Regular Expressions Under the Process Interpretation

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(Partly) Joint work with
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Overview

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The Language Interpretation L (Kleene)

$$0 \xmapsto{L} \text{empty set } \emptyset$$

$$1 \xmapsto{L} \{\lambda\} \qquad (\lambda \text{ the empty word})$$

$$a \xmapsto{L} \{a\}$$

$$e + f \xmapsto{L} \text{union of } L(e) \text{ and } L(f)$$

$$e \cdot f \xmapsto{L} \text{element-wise concatenation of } L(e) \text{ and } L(f)$$

$$e^{*} \xmapsto{L} \text{set of “words over of } L(e)\text{”}$$

The Process Interpretation $\boldsymbol{P}$ (Milner)

$\mathbf{0} \quad \overset{\boldsymbol{P}}{\longmapsto} \quad$ deadlock δ

$\mathbf{1} \quad \overset{\boldsymbol{P}}{\longmapsto} \quad$ empty process ϵ

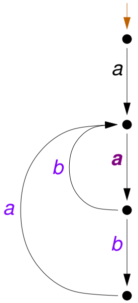
$a \quad \overset{\boldsymbol{P}}{\longmapsto} \quad$ atomic action a

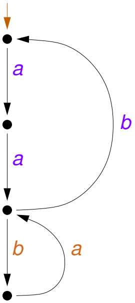
$e + f \quad \overset{\boldsymbol{P}}{\longmapsto} \quad$ alternative composition between $\boldsymbol{P}(e)$ and $\boldsymbol{P}(f)$

$e \cdot f \quad \overset{\boldsymbol{P}}{\longmapsto} \quad$ sequential composition of $\boldsymbol{P}(e)$ and $\boldsymbol{P}(f)$

$e^{*} \quad \overset{\boldsymbol{P}}{\longmapsto} \quad$ unbounded iteration of $\boldsymbol{P}(e)$

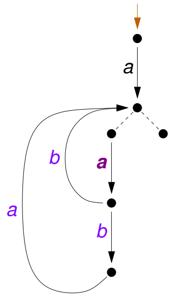
The Process Interpretation P

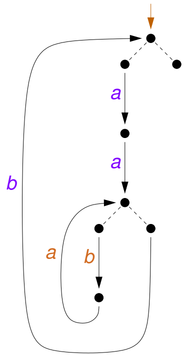




$$P\left(a\cdot\left(a\cdot(b+b\cdot a)\right)^{*}\cdot 0\right) \qquad P\left(\left(a\cdot a\cdot(b\cdot a)^{*}\cdot b\right)^{*}\cdot 0\right)$$

The Process Interpretation *P*





$$P\big(a\cdot(a\cdot(b+b\cdot a))^{*}\cdot 0\big) \qquad P\big((a\cdot a\cdot(b\cdot a)^{*}\cdot b)^{*}\cdot 0\big)$$

The Process Interpretation P (Transition System)

$$\frac{}{P(a) \xrightarrow{a} 1} \qquad \frac{}{1\downarrow}$$

$$\frac{P(e) \xrightarrow{a} P(e')}{P(e+f) \xrightarrow{a} P(e')} \qquad \frac{P(e)\downarrow}{P(e+f)\downarrow}$$

$$\frac{P(f) \xrightarrow{a} P(f')}{P(e+f) \xrightarrow{a} P(f')} \qquad \frac{P(f)\downarrow}{P(e+f)\downarrow} \qquad \frac{P(e)\downarrow \quad P(f)\downarrow}{P(e \cdot f)\downarrow}$$

$$\frac{P(e) \xrightarrow{a} P(e')}{P(e \cdot f) \xrightarrow{a} P(e' \cdot f)} \qquad \frac{P(e)\downarrow \quad P(f) \xrightarrow{a} P(f')}{P(e \cdot f) \xrightarrow{a} P(f')}$$

$$\frac{P(e) \xrightarrow{a} P(e')}{P(e^*) \xrightarrow{a} P(e' \cdot e^*)} \qquad \frac{}{P(e^*)\downarrow}$$

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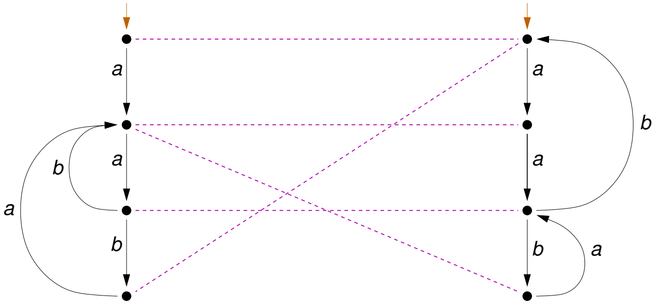
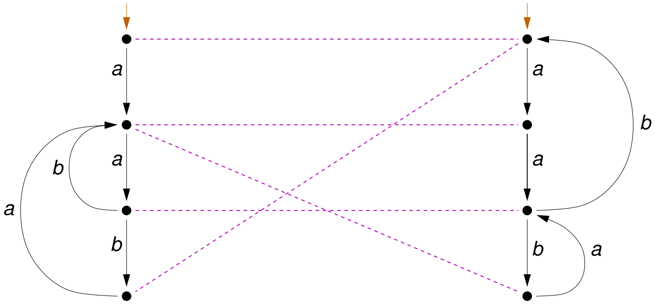
$$\frac{P(e) \xrightarrow{a} P(e')}{P(e^*) \xrightarrow{a} P(e' \cdot e^*)} \qquad \frac{}{P(e^*){\downarrow}}$$

