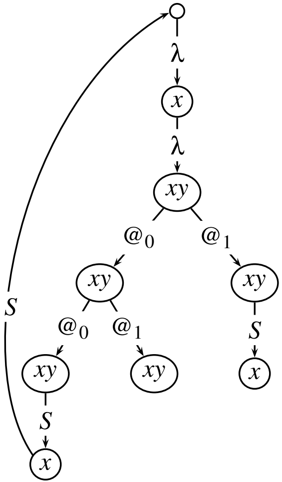
Example ($M = \lambda xy.M\,x\,y$)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{((\,)M)^u}{(\lambda x)M}\,\mathrm{S} \qquad \dfrac{}{(\lambda x)x}\,0}{(\lambda x)M\,x}\,@}{(\lambda xy)M\,x}\,\mathrm{S} \qquad \dfrac{}{(\lambda xy)y}\,0}{(\lambda xy)M\,x\,y}\,@}{(\lambda x)\lambda y.M\,x\,y}\,\lambda}{(\,)M}\,\lambda}{(\,)M}\,\mathrm{FIX}, u$$

Proof system $\mathbf{Reg}^{+}$ for strong regularity

Example ($M = \lambda xy.M\,x\,y$)

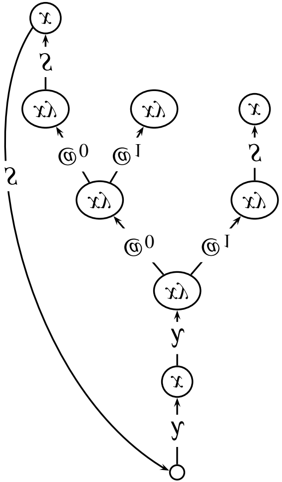
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(()M)^u}{(\lambda x)M}\,\mathsf{S} \qquad \dfrac{}{(\lambda x)x}\,0}{(\lambda x)M\,x}\,@}{(\lambda xy)M\,x}\,\mathsf{S} \qquad \dfrac{}{(\lambda xy)y}\,0}{(\lambda xy)M\,x\,y}\,@}{(\lambda x)\lambda y.M\,x\,y}\,\lambda}{()M}\,\lambda}{()M}\,\text{FIX}, u$$



Proof system $\mathbf{Reg}^{+}$ for strong regularity

Example ($M = \lambda xy.M\,x\,y$)

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{((\,)M)^{u}}{(\lambda x)M}\,\mathsf{S}
        \qquad
        \dfrac{}{(\lambda x)x}\,0
      }{(\lambda x)M\,x}\,@
    }{(\lambda xy)M\,x}\,\mathsf{S}
    \qquad
    \dfrac{}{(\lambda xy)y}\,0
  }{
    \dfrac{
      \dfrac{(\lambda xy)M\,x\,y}{(\lambda x)\lambda y.M\,x\,y}\,\lambda
    }{(\,)M}\,\lambda
  }\,@
}{(\,)M}\,\text{FIX}, u
$$



Proof system **Reg**$^+$ for strong regularity

$$\frac{}{(\lambda\vec{x}y)y}\;0 \qquad \frac{(\lambda\vec{x}y)M_0}{(\lambda\vec{x})\lambda y.M_0}\;\lambda$$

$$\frac{(\lambda\vec{x})M_0 \qquad (\lambda\vec{x})M_1}{(\lambda\vec{x})M_0\,M_1}\;@$$

$$\frac{(\lambda x_1\ldots x_{n-1})M}{(\lambda x_1\ldots x_n)M}\;\text{S (if the binding } \lambda x_n \text{ is vacuous)}$$

$$\frac{\begin{matrix}[(\lambda\vec{x})M]^u \\ \mathcal{D}_0 \\ (\lambda\vec{x})M\end{matrix}}{(\lambda\vec{x})M}\;\text{FIX}, u \quad (\text{if } |\mathcal{D}_0| \geq 1)$$

