On the Star Height of Regular Expressions Under Bisimulation

J.C.M. Baeten\textsuperscript{1}  F. Corradini\textsuperscript{2}  C. Grabmayer\textsuperscript{3}

\textsuperscript{1}Technische Universiteit Eindhoven  
josb@win.tue.nl

\textsuperscript{2}Universitá di Camerino  
flavio.corradini@unicam.it

\textsuperscript{3}Vrije Universiteit Amsterdam  
clemens@cs.vu.nl

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Overview

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   - Star Height of Regular Languages
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4 Summary
The Process Interpretation $P$ (Milner)

\[
\begin{align*}
0 & \xrightarrow{P} \text{deadlock } \delta \\
1 & \xrightarrow{P} \text{empty process } \epsilon \\
a & \xrightarrow{P} \text{atomic action } a \\
e + f & \xrightarrow{P} \text{alternative composition between } P(e) \text{ and } P(f) \\
e \cdot f & \xrightarrow{P} \text{sequential composition of } P(e) \text{ and } P(f) \\
e^* & \xrightarrow{P} \text{unbounded iteration of } P(e)
\end{align*}
\]
The Process Interpretation

\[ P(a \cdot (a \cdot (b + b \cdot a))^* \cdot 0) \quad P\left( (a \cdot a \cdot (b \cdot a)^* \cdot b)^* \cdot 0 \right) \]
The Process Interpretation $P$

\[
P(a \cdot (a \cdot (b + b \cdot a))^* \cdot 0)
P((a \cdot a \cdot (b \cdot a)^* \cdot b)^* \cdot 0)
\]
Regular Expressions under Bisimulation

\[ P\left( a(a(b + ba))^* \cdot 0 \right) \Leftrightarrow P\left( (aa(ba)^*a)^* \cdot 0 \right) \]
Regular Expressions under Bisimulation

\( (a(a(b + ba))^* \cdot 0 \) \( \Leftrightarrow_P \) \( (aa(ba)^*a)^* \cdot 0 \)
The Process Interpretation $P$ (Transition System)

\[
\begin{align*}
P(a) \xrightarrow{a} 1 \\
P(e) \xrightarrow{a} P(e') \\
P(e + f) \xrightarrow{a} P(e') \\
P(f) \xrightarrow{a} P(f') \\
P(e + f) \xrightarrow{a} P(f') \\
P(e \cdot f) \xrightarrow{a} P(e' \cdot f) \\
P(e^*) \xrightarrow{a} P(e' \cdot e^*) \\
P(e^*) \downarrow \\
\end{align*}
\]
The Process Interpretation $P$ (Transition System)

$P(a) \xrightarrow{a} 1$

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$P(e + f) \xrightarrow{a} P(f')$

$P(e \cdot f) \xrightarrow{a} P(e' \cdot f)$

$P(e) \xrightarrow{a} P(e')$

$P(e^*) \xrightarrow{a} P(e' \cdot e^*)$

$P(e) \xrightarrow{a} P(e')$

$P(e^*) \xrightarrow{a} P(e' \cdot e^*)$
Regular Expressions under Bisimulation

“Two-exit iteration”

\[ \not \in \text{im}(P) \]
Properties of the Process Interpretation $P$

- There are finite transition graphs that are *not isomorphic* to any process graph $P(e)$ in the image of $P$.
- *What is more*: there are finite transition graphs that are *not bisimilar* to any process graph $P(e)$ in the image of $P$.
- Identities $e \Leftrightarrow_P f$ under $P$ also hold as identities $e =_L f$ under the language interpretation $L$. The converse is false:

\[
\begin{align*}
    a \cdot (b + c) & \not\Leftrightarrow_P a \cdot b + a \cdot c
\end{align*}
\]
Milner’s Questions (1984)

1. Is a variant of Salomaa’s axiomatisation for language equality complete for $\Leftrightarrow_p$?
   – To my knowledge: Yet unsolved. (Partial & related results by Sewell; Fokkink; Corradini/De Nicola/Labella; C.G.)

2. What structural property characterises the finite-state proc’s that are bisimilar to proc’s in the image of $P$?
   – Definiability by “well-behaved” specifications ([BC05]); this is decidable ([BCG05]).

3. Does “minimal star height” over single-letter alphabets define a hierarchy modulo $\Leftrightarrow_p$?
   – Yes! (Hirshfeld and Moller, 1999).
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1. **Is a variant of Salomaa’s axiomatisation for language equality complete for \( \equiv_P \)?**
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   - Yes! (Hirshfeld and Moller, 1999).
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   - Definiability by “well-behaved” specifications ([BC05]); this is decidable ([BCG05]).