Regular Expressions under Bisimulation



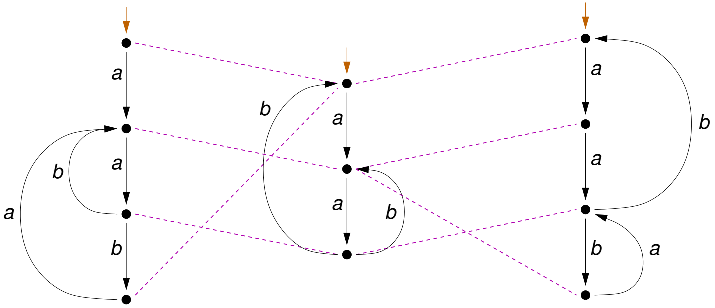
$$\boldsymbol{P}\big(a(a(b+ba))^*.0\big) \quad \Leftrightarrow \quad \boldsymbol{P}\big((aa(ba)^*a)^*.0\big)$$

Regular Expressions under Bisimulation



$$\big(a(a(b+ba))^*.0 \quad \underline{\leftrightarrow}_{P} \quad (aa(ba)^*a)^*.0$$

Regular Expressions under Bisimulation

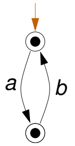


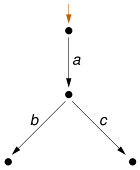
"*Two-exit iteration*"

$\notin im(\boldsymbol{P})$

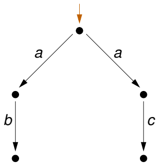
Properties of the Process Interpretation P

- There are finite transition graphs that are *not isomorpic* to any process graph $P(e)$ in the image of P.
- *What is more:* there are finite transition graphs that are *not bisimilar* to any process graph $P(e)$ in the image of P.
- Identities $e \leftrightarrow_P f$ under P also hold as identities $e =_L f$ under the language intepretation L. The converse is false:





$\not\leftrightarrow$



$$a \cdot (b + c) \quad \not\leftrightarrow_P \quad a \cdot b + a \cdot c$$

Milner's Questions (1984)

1. *Is a variant of Salomaa's axiomatization for language equivalence $=_L$ complete for $\leftrightarrow_P$?*

   – To my knowledge: Yet unsolved. (Partial & related results by Sewell; Fokkink; Corradini/De Nicola/Labella; G.)

2. *What structural property characterises the finite-state proc's that are bisimilar to proc's in the image of P ?*

   – Definiability by "well-behaved" specifications ([BC05]); this is decidable ([BCG05]).

3. *Does "minimal star height" over single-letter alphabets define a hierarchy modulo $\leftrightarrow_P$?*

   – Yes! (Hirshfeld and Moller, 1999).

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The Axiom System **REG** for $=_L$ (Salomaa's Axiomatization $\mathbf{F_1}$ reversed)

Axioms :

(B1)	$x + (y + z) = (x + y) + z$	(B7)	$x \cdot 1 = x$
(B2)	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	(B8)	$x \cdot 0 = 0$
(B3)	$x + y = y + x$	(B9)	$x + 0 = x$
(B4)	$(x + y) \cdot z = x \cdot z + y \cdot z$	(B10)	$x^* = 1 + x \cdot x^*$
(B5)	$x \cdot (y + z) = x \cdot y + x \cdot z$	(B11)	$x^* = (1 + x)^*$
(B6)	$x + x = x$		

Inference rules : equational logic *plus*

$$\frac{e = f \cdot e + g}{e = f^* \cdot g}\ \text{FIX}\ (\text{if } \lambda \notin \boldsymbol{L}(f))$$

Sound and Unsound Axioms of **REG** w.r.t. $\underline{\leftrightarrow}_P$

(B1) $x + (y + z) = (x + y) + z$

(B2) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

(B3) $x + y = y + x$

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Also sound are:

$$0 \cdot x = 0 \qquad \frac{e = f \cdot e + g}{e = f^* \cdot g}\ \text{FIX (if } \lambda \notin L(f))$$

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Milner's Adaptation for $\underline{\leftrightarrow}_P$: $\mathbf{BPA}^*_{0,1}$ +1-RSP

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Overview

The Expressibility Problem

A finite-state process p is called
expressible as a regular expression under $\boldsymbol{P}$

iff

there exists $e \in RegExps$ such that $p \leftrightarrows \boldsymbol{P}(e)$.

The Expressibility Problem for $\boldsymbol{P}$
Instance: p a finite-state process
Question: Is p expressible as a regular expression under $\boldsymbol{P}$?

Well-Behaved Specifications (Motivation): A Correspondence Theorem

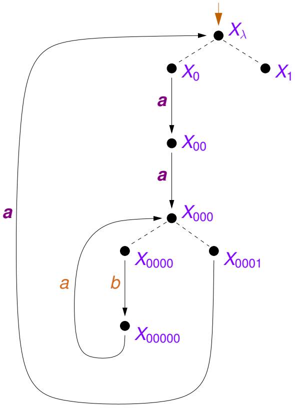
Theorem ([BC05])

Expressibility as a regular expression under ***P***
is equivalent to
definability by a well-behaved specification:

For all processes p ,

$$(\exists\, e \in \mathit{RegExps})\left[\, p \leftrightarrow \boldsymbol{P}(e) \,\right] \quad\Leftrightarrow\quad (\exists\, \mathcal{E} \in \mathit{WBSpecs})\left[\, p \text{ is a solution of } \mathcal{E} \,\right]$$

