Relating Proof Systems for Recursive Types

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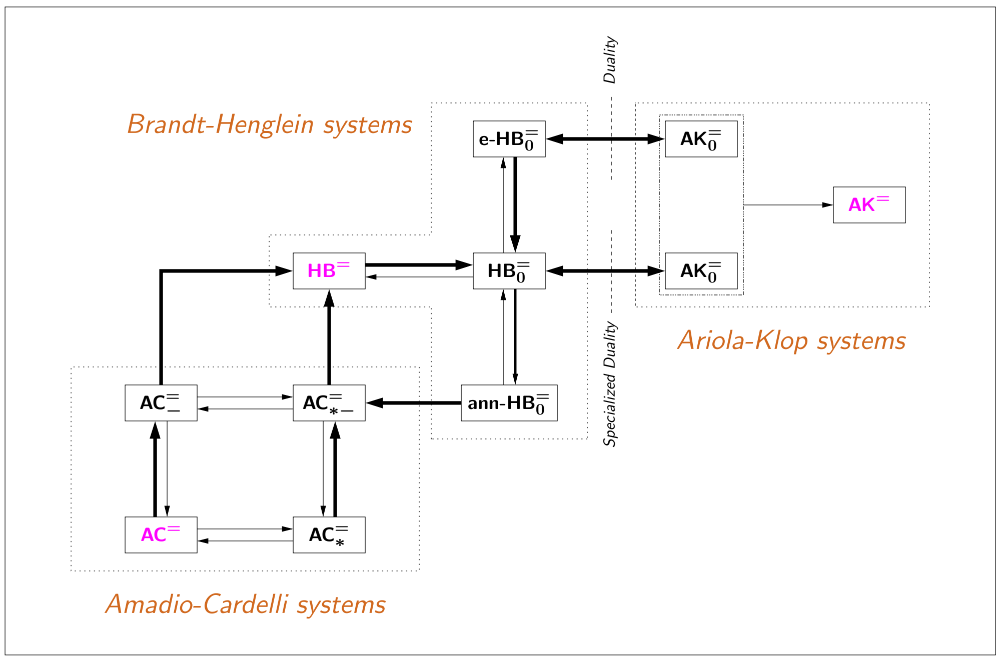

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Types and Recursive Types

Types

– *Basic types*: Int (integers), Real (reals), Bool (booleans).

– *Composite types*: Int $\times$ Int (pairs), Real $\longrightarrow$ Int (functions),
Bool $+$ Int (elements of either).

Recursive Types

$$\textbf{List} \;=\; \text{Empty} + (\text{Int} \times \textbf{List}) \qquad \text{(type of \textit{integer lists})}$$

because: $() \in$ Empty

e.g. $(5, 8, 13) \triangleq \langle 5, (8, 13) \rangle \in$ Int $\times$ **List**

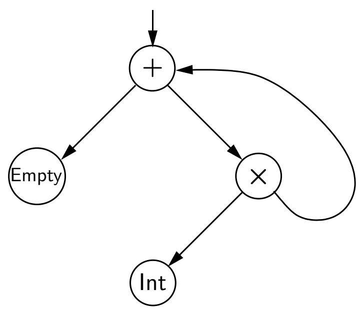
Notation: $\textbf{List} = \mu\alpha.\,(\text{Empty} + (\text{Int} \times \alpha))$.