3. **Does “minimal star height” over single-letter alphabets define a hierarchy modulo \( \Leftrightarrow_p \)?**
   - Yes! (Hirshfeld and Moller, 1999).
The *star height* $sh(e)$ of a regular expression $e$ is the maximum number of nested stars in $e$.

For example: $sh((a + b)c) = 0$, $sh((a(ba)^*a)^*dc^*) = 2$.

**Definition**

The (restricted) *star height* $sh(L)$ of a regular language $L$ is the least natural number $n$ such that $sh(e) = n$ for some regular expression $e$ that represents $L$.

**Generalised Star Height**: concerning *generalised regular expressions* in which *complementation* and *intersection* may occur.
Every regular language over a single-letter alphabet has star height 1 at most.

There are regular languages with any preassigned star height (Eggan, 1963);
... even over a two-letter alphabet (McNaughton, 1965, Dejean/Schützenberger, 1966);

There exists an algorithm for computing the star height of the regular language given by a regular expression (Hashiguchi, 1983). (The (Restricted) Star Height Problem is solvable).
Definition

The **minimal star height** $msh(e)$ (under $P$) of a regular expression $e$ is the least natural number $n$ such that there exists a regular expression $e_{\text{min}}$ with $sh(e_{\text{min}}) = n$ and $e_{\text{min}} \leftrightarrow_P e$.

Remark. For all $e \in RegExps$ it holds: $sh(L(e)) \leq msh(e)$.
Results for Minimal Star Height under $P$?

1. For every $n \in \mathbb{N}$, there exists a regular expression $f_n$ over the single-letter alphabet such that the minimal star height of $f_n$ is $n$ (Hirshfeld/Moller, 2000).

2. Consequently: For the set regular expressions over a non-empty alphabet, “minimal star height under $P$” defines a proper hierarchy.

3. Is the Star-Height Problem under $P$ solvable?

The Star Height Problem under $P$

Instance: $e \in \text{RegExps}(\Sigma)$

Question: What is the minimal star height of $e$ under $P$?
Well-Behaved Specifications (Motivation):
A Correspondence Theorem

**Theorem ([BC05])**

*Expressibility as a regular expression under $P$ is equivalent to definability by a well-behaved specification:*

For all processes $p$, 

$$(\exists e \in \text{RegExp}) \left[ p \iff P(e) \right] \iff (\exists \mathcal{E} \in \text{WBSpecs}) \left[ p \text{ is a solution of } \mathcal{E} \right]$$
Well-Behaved Specifications (Example)

\[ P((aa(ba)^*a)^*.0) \]

\[ X_\lambda = 1 \cdot X_0 + 1 \cdot X_1 \]
\[ X_0 = a \cdot X_{00} \]
\[ X_{00} = a \cdot X_{0000} \]
\[ X_{0000} = 1 \cdot X_{00000} + 1 \cdot X_{00001} \]
\[ X_{00000} = b \cdot X_{000000} \]
\[ X_{000000} = a \cdot X_{0000} \]
\[ X_{00001} = a \cdot X_\lambda \]
\[ X_1 = 0 \]
Well-Behaved Specifications (Some Intuition, I)

$X_\xi, X_\lambda \ldots$ well-behaved variables
($X_\xi$ “does not return” to a recursion variable above itself)

$X_\sigma$ is a cycling variable
(Some recursion variable below $X_\sigma$ “returns to” $X_\sigma$)
Well-Behaved Specifications (Some Intuition, II)

\[ X_\sigma, X_\rho \ldots \text{cycling variables} \]

\[ X_\xi \text{ cycles back to } X_\sigma \]

(The nearest return of \( X_\xi \) to a rec.var. above is to \( X_\sigma \))

\[ X_\sigma \text{ cycles back to } X_\rho \]

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Star Height of Regular Expressions under Bisimulation
Lemma (Definability by well-behaved spec’s [BC05])

The processes represented by regular expressions under $P$ are definable by well-behaved specifications. Moreover: there is an effectively computable mapping $Spec : RegExps(\Sigma) \rightarrow WBSpecs(\Sigma)$ such that

$Spec(e)$ for all $e \in RegExps(\Sigma)$,

$P(e)$ is a solution of $Spec(e)$. 