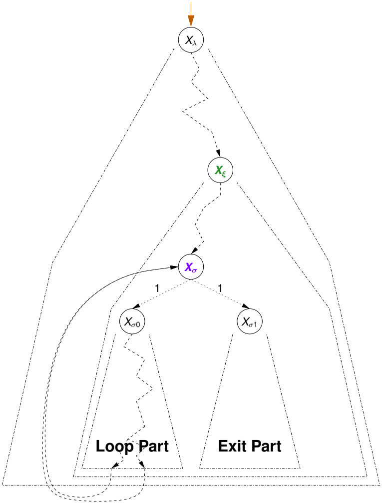
Well-Behaved Specifications (Example)



$$X_\lambda = 1 \cdot X_0 + 1 \cdot X_1$$
$$X_0 = \boldsymbol{a} \cdot X_{00}$$
$$X_{00} = \boldsymbol{a} \cdot X_{000}$$
$$X_{000} = 1 \cdot X_{0000} + 1 \cdot X_{0001}$$
$$X_{0000} = b \cdot X_{00000}$$
$$X_{00000} = a \cdot X_{000}$$
$$X_{0001} = \boldsymbol{a} \cdot X_\lambda$$
$$X_1 = 0$$

$$\boldsymbol{P}((\boldsymbol{aa}(ba)^{*}\boldsymbol{a})^{*}.0)$$

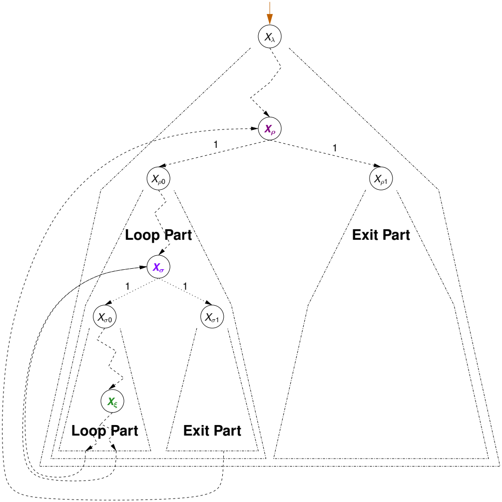
Well-Behaved Specifications (Some Intuition, I)



X_ξ, X_λ … *well-behaved variables*
(X_ξ “does not return” to a recursion variable above itself)

X_σ is a *cycling variable*
(Some recursion variable below X_σ “returns to” X_σ)

Well-Behaved Specifications (Some Intuition, II)



X_σ, X_ρ ... cycling variables

X_ξ *cycles back to* X_σ

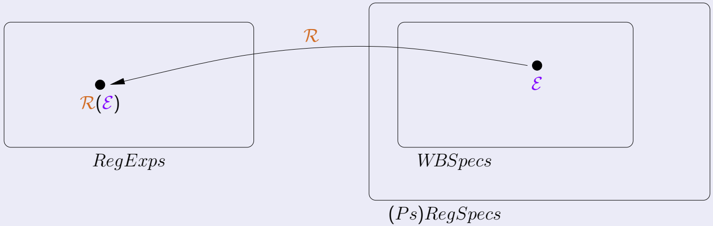
(The nearest return of X_ξ to a rec.var. above is to X_σ)

X_σ *cycles back to* X_ρ

Solvability Lemma

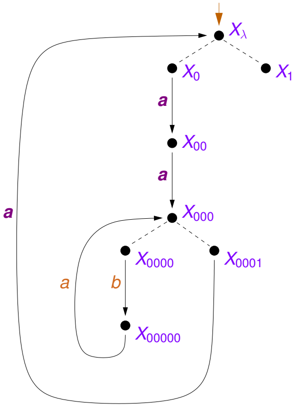
Lemma (Solvability of well-behaved spec's [BC05])

Every well-behaved specification is solved by a process represented by a regular expression.
Moreover: there is an effectively computable mapping $\mathcal{R} : \mathit{WBSpecs}(\Sigma) \to \mathit{RegExps}(\Sigma)$ *such that*



P$(\mathcal{R}(\mathcal{E}))$ *is a solution of* $\mathcal{E}$, *for all* $\mathcal{E} \in \mathit{WBSpecs}(\Sigma))$.

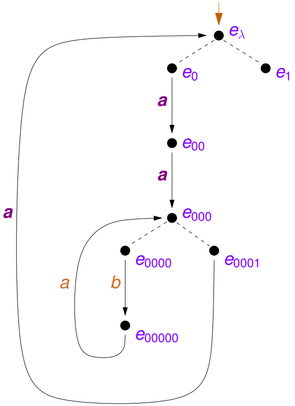
Well-Behaved Specifications (Example)



$P((aa(ba)^*a)^*.0)$

$$
\begin{aligned}
X_\lambda &= 1 \cdot X_0 + 1 \cdot X_1 \\
X_0 &= a \cdot X_{00} \\
X_{00} &= a \cdot X_{000} \\
X_{000} &= 1 \cdot X_{0000} + 1 \cdot X_{0001} \\
X_{0000} &= b \cdot X_{00000} \\
X_{00000} &= a \cdot X_{000} \\
X_{0001} &= a \cdot X_\lambda \\
X_1 &= 0
\end{aligned}
$$

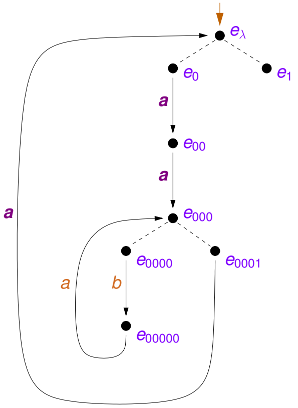
Solving a Well-Behaved Specification (Example, I/III)



$$P((aa(ba)^*a)^*.0)$$

$$
\begin{aligned}
e_\lambda &= 1 \cdot e_0 + 1 \cdot e_1 \\
e_0 &= a \cdot e_{00} \\
e_{00} &= a \cdot e_{000} \\
e_{000} &= 1 \cdot e_{0000} + 1 \cdot e_{0001} \\
e_{0000} &= b \cdot e_{00000} \\
e_{00000} &= a \cdot e_{000} \\
e_{0001} &= a \cdot e_\lambda \\
e_1 &= 0
\end{aligned}
$$