Recursive Type Equality

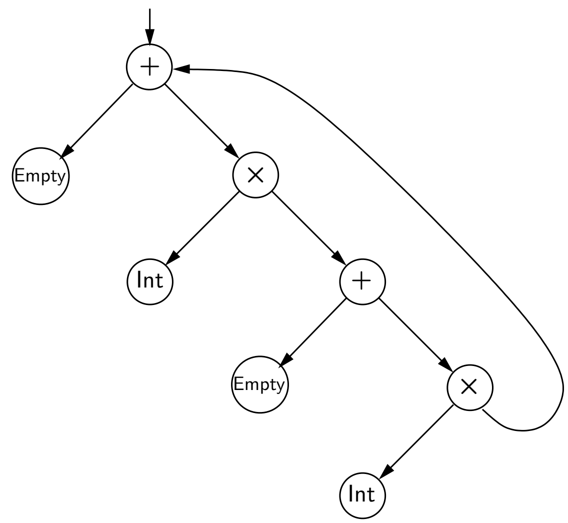
Example. The recursive types

$$\mathbf{List_1} \equiv \mu\alpha.\,(\mathsf{Empty} + \mathsf{Int} \times \alpha)\,,\ \ \mathbf{List_2} \equiv \mu\beta.\,(\mathsf{Empty} + \mathsf{Int} \times (\mathsf{Empty} + \mathsf{Int} \times \beta))$$

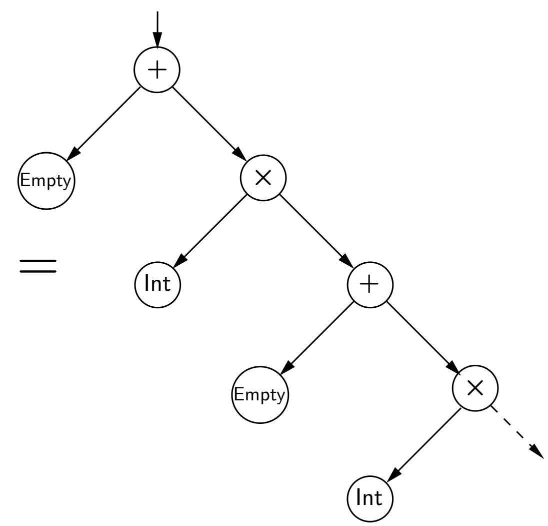
can be visualized as the different cyclic term graphs



and



and have the same *tree unfolding* $\mathbf{Tree}(\mathbf{List_1}) = \mathbf{Tree}(\mathbf{List_2}) =$



$\mathbf{List_1}$ and $\mathbf{List_2}$ are related by *recursive type equality* (notation: $\mathbf{List_1} =_\mu \mathbf{List_2}$). Recursive types that are linked in this way satisfy the same recursive equations.

Proof Systems for Recursive Type Equality

- Sound and complete axiomatisations of $=_\mu$:
  - **AC**$^=$ by Amadio and Cardelli (1993) is *of "traditional form"*.
  - **HB**$^=$ by Henglein and Brandt (1998) is *coinductively motivated*.

$$\tau =_\mu \sigma \iff \vdash_{\mathbf{AC}^=} \tau = \sigma$$
$$\iff \vdash_{\mathbf{HB}^=} \tau = \sigma \,.$$

- A system on which "consistency-checking" can be based:
  - **AK**$^=$, by Ariola and Klop (1995), a *"syntactic-matching"* system.

$\tau =_\mu \sigma \iff$ no "contradiction" is derivable in **AK**$^=$ from the assumption $\tau = \sigma$.

Specific Rules in $\mathbf{AC}^{=}$, $\mathbf{HB}^{=}$, and $\mathbf{AK}^{=}$

- in $\mathbf{AC}^{=}$: $$\frac{\sigma_1 = \tau[\sigma_1/\alpha] \qquad \sigma_2 = \tau[\sigma_2/\alpha]}{\sigma_1 = \sigma_2}\ \mathsf{UFP} \quad (\text{if } \alpha \downarrow \tau)$$

- in $\mathbf{HB}^{=}$: $$\frac{\begin{array}{c}[\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2]^{u} \\ \mathcal{D}_1 \\ \tau_1 = \sigma_1\end{array} \qquad \begin{array}{c}[\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2]^{u} \\ \mathcal{D}_2 \\ \tau_2 = \sigma_2\end{array}}{\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}\ \mathsf{ARROW/FIX}, u$$

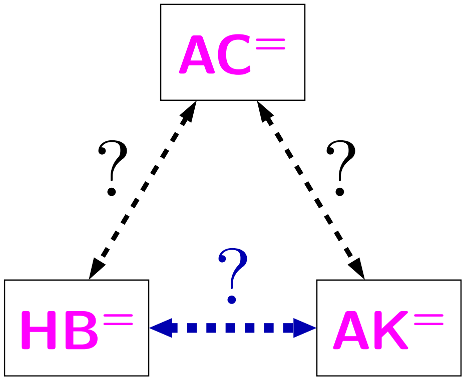
- in $\mathbf{AK}^{=}$: $$\frac{\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}{\tau_i = \sigma_i}\ \mathsf{DECOMP}\ (\text{for } i \in \{1,2\})$$

Present in all systems: REFL, SYMM, TRANS, (FOLD/UNFOLD).