Proof system $\mathbf{Reg}_0^+$ for strong regularity

$$\frac{}{(\lambda \vec{x} y) y}\ 0 \qquad \frac{(\lambda \vec{x} y) M_0}{(\lambda \vec{x}) \lambda y . M_0}\ \lambda$$

$$\frac{(\lambda \vec{x}) M_0 \qquad (\lambda \vec{x}) M_1}{(\lambda \vec{x}) M_0 M_1}\ @$$

$$\frac{(\lambda x_1 \ldots x_{n-1}) M}{(\lambda x_1 \ldots x_n) M}\ \mathsf{S} \text{ (if the binding } \lambda x_n \text{ is vacuous)}$$

$$\frac{\begin{array}{c}[(\lambda \vec{x}) M]^u \\ \mathcal{D}_0 \\ (\lambda \vec{x}) M\end{array}}{(\lambda \vec{x}) M}\ \mathrm{FIX}, u$$

(if $|\mathcal{D}_0| \geq 1$, and, on threads in $\mathcal{D}_0$ down from assumptions $((\lambda \vec{x}) M)^u$, prefix-lengths are $\geq |\vec{x}|$)

Proof system **ann-Reg$_0^+$** for λ_{letrec}-expressibility

$$\frac{}{(\lambda\vec{x}y)\; y : y}\; 0 \qquad \frac{(\lambda\vec{x}y)\; L : M}{(\lambda\vec{x})\; \lambda y.L : \lambda y.M}\; \lambda$$

$$\frac{(\lambda\vec{x})\; L_0 : M_0 \qquad (\lambda\vec{x})\; L_1 : M_1}{(\lambda\vec{x})\; L_0\, L_1 : M_0\, M_1}\; @$$

$$\frac{(\lambda x_1 \ldots x_{n-1})\; L : M}{(\lambda x_1 \ldots x_n)\; L : M}\; \text{S (if the binding } \lambda x_n \text{ is vacuous)}$$

$$\frac{\begin{array}{c}[(\lambda\vec{x})\; \mathsf{c}_u : M]^u \\ \mathcal{D}_0 \\ (\lambda\vec{x})\; L[u := \mathsf{c}_u] : M\end{array}}{(\lambda\vec{x})\; (\text{letrec}\; u = L \;\text{in}\; u) : M}\; \text{FIX}, u$$

(if $|\mathcal{D}_0| \geq 1$, and, on threads in $\mathcal{D}_0$ down from assumptions $((\lambda\vec{x})M)^u$, prefix-lengths are $\geq |\vec{x}|$)

Proof system **Reg**$^{+}$ for strong regularity

Example ($M = \lambda xy.M\,x\,y$)

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{((\,)M)^{u}}{(\lambda x)M}\;\mathrm{S}
          \qquad
          \dfrac{}{(\lambda x)x}\;0
        }{(\lambda x)M\,x}\;@
      }{(\lambda xy)M\,x}\;\mathrm{S}
      \qquad
      \dfrac{}{(\lambda xy)y}\;0
    }{(\lambda xy)M\,x\,y}\;@
  }{(\lambda x)\lambda y.M\,x\,y}\;\lambda
}{
  \dfrac{(\,)M}{(\,)M}\;\mathrm{FIX},u
}\;\lambda
$$

Proof system $\mathbf{Reg}_0^+$ for strong regularity

Example ($M = \lambda xy.M\,x\,y$)

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{((\,)M)^u}{(\lambda x)M}\,\mathsf{S}
          \qquad
          \dfrac{}{(\lambda x)x}\,0
        }{(\lambda x)M\,x}\,@
      }{(\lambda xy)M\,x}\,\mathsf{S}
      \qquad
      \dfrac{}{(\lambda xy)y}\,0
    }{(\lambda xy)M\,x\,y}\,@
  }{(\lambda x)\lambda y.M\,x\,y}\,\lambda
}{
  \dfrac{(\,)M}{(\,)M}\,\mathrm{FIX},u
}\,\lambda
$$