Solving a Well-Behaved Specification (Example, II/III)



$P((aa(ba)^*a)^*.0)$

$$e_{00000} = a \cdot e_{000}$$

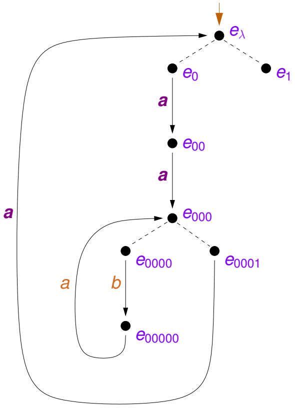
$$e_{0000} = b \cdot e_{00000} = b \cdot a \cdot e_{000}$$

$$e_{0001} = a \cdot e_\lambda$$

$$e_{000} = 1 \cdot e_{0000} + 1 \cdot e_{0001}$$
$$= b \cdot a \cdot e_{000} + a \cdot e_\lambda$$
$$\Rightarrow \quad e_{000} = (b \cdot a)^* \cdot a \cdot e_\lambda$$

(by 1-RSP)

Solving a Well-Behaved Specification (Example, III/III)



$P((aa(ba)^*a)^*.0)$

$$e_{000} = (b \cdot a)^* \cdot a \cdot e_\lambda$$

$$e_{00} = a \cdot e_{000} = a \cdot (b \cdot a)^* \cdot a \cdot e_\lambda$$

$$e_0 = a \cdot e_{00} = a \cdot a \cdot (b \cdot a)^* \cdot a \cdot e_\lambda$$

$$e_1 = 0$$

$$\begin{aligned} e_\lambda &= 1 \cdot e_0 + 1 \cdot e_1 \\ &= 1 \cdot a \cdot a \cdot (b \cdot a)^* \cdot a \cdot e_\lambda + 1 \cdot 0 \\ &= a \cdot a \cdot (b \cdot a)^* \cdot a \cdot e_\lambda + 0 \end{aligned}$$

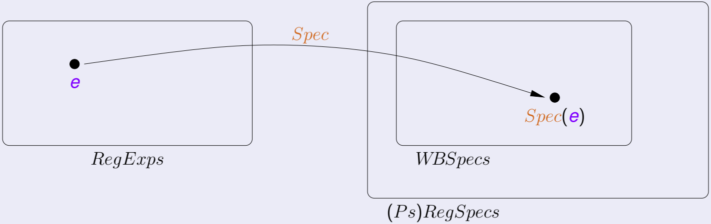
$$\Rightarrow \; e_\lambda = (a \cdot a \cdot (b \cdot a)^* \cdot a)^* \cdot 0$$

(by 1-RSP)

Definability Lemma

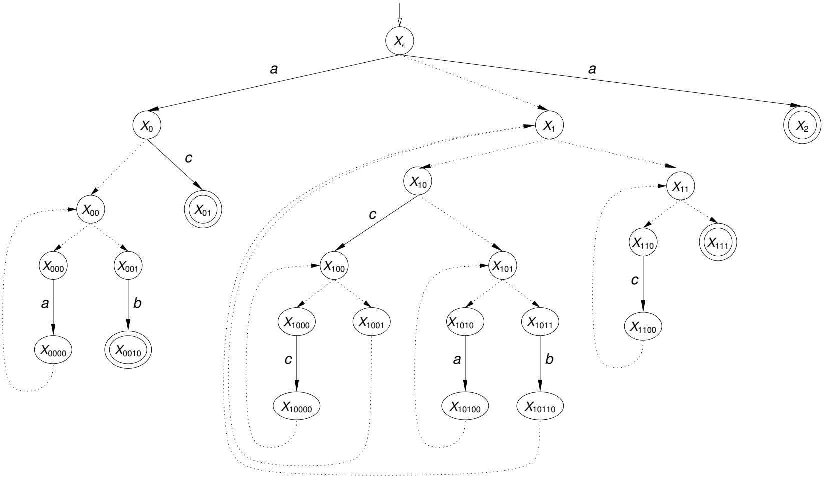
Lemma (Definability by well-behaved spec's [BC05])

*The processes represented by regular expressions under **P** are definable by well-behaved specifications.*
Moreover: there is an effectively computable mapping
$Spec : RegExps(\Sigma) \to WBSpecs(\Sigma)$ *such that*

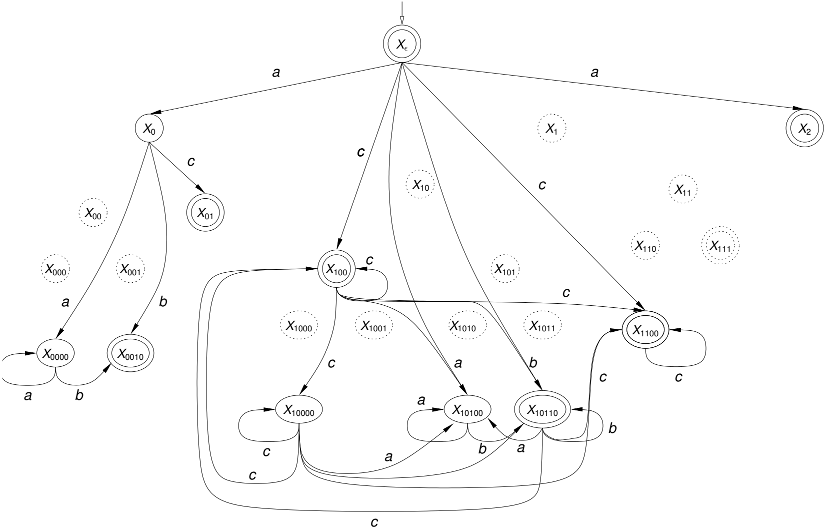


for all $e \in RegExps(\Sigma)$, **P**(e) *is a solution of* $Spec(e)$.