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Lemma (Solvability of well-behaved spec's [BC05])

Every well-behaved specification is solved by a process represented by a regular expression. Moreover: there is an effectively computable mapping \( \mathcal{R} : WBSpecs(\Sigma) \rightarrow RegExps(\Sigma) \) such that

\[ P(\mathcal{R}(\mathcal{E})) \text{ is a solution of } \mathcal{E}, \quad \text{for all } \mathcal{E} \in WBSpecs(\Sigma). \]
The Correspondence Theorem

**Theorem ([BC05])**

Expressibility as a regular expression under $P$ is equivalent to definability by a well-behaved specification:

For all processes $p$,

$$(\exists e \in \text{RegExps}) \left[ p \Leftrightarrow P(e) \right] \Leftrightarrow (\exists E \in \text{WBSpecs}) \left[ p \text{ is a solution of } E \right]$$
Reducible Well-Behaved Specifications (Example)

\[ \langle X_\sigma | \mathcal{E} \rangle \Leftrightarrow \langle X_{\sigma_0} | \mathcal{E} \rangle \]

\(X_\sigma, X_{\sigma_0}\) are well-behaved
Lemma (Reducibility of well-behaved spec’s [BCG05])

Let $\mathcal{E}$ be a well-behaved specification that has a finite-state process $p$ with $n$ states and maximal branching degree $k$ as a solution.

Then $\mathcal{E}$ is equivalent to a well-behaved specification $\mathcal{E}_{\text{red}}$ with

- depth less or equal to $(n + 1)^3 \cdot 2^{3k}$, and
- less or equal to $k$ summands in each defining equation.

Theorem ([BCG05])

Expressibility by a regular expression under the process interpretation is decidable.
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Solution of the Star Height Problem

Four Steps:

1. Introduction of the notion “star height” for well-behaved specifications.

2. Refined versions of the Definability, Solvability, and Reducibility Lemmas.

3. The algorithm $\text{CSH}$ for computing the minimal star height under $P$ of a regular expression.

4. Correctness Proof for the algorithm $\text{CSH}$. 
Definition

The **star height** $sh(\mathcal{E})$ of a well-behaved specification $\mathcal{E}$ is the maximum number of nested cycling variables in $\mathcal{E}$.

\[
\begin{align*}
sh(\mathcal{E}_1) &= 1 \\
sh(\mathcal{E}_2) &= 2
\end{align*}
\]
Lemma (Definability by well-behaved spec’s)

There is an effectively computable mapping
\( Spec : \text{RegExprs} \rightarrow \text{WBSpecs} \) such that

\[
\text{for all } E \in \text{WBSpecs}, \quad P(e) \text{ is a solution of } Spec(e),
\]
\[
\text{and } sh(Spec(e)) = sh(e),
\]
Lemma (Solvability of well-behaved spec’s)

There is an effectively computable mapping
\( \mathcal{R} : \text{WBSpecs} \to \text{RegExps} \) such that

\[ P(\mathcal{R}(\varepsilon)) \text{ is a solution of } \varepsilon, \]
\[ \text{and } \text{sh}(\mathcal{R}(\varepsilon)) = \text{sh}(\varepsilon), \]

for all \( \varepsilon \in \text{WBSpecs} \).
Refined Reducibility Lemma

Lemma (Reducibility of well-behaved spec’s)

Let $E$ be a well-behaved specification that has a finite-state process $p$ with $n$ states and maximal branching degree $k$ as a solution.

Then $E$ is equivalent to a well-behaved specification $E_{\text{red}}$ with

- depth less or equal to $(n + 1)^3 \cdot 2^{3k}$,
- less or equal to $k$ summands in each defining equation,
- and $sh(E_{\text{red}}) \leq sh(E)$.
The Algorithm CSH (Step CSH1)

\[ P(e) \text{ solves } \mathcal{E}, \quad sh(\mathcal{E}) = sh(e) \]
(by the Ref.Def.Lemma)
The Algorithm CSH (Step CSH2)

\[ P(e) \text{ solves } \mathcal{E}, \quad \text{sh}(\mathcal{E}) = \text{sh}(e) \]

(by the Ref. Def. Lemma)

\[ P(e) : n \text{ states, } \leq k \text{ branch.degr.} \]
The Algorithm CSH (Step CSH3)

\[ P(e_{\text{min}}) \text{ solves } \mathcal{E}_{\text{min}}, \quad sh(e_{\text{min}}) = \]
\[ = sh(\mathcal{E}_{\text{min}}) \quad \text{(by the Ref.Solv.Lemma)} \]
\[ e_{\text{min}} \Leftrightarrow_P e \]
\[ \text{min. star height of } e = sh(e_{\text{min}}) = n_0 \]