Proof system **ann-Reg**$_0^+$ for λ_{letrec}-expressibility

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{((\,)\; c_u : M)^u}{(\lambda x)\; c_u : M}\,\mathrm{S} \qquad \dfrac{}{(\lambda x)\; x : x}\,0}{(\lambda x)\; c_u\, x : M\, x}\,@}{(\lambda xy)\; c_u\, x : M\, x}\,\mathrm{S} \qquad \dfrac{}{(\lambda xy)\; y : y}\,0}{(\lambda xy)\; c_u\, x\, y : M\, x\, y}\,@
$$

$$
\dfrac{\dfrac{\dfrac{(\lambda xy)\; c_u\, x\, y : M\, x\, y}{(\lambda x)\; \lambda y . c_u\, x\, y : M\, x\, y}\,\lambda}{(\,)\; \lambda xy . c_u\, x\, y : \lambda xy . M\, x\, y}\,\lambda}{(\,)\; (\text{letrec}\; u = \lambda xy . u\, x\, y \;\text{in}\; u) : M}\,\text{FIX}, u
$$

Main Result

Theorem

An infinite λ-term is λ_{letrec}-expressible if and only if it is strongly regular.

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An infinite λ-term is λ_{letrec}-expressible if and only if it is strongly regular.

Proof.

$\Leftarrow$: For every infinite λ-term M:

$$
\begin{aligned}
M \text{ is strongly regular} \;&\Longrightarrow\; \vdash_{\mathbf{Reg}^+} ()\, M \\
&\Longrightarrow\; \vdash_{\mathbf{Reg}_0^+} ()\, M \\
&\Longrightarrow\; \vdash_{\mathbf{ann\text{-}Reg}_0^+} ()\, L : M \\
&\Longrightarrow\; L \to_{\triangledown}^{\omega} M \\
&\Longrightarrow\; L \text{ expresses } M
\end{aligned}
$$

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Theorem

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Proof.

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&\Longrightarrow \vdash_{\mathbf{Reg}_0^+} ()\, M \\
&\Longrightarrow \vdash_{\mathbf{ann\text{-}Reg}_0^+} ()\, L : M \\
&\Longrightarrow L \to_{\triangledown}^{\omega} M \\
&\Longrightarrow L \text{ expresses } M
\end{aligned}
$$

$\Rightarrow$:

- define: ***Reg***$^+$-observations (& observ. graphs) for λ_{letrec}-terms
- show: for every λ_{letrec}-term L, the $\mathbb{S}^+_{eag}$-observ. graph of L is finite
- show: observation graph of λ_{letrec}-term L is finite
  $\Rightarrow$ observation graph of the unfolding $\mathcal{U}(L)$ of L is finite.

Main Result

Theorem

An infinite λ-term is λ_{letrec}-expressible if and only if it is strongly regular.

Proof.

$\Leftarrow$: For every infinite λ-term M:

$$
\begin{aligned}
M \text{ is strongly regular} &\implies \vdash_{\mathbf{Reg}^+} ()M \\
&\implies \vdash_{\mathbf{Reg}_0^+} ()M \\
&\implies \vdash_{\mathbf{ann\text{-}Reg}_0^+} ()\, L : M \\
&\implies L \to^{\omega}_{\triangledown} M \\
&\implies L \text{ expresses } M
\end{aligned}
$$

$\Rightarrow$:

- define: ***Reg***$^+$-observations (& observ. graphs) for λ_{letrec}-terms
- show: for every λ_{letrec}-term L, the $\mathbb{S}^+_{\text{eag}}$-observ. graph of L is finite
- show: observation graph of λ_{letrec}-term L is finite
  $\Rightarrow$ observation graph of the unfolding $\mathcal{U}(L)$ of L is finite.

Main theorem (extended)

Theorem

For all infinite λ-terms the following are equivalent:

(i) *M is λ_{letrec}-expressible.*

(ii) *M is strongly regular.*

(iii) *M only contains finite binding–capturing chains.*