Example: $Spec(a(a^*b + c) + (c^* + a^*b)^* + a)$



Example: Canonical Solution of *Spec*$(a(a^*b + c) + \ldots)$



The Correspondence Theorem

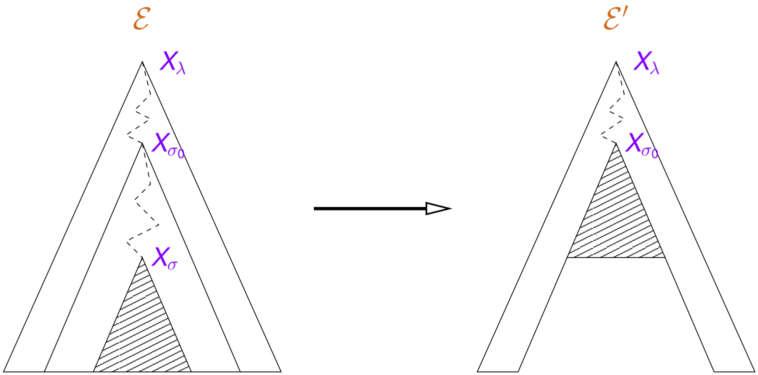
Theorem ([BC05])

Expressibility as a regular expression under ***P***
is equivalent to
definability by a well-behaved specification:

For all processes p ,

$$(\exists\, e \in \mathit{RegExps})\,\big[\, p \leftrightarrows \boldsymbol{P}(e) \,\big] \;\Leftrightarrow\; (\exists\, \mathcal{E} \in \mathit{WBSpecs})\,\big[\, p \text{ is a solution of } \mathcal{E} \,\big]$$

Reducible Well-Behaved Specifications (Example)



$\langle X_\sigma \,|\, \mathcal{E} \rangle \Leftrightarrow \langle X_{\sigma_0} \,|\, \mathcal{E} \rangle$

X_σ, X_{σ_0} are well-behaved

Reducibility Lemma, Decidability Theorem

Lemma (Reducibility of well-behaved spec's [BCG05])

Let $\mathcal{E}$ be a well-behaved specification that has a finite-state process p with n states and maximal branching degree k as a solution.

Then $\mathcal{E}$ is equivalent to a well-behaved specification $\mathcal{E}_{\boldsymbol{red}}$ with

- *depth less or equal to $(n+1)^3 \cdot 2^{3k}$, and*
- *less or equal to k summands in each defining equation.*

Theorem ([BCG05])

*Expressibility by a regular expression under the process interpretation is decidable. In other words, the expressibility problem under **P** is algorithmically solvable.*

Overview

The Star Height Problem

The Star Height Problem for *P*

Instance: $e \in RegExps(\Sigma)$

Question: What is the minimal star height of e under P?

Milner's Star-Height Question

Does "minimal star height" modulo $\leftrightarrows_P$ define a hierarchy also over single-letter alphabets?

Star Height, and Star Height of Regular Languages

The *star height* $sh(e)$ of a regular expression e is the maximum number of nested stars in e.

For example: $sh(\,(a+b)c\,) = 0\,,\;\; sh(\,(a(ba)^{*}a)^{*}dc^{*}\,) = 2\,.$

Definition

The *(restricted) star height* $sh(L)$ of a regular language L is the least natural number n such that $sh(e) = n$ for some regular expression e that represents L.

Generalised Star Height: concerning
generalised regular expressions in which
complementation and *intersection* may occur.

Classical Results on (Restricted) Star Height

1. *Every regular language over a single-letter alphabet has star height 1 at most.*

2. *There are regular languages with any preassigned star height* (Eggan, 1963);
   ... even over a two-letter alphabet (McNaughton, 1965, Dejean/Schützenberger, 1966);

3. *There exists an algorithm for computing the star height of the regular language given by a regular expression* (Hashiguchi, 1983).
   (The (Restricted) Star Height Problem is solvable).

Minimal Star Height under *P*

Definition

The *minimal star height* $msh(e)$ *(under **P**)* of a regular expression e is the least natural number n such that there exists a regular expression e_{min} with $sh(e_{min}) = n$ and $e_{min} \leftrightarrows_{P} e$.

Remark. For all $e \in RegExps$ it holds: $sh(L(e)) \leq msh(e)$.

Results for Minimal Star Height under *P*

1. *For every $n \in \mathbb{N}$, there exists a regular expression f_n over the single-letter alphabet such that the minimal star height of f_n is n* (Hirshfeld/Moller, 2000).
2. Consequently: *For the set regular expressions over a non-empty alphabet, "minimal star height under **P**" defines a proper hierarchy.*
3. The Star-Height Problem under ***P*** is solvable ([BCG05]).

The Star Height Problem under *P*
Instance: $e \in RegExps(\Sigma)$
Question: What is the minimal star height of e under ***P***?

Overview

The Axiomatization Problem(s)

1. Is Milner's adaptation $\mathbf{BPA}^{*}_{\mathbf{0},\mathbf{1}} + \text{1-RSP}$ of Salomaa's complete axiomatization $\mathbf{F_1}$ for $=_L$ complete for $\leftrightarrows_P$?