\[ P(e) \text{ solves } \mathcal{E}, \quad sh(\mathcal{E}) = sh(e) \]
\[ \text{(by the Ref.Def.Lemma)} \]
\[ P(e) : n \text{ states, } \leq k \text{ branch.degr.} \]

\[ \mathcal{E}_{\text{min}} \sim \mathcal{E}, \quad sh(\mathcal{E}_{\text{min}}) = \text{min!} \]
Let \( f \in \text{RegExps} \) be arbitrary with \( f \equiv_{P} e \). Then it follows for \( \mathcal{F} = \text{Spec}(f) \) by the Ref.Def.Lem.: \( \text{sh}(\mathcal{F}) = \text{sh}(f) \), \( P(f) \) and also \( P(e) \) are solutions of \( \mathcal{F} \). By the Ref.Red.Lem., a reduced specification \( \mathcal{F}_{\text{red}} \) exists with solution \( P(e) \) and \( \text{sh}(\mathcal{F}) \geq \text{sh}(\mathcal{F}_{\text{red}}) \). Then by the choice of \( \mathcal{E}_{\text{min}} \) and \( e_{\text{min}} \) it follows:

\[
\text{sh}(f) = \text{sh}(\mathcal{F}) \geq \text{sh}(\mathcal{F}_{\text{red}}) \geq \text{sh}(\mathcal{E}_{\text{min}}) = \text{sh}(e_{\text{min}}) = n_{0}.
\]

Hence: \( n_{0} = \text{sh}(e_{\text{min}}) = m\text{sh}(e) \).
Let $f \in \text{RegExps}$ be arbitrary with $f \leftrightarrow_{\mathcal{P}} e$. Then it follows for $\mathcal{F} = \text{Spec}(f)$ by the Ref.Def.Lem.: $\text{sh}(\mathcal{F}) = \text{sh}(f)$, $\mathcal{P}(f)$ and also $\mathcal{P}(e)$ are solutions of $\mathcal{F}$. By the Ref.Red.Lem., a reduced specification $\mathcal{F}_{\text{red}}$ exists with solution $\mathcal{P}(e)$ and $\text{sh}(\mathcal{F}) \geq \text{sh}(\mathcal{F}_{\text{red}})$. Then by the choice of $\mathcal{E}_{\text{min}}$ and $e_{\text{min}}$ it follows:

$$\text{sh}(f) = \text{sh}(\mathcal{F}) \geq \text{sh}(\mathcal{F}_{\text{red}}) \geq \text{sh}(\mathcal{E}_{\text{min}}) = \text{sh}(e_{\text{min}}) = n_0.$$  

Hence: $n_0 = \text{sh}(e_{\text{min}}) = m\text{sh}(e)$. 

\[\begin{array}{c}
\text{RegExps} \\
\mathcal{F} = \text{Spec}(f) \\
\mathcal{E} = \text{Spec}(e) \\
\mathcal{E}_{\text{min}} = R(\mathcal{E}_{\text{min}}) \\
e_{\text{min}} = R(e_{\text{min}}) \\
\mathcal{F}_{\text{red}} \\
\end{array}\]
Let \( f \in \text{RegExps} \) be arbitrary with \( f \not\equiv \!\!_{P} e \). Then it follows for \( \mathcal{F} = \text{Spec}(f) \) by the Ref.Def.Lem.: \( \text{sh}(\mathcal{F}) = \text{sh}(f) \), \( P(f) \) and also \( P(e) \) are solutions of \( \mathcal{F} \). By the Ref.Red.Lem., a reduced specification \( \mathcal{F}_{\text{red}} \) exists with solution \( P(e) \) and \( \text{sh}(\mathcal{F}) \geq \text{sh}(\mathcal{F}_{\text{red}}) \). Then by the choice of \( \mathcal{E}_{\text{min}} \) and \( e_{\text{min}} \) it follows:

\[
\text{sh}(f) = \text{sh}(\mathcal{F}) \geq \text{sh}(\mathcal{F}_{\text{red}}) \geq \text{sh}(\mathcal{E}_{\text{min}}) = \text{sh}(e_{\text{min}}) = n_{0} .
\]

Hence: \( n_{0} = \text{sh}(e_{\text{min}}) = m\text{sh}(e) \).
Let $f \in \text{RegExps}$ be arbitrary with $f \Leftrightarrow_P e$. Then it follows for $\mathcal{F} = \text{Spec}(f)$ by the Ref.Def.Lem.: $sh(\mathcal{F}) = sh(f)$, $P(f)$ and also $P(e)$ are solutions of $\mathcal{F}$. By the Ref.Red.Lem., a reduced specification $\mathcal{F}_{\text{red}}$ exists with solution $P(e)$ and $sh(\mathcal{F}) \geq sh(\mathcal{F}_0)$. Then by the choice of $\mathcal{E}_{\text{min}}$ and $e_{\text{min}}$ it follows:

$$sh(f) = sh(\mathcal{F}) \geq sh(\mathcal{F}_{\text{red}}) \geq sh(\mathcal{E}_{\text{min}}) = sh(e_{\text{min}}) = n_0.$$ 