Questions investigated

- *Main Question*: What kind of proof-theoretic relationships do exist between the systems $\mathbf{AC}^=$, $\mathbf{HB}^=$, and $\mathbf{AK}^=$?
  - An initial observation suggested a close connection between $\mathbf{HB}^=$ and $\mathbf{AK}^=$. Can this be made *precise*?
  - Can the “traditional” proofs in $\mathbf{AC}^=$ be transformed into the “coinductive” proofs in $\mathbf{HB}^=$?

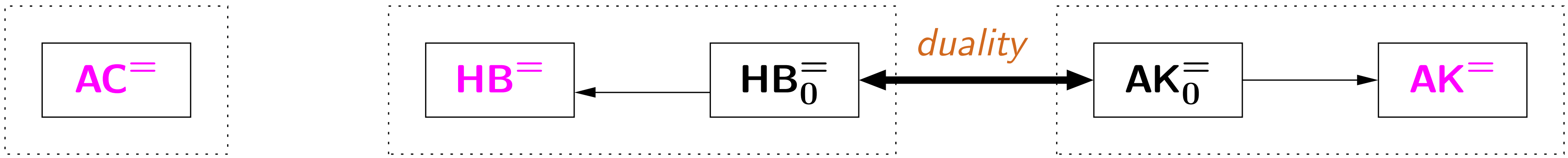
    And *vice versa*: Does there exist a transformation of proofs in $\mathbf{HB}^=$ into proofs in $\mathbf{AC}^=$?



Answers offered

- Introduction of variant systems $\mathbf{HB}_{\mathbf{0}}^{=}$ and $\mathbf{AK}_{\mathbf{0}}^{=}$ with *subformula properties*.

- A **network** of proof-transformations:
  - A *duality* via *mirroring* between derivations in $\mathbf{HB}_{\mathbf{0}}^{=}$ and "consistency-unfoldings" in $\mathbf{AK}_{\mathbf{0}}^{=}$.



Answers offered: A duality between $\mathbf{HB}_0^{=}$ and $\mathbf{AK}_0^{=}$

$$
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\dfrac{}{\mathsf{E}=\mathsf{E}}\,(\text{REFL})
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\dfrac{}{\mathsf{I}=\mathsf{I}}\,(\text{REFL})
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\dfrac{}{\mathsf{I}=\mathsf{I}}\,(\text{REFL})
\quad
\dfrac{(\mathsf{E}+\mathsf{List}_1^*\times\mathsf{I}=\mathsf{E}+\mathsf{I}\times(\mathsf{E}+\mathsf{I}\times\mathsf{List}_2^*))^{\boldsymbol{u}}}{\mathsf{List}_1^*=\mathsf{List}_2^*}
}{\mathsf{I}\times\mathsf{List}_1^*=\mathsf{I}\times\mathsf{List}_2^*}\,\times
}{\mathsf{E}+\mathsf{I}\times\mathsf{List}_1^*=\mathsf{E}+\mathsf{I}\times\mathsf{List}_2^*}\,+
}{\mathsf{List}_1^*=\mathsf{E}+\mathsf{I}\times\mathsf{List}_2^*}\,\text{FOLD}_{l/r}
}{\mathsf{I}\times\mathsf{List}_1^*=\mathsf{I}\times(\mathsf{E}+\mathsf{I}\times\mathsf{List}_2^*)}\,\times
}{
\dfrac{\mathsf{E}+\mathsf{I}\times\mathsf{List}_1^*=\mathsf{E}+\mathsf{I}\times(\mathsf{E}+\mathsf{I}\times\mathsf{List}_2^*)}{\underbrace{\mu\alpha.\,(\mathsf{E}+\mathsf{I}\times\alpha)}_{\equiv\ \mathsf{List}_1^*}=\underbrace{\mu\alpha.\,(\mathsf{E}+\mathsf{I}\times(\mathsf{E}+\mathsf{I}\times\alpha))}_{\equiv\ \mathsf{List}_2^*}}\,\text{FOLD}_{l/r}
}\,+/\text{FIX},\ \boldsymbol{u}
$$