   Is there a finite extension of $\mathbf{BPA}^{*}_{\mathbf{0},\mathbf{1}} + \text{1-RSP}$ (by additional axioms or rules) that is complete for $\leftrightarrows_P$?

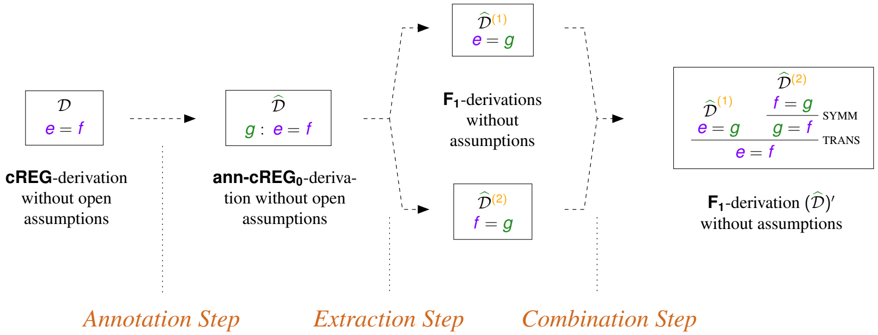
2. Is there a natural-deduction style or sequent-style proof system that is complete for $\leftrightarrows_P$?

Inspiration: A Coinductive/Proof-Theoretic Completeness Proof

In [G05] a coinductive/proof-theoretic proof is given for the completeness of Salomaa's axiomatisation $\mathbf{F_1}$ w.r.t. $=_L$:

1. A characterisation of $=_L$ by a "finitary coinduction principle" (based on "Brzozowski derivatives"):
$$e =_L f \iff e \sim_{\text{fin}} f\,.$$
2. A natural-deduction system **cREG** that is sound and complete with respect to $=_L$ (reminiscent of a system by Brandt/Henglein, 1998).
3. A proof-transformation from **cREG** to Salomaa's complete axiomatisation $\mathbf{F_1}$ of $=_L$.

The Proof-Transformation from **cREG** to $\mathbf{F_1}$



A Coinductive/Proof-Theoretic Completeness Proof for an Extension of Milner's system

Here we describe a similar coinductive/proof-theoretic completeness proof w.r.t. $\leftrightarrows_P$ for an extension of $\mathbf{BPA}^*_{0,1}+$1-RSP by a more powerful rule USP:

1. A characterisation of $\leftrightarrows_P$ by a "finitary coinduction principle" (based on "Antimirov derivatives"):
$$e \leftrightarrows_P f \quad \Longleftrightarrow \quad e \sim_{\text{fin}} f\,.$$
2. A natural-deduction system $\mathbf{c\text{-}BPA}^*_{0,1}$ that is sound and complete with respect to $\leftrightarrows_P$.
3. A proof-transformation from $\mathbf{c\text{-}BPA}^*_{0,1}$ to an extension $\mathbf{BPA}^*_{0,1}+$USP of Milner's system $\mathbf{BPA}^*_{0,1}+$1-RSP.

Antimirov and Brzozowksi Derivatives

Brzozowski deriv's (1963) Antimirov's partial derivatives (1995)

$$(\cdot)_{\cdot} : \mathcal{R}(\Sigma) \times \Sigma \to \mathcal{R}(\Sigma) \qquad \partial : \mathcal{R}(\Sigma) \times \Sigma \to \mathcal{P}_{\mathsf{f}}(\mathcal{R}(\Sigma))$$

$$\langle e, a \rangle \mapsto e_a \qquad \langle e, a \rangle \mapsto \partial_a(e)$$

- Brzozowski der's mimic language derivatives on a synatactic level: $\boldsymbol{L}(e_a) = (\boldsymbol{L}(e))_a \; (=_{\text{def}} \{ v \mid a.v \in \boldsymbol{L}(e) \})$.
- Partial der's are mathematically motivated refinements.
- Both defined syntactically by ind. on the size of reg. expr's.
- Relationship: F.a. $e \in RegExps(\Sigma)$, $e_a \equiv_{\mathbf{ACI}} \sum_{e' \in \partial_a(e)} e'$
- *Every regular expression has only finitely many Brzozowski (Antimirov) derivatives.*

The Coalgebra Induced by Partial Derivatives

Antimirov's partial derivatives induce an F-coalgebra $(\mathit{RegExps}(\Sigma), \langle o, t\rangle)$, for the functor $F(X) = 2 \times \mathcal{P}_{\mathrm{f}}(\Sigma \times X)$ by:

$$\langle o, t\rangle : \mathit{RegExps}(\Sigma) \longmapsto 2 \times \mathcal{P}_{\mathrm{f}}(\Sigma \times \mathit{RegExps}(\Sigma)), \text{ where}$$

$$o : \mathit{RegExps}(\Sigma) \longrightarrow 2$$
$$e \longmapsto o(e) =_{\mathrm{def}} \begin{cases} 0 & \ldots \boldsymbol{P}(e)\not\downarrow \;\; (\lambda \notin \boldsymbol{L}(e)) \\ 1 & \ldots \boldsymbol{P}(e)\downarrow \;\; (\lambda \in \boldsymbol{L}(e)) \end{cases}$$

$$t : \mathit{RegExps}(\Sigma) \longrightarrow \mathcal{P}_{\mathrm{f}}(\Sigma \times \mathit{RegExps}(\Sigma))$$
$$e \longmapsto t(e) =_{\mathrm{def}} \{\langle a, e'\rangle \mid a \in \Sigma,\ e' \in \partial_a e\}\ .$$

$\sim$: bisimilarity on this coalgebra;

$e \sim_{\mathrm{fin}} f$: there is a finite bisimulation between e and f.

Relationship with the Process Interpretation *P*

Lemma

For all $e, f \in RegExps(\Sigma)$ *and* $a \in \Sigma$ *:*

$$\left[\, P(e) \xrightarrow{a} P(f) \iff f \in \partial_a(e) \,\right].$$

A *finitary coinduction principle* (*finite bisimulation principle*):

Theorem

For all $e, f \in RegExps(\Sigma)$ *:*

$$e \leftrightarrows_{P} f \iff e \sim_{fin} f \text{ in } (RegExps(\Sigma), \langle o, t \rangle).$$

The Proof System $\mathbf{c\text{-}BPA}^*_{0,1}$

Inference rule in $\mathbf{c\text{-}BPA}^*_{0,1}$: (Given $\Sigma = \{a_1, \ldots, a_n\}$).

$$\frac{\ldots \quad \begin{array}{c}[e = f]^{\boldsymbol{u}} \\ \mathcal{D}_1^{(i)} \\ e_1^{(i)} = f_1^{(i)}\end{array} \quad \ldots \quad \begin{array}{c}[e = f]^{\boldsymbol{u}} \\ \mathcal{D}_{m_i}^{(i)} \\ e_{m_i}^{(i)} = f_{m_i}^{(i)}\end{array} \quad \ldots}{e = f} \;\text{c-COMP}, \boldsymbol{u} \text{ (if (*))}$$

where (*) demands:

- $o(e) = o(f)$ holds, and
- $\partial_{a_i} e = \{e_1^{(i)}, \ldots, e_{m_i}^{(i)}\}$ and $\partial_{a_i} f = \{f_1^{(i)}, \ldots, f_{m_i}^{(i)}\}$
  (for all $i \in \{1, \ldots, n\}$).

A Derivation in $\mathbf{c\text{-}BPA}^*_{0,1}$

For $e =_{\text{def}} 1\cdot(a\cdot a\cdot(b\cdot a)^*\cdot b)^*\cdot 0$ and $f =_{\text{def}} a\cdot(a\cdot(b+b\cdot a)^*)\cdot 0$, for which $e \leftrightarrows_P f$ holds, we find the following proof in $\mathbf{c\text{-}BPA}^*_{0,1}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{(e_1 = f_1)^{u}}{e = f_3}\,\text{COMP} \qquad \dfrac{(e_2 = f_2)^{v}}{e_3 = f_1}\,\text{COMP}}{e_2 = f_2}\,\text{c-COMP},\, v}{e_1 = f_1}\,\text{c-COMP},\, u}{e = f}\,\text{COMP}$$

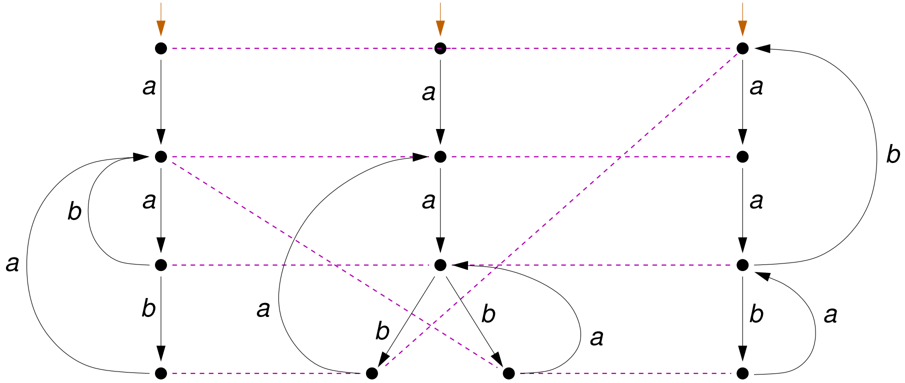
where, in particular,

$$e_2 \equiv 1\cdot(b\cdot a)^*\cdot b\cdot(a\cdot a\cdot(b\cdot a)^*\cdot b)^*\cdot 0\,,$$

$$f_2 \equiv 1\cdot(b + b\cdot a)\cdot(a\cdot(b + b\cdot a))^*\cdot 0\,,$$

$$\partial_b(e_2) = \{e, e_3\} \text{ and } \partial_b(f_2) = \{f_1, f_3\}\,.$$

A Derivation in $\mathbf{c\text{-}BPA}^*_{0,1}$ (the Intuition)



Completeness of $\mathbf{c\text{-}BPA}^*_{0,1}$

Theorem

$\mathbf{c\text{-}BPA}^*_{0,1}$ *is sound and complete w.r.t.* $\leftrightarrows_P$:

$$(\forall e, f \in RegExps(\Sigma)) \left[\vdash_{\mathbf{c\text{-}BPA}^*_{0,1}} e = f \iff e \leftrightarrows_P f \right].$$

Proof.

By the finitary coinduction principle for $\leftrightarrows_P$. □

Reconstructing Regular Expressions from Partial Derivatives

Lemma

Let $\Sigma = \{a_1, \ldots, a_n\}$ *. Then for all* $e \in RegExps(\Sigma)$ *it holds:*

$$\vdash_{\mathbf{BPA}^*_{0,1}} e = o(e) + \sum_{i=1}^{n} \sum_{e' \in \partial_{a_i}(e)} a_i . e' \ .$$

(This statement is reminiscent of the *fundamental theorem of calculus* that links differentiation and integration.)