derivation in $\mathbf{HB}_0^{=}$

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consistency-unfolding in $\mathbf{AK}_0^{=}$

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\mathsf{List}_1^*=\mathsf{List}_2^* & \\
\hline\hline
(\mathsf{E}+\mathsf{I}\times\mathsf{List}_1^*=\mathsf{E}+\mathsf{I}\times(\mathsf{E}+\mathsf{I}\times\mathsf{List}_2^*))^{\boldsymbol{u}} & \text{FOLD}_{l/r} \\
\hdashline
\mathsf{E}=\mathsf{E} \qquad \mathsf{I}\times\mathsf{List}_1^*=\mathsf{I}\times(\mathsf{E}+\mathsf{I}\times\mathsf{List}_2^*) & +\text{-DECOMP} \\
\hdashline
\mathsf{I}=\mathsf{I} \qquad \mathsf{List}_1^*=\mathsf{E}+\mathsf{I}\times\mathsf{List}_2^* & \times\text{-DECOMP} \\
\hline
\mathsf{E}+\mathsf{I}\times\mathsf{List}_1^*=\mathsf{E}+\mathsf{I}\times\mathsf{List}_2^* & \text{FOLD}_{l/r} \\
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\mathsf{E}=\mathsf{E} \qquad \mathsf{I}\times\mathsf{List}_1^*=\mathsf{I}\times\mathsf{List}_2^* & +\text{-DECOMP} \\
\hdashline
\mathsf{I}=\mathsf{I} \qquad \mathsf{List}_1^*=\mathsf{List}_2^* & \times\text{-DECOMP} \\
\hline\hline
(\mathsf{E}+\mathsf{List}_1^*\times\mathsf{I}=\mathsf{E}+\mathsf{I}\times(\mathsf{E}+\mathsf{I}\times\mathsf{List}_2^*))^{\boldsymbol{u}} &
\end{array}
$$

Answers offered

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- A **network** of proof-transformations:
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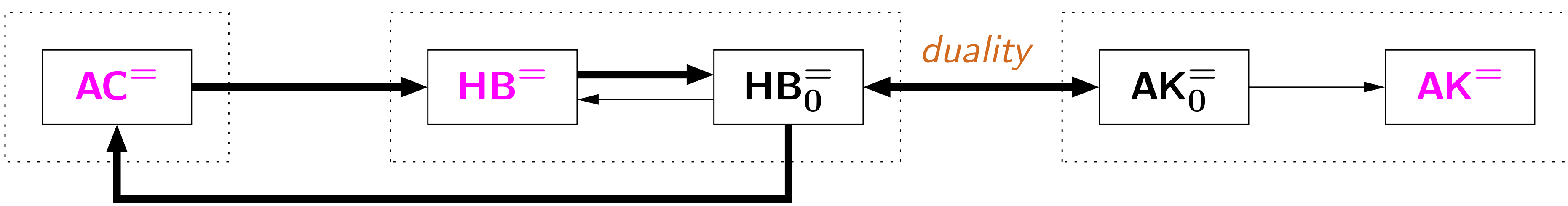
  - A proof-transformation from $\mathbf{AC}^{=}$ to $\mathbf{HB}^{=}$.
  - A proof-transformation from $\mathbf{HB}^{=}$ via $\mathbf{HB}_{\mathbf{0}}^{=}$ to $\mathbf{AC}^{=}$.



The found network of proof-transformations