Unique Solvability Principle(s)

$$\text{1-RSP}\ \frac{x = f.x + g}{x = f^{*}.g}\ \ (\text{if } \lambda \notin \boldsymbol{L}(f))$$

$$\text{1-USP}\ \frac{x = f.x + g \qquad y = f.y + g}{x = y}\ \ (\text{if } \lambda \notin \boldsymbol{L}(f))$$

$$\text{USP}\ \frac{\{\ x_j = E_j(x_1, \ldots, x_m)\ \}_{j=1}^{m} \qquad \{\ y_j = E_j(y_1, \ldots, y_m)\ \}_{j=1}^{m}}{x_i = y_i}$$

where, for all $i \in \{1, \ldots, m\}$,

$E_j(x_1, \ldots, x_m)$ is of the form $[1+] \sum_{k=1}^{m_j} a_{l_k}.x_{l_{j,k}}$.

Transforming into $\mathbf{BPA}^*_{\mathbf{0},\mathbf{1}}+$ USP-der's (Example)

By the "expr's reconstr. lemma", one finds that in the example the vectors $\langle e, e_1, e_2, e, e_3\rangle$ and $\langle f, f_1, f_2, f_3, f_1\rangle$ of reg. expr's satisfy the same system of equations. This enables to extract from the proof in $\mathbf{c\text{-}BPA}^*_{\mathbf{0},\mathbf{1}}$ a proof in $\mathbf{BPA}^*_{\mathbf{0},\mathbf{1}}+$USP:

$$e \overset{\vdots}{=} a.e_1 \qquad e_3 \overset{\vdots}{=} a.e_2 \qquad\qquad f_3 \overset{\vdots}{=} a.f_1 \qquad f_1 \overset{\vdots}{=} a.f_2$$

$$e_2 \overset{\vdots}{=} b.e + b.e_3 \qquad\qquad f_2 \overset{\vdots}{=} b.f_3 + b.f_1$$

$$e_1 \overset{\vdots}{=} a.e_2 \qquad\qquad f_1 \overset{\vdots}{=} a.f_2$$

$$\frac{e \overset{\vdots}{=} a.e_1 \qquad\qquad f \overset{\vdots}{=} a.f_1}{e = f}\ \text{USP}$$

Completeness of $\mathbf{BPA}^{*}_{0,1}$+USP

Theorem

$\mathbf{BPA}^{*}_{0,1}$+USP *is sound and complete w.r.t.* $\leftrightarrows_{P}$:

$$(\forall e, f \in RegExps)\ \big[\ \vdash_{\mathbf{BPA}^{*}_{0,1}+\mathrm{USP}} e = f \iff e \leftrightarrows_{P} f\ \big]\ .$$

Remaining Question (equivalent to Milner's first question):

Is $\mathbf{BPA}^{*}_{0,1}$+1-USP *complete for* $\leftrightarrows_{P}$ *?*

Local Summary

- Antimirov's partial derivatives guide the operational behaviour of regular expressions under ***P***.
- The complete proof system $\mathbf{c\text{-}BPA}^*_{0,1}$ for $\Leftrightarrow_P$ which is based on a "finitary coinduction principle" for $\Leftrightarrow_P$.
- Replacing 1-RSP in Milner's system $\mathbf{BPA}^*_{0,1}$+1-RSP by the *unique solvability principle* USP gives the complete axiomatization $\mathbf{BPA}^*_{0,1}$+USP for $\Leftrightarrow_P$.

Global Summary

- The **Expressibility Problem** for ***P*** is solvable.
- The **Star-Height Problem** for ***P*** is solvable.
- Concerning the **Axiomatisation Problem** for $\Leftrightarrow_{\boldsymbol{P}}$:
  - There is a coinductively motivated, natural-deduction system $\mathbf{c\text{-}BPA}^{*}_{\mathbf{0},\mathbf{1}}$ that is complete for $\Leftrightarrow_{\boldsymbol{P}}$.
  - The system $\mathbf{BPA}^{*}_{\mathbf{0},\mathbf{1}} + \mathrm{USP}$ is complete for $\Leftrightarrow_{\boldsymbol{P}}$ (USP is a *unique solvability principle* for linear systems of equations).
  - Milner's question: "Is $\mathbf{BPA}^{*}_{\mathbf{0},\mathbf{1}} + \text{1-RSP}$ complete for $\Leftrightarrow_{\boldsymbol{P}}$?" is (to my knowledge) still unanswered.

A Further Partial Result. Questions.

Let $\rightarrow_P$ denote the relation functional bisimulation on well-behaved specifications, and $\leftarrow_P$ its converse.

Theorem

Let $e, f \in RegExps(\Sigma)$. Then it holds:

$$Spec(e)\ (\rightarrow_P \cup \leftarrow_P)^*\ Spec(f) \implies \vdash_{\mathbf{BPA}^*_{0,1}+\text{1-RSP}} e = f \qquad (1)$$

Questions:

1. Does the converse of (1) hold? (My Conjecture is: No)
2. What relation on corresponding well-behaved spec's does provability in $\mathbf{BPA}^*_{0,1}$+1-RSP induce?
   (Having a grip on this relation could help to prove/disprove completeness of $\mathbf{BPA}^*_{0,1}$+1-RSP w.r.t. $\leftrightarrow_P$.)

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A Further Partial Result. Questions.

Let $\rightharpoondown_P$ denote the relation functional bisimulation on well-behaved specifications, and $\leftharpoondown_P$ its converse.

Theorem

Let $e, f \in RegExps(\Sigma)$. *Then it holds:*

$$Spec(e)\ (\rightharpoondown_P \cup \leftharpoondown_P)^*\ Spec(f) \implies \vdash_{\mathbf{BPA}^*_{0,1}+\text{1-RSP}} e = f \qquad (1)$$

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