Hence: $n_0 = sh(e_{\text{min}}) = msh(e)$.
Let $f \in \text{RegExps}$ be arbitrary with $f \Leftrightarrow P e$. Then it follows for $\mathcal{F} = \text{Spec}(f)$ by the Ref.Def.Lem.: $sh(\mathcal{F}) = sh(f)$, $P(f)$ and also $P(e)$ are solutions of $\mathcal{F}$. By the Ref.Red.Lem., a reduced specification $\mathcal{F}_{\text{red}}$ exists with solution $P(e)$ and $sh(\mathcal{F}) \geq sh(\mathcal{F}_0)$. Then by the choice of $\mathcal{E}_{\text{min}}$ and $e_{\text{min}}$ it follows:

$$sh(f) = sh(\mathcal{F}) \geq sh(\mathcal{F}_{\text{red}}) \geq sh(\mathcal{E}_{\text{min}}) = sh(e_{\text{min}}) = n_0.$$  

Hence: $n_0 = sh(e_{\text{min}}) = msh(e)$.  

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The Star Height Problem is Solvable

**Theorem**

*The star-height problem for the process interpretation is solvable: there is an algorithm that, for all regular expressions $e$, on input $e$ computes the minimal star height of $e$.***

**Remark.** This is a theoretical result, which (on its own) does not allow to show a better than double-exponential time bound on a naive decision algorithm extracted from the proof.
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Milner’s Conjecture on Star-Height

Let the sequence \( \{f_n\} \) be defined as

\[
f_1 = a^* \quad f_{n+1} = (f_n \cdot a)^* .
\]

Then \( P(f_1), P(f_2), P(f_3), P(f_4) \) are of the forms:

Conjecture (Milner). *The minimal star height of \( f_n \) is \( n \).*
Well-Behaved Spec’s with solutions $P(f_1)$, $P(f_2)$, $P(f_3)$

\[
X_\lambda^{(1)} = 1 \cdot X_0^{(1)} + 1 \cdot X_1^{(1)}
\]

\[
X_0^{(1)} = a \cdot X_\lambda^{(\lambda)}
\]

\[
X_1^{(1)} = 1
\]
Alternative Proof of Milner’s Conjecture (I)

**Theorem (Hirshfeld and Moller, 2000)**

For every \( n \), there exists a regular expression \( f_n \) over the single-letter alphabet \( \{a\} \) such that the minimal star height of \( f_n \) under the process interpretation is \( n \). This is witnessed by the sequence \( \{f_n\}_n \) in Milner’s Conjecture.

**Lemma (Main Lemma)**

For every well-behaved specification \( \mathcal{E} \) that has \( P(f_n) \) as a solution, \( sh(\mathcal{E}) \geq n \) holds.

**Hint at the Proof.**

A careful analysis of well-behaved specifications \( \mathcal{E} \) that have \( P(f_n) \) as a solution.
Alternative Proof of Milner’s Conjecture (I)

Theorem (Hirshfeld and Moller, 2000)

For every $n$, there exists a regular expression $f_n$ over the single-letter alphabet $\{a\}$ such that the minimal star height of $f_n$ under the process interpretation is $n$. This is witnessed by the sequence $\{f_n\}_n$ in Milner’s Conjecture.

Lemma (Main Lemma)

For every well-behaved specification $\mathcal{E}$ that has $P(f_n)$ as a solution, $sh(\mathcal{E}) \geq n$ holds.

Hint at the Proof.

A careful analysis of well-behaved specifications $\mathcal{E}$ that have $P(f_n)$ as a solution.
Alternative Proof of Milner’s Conjecture (II)

Lemma (Main Lemma)

For every well-behaved specification $\mathcal{E}$ that has $P(f_n)$ as a solution, $sh(\mathcal{E}) \geq n$ holds.

(Alternative) Proof of the Theorem.

Let $n \in \mathbb{N} \setminus \{0\}$ arbitrary. It suffices to show that $sh(f_n) = n$.

Let $e \in \text{RegExps}\{a\}$ such that $e \leftrightarrow_P f_n$. Then by the Ref.Def.Lemma there exists a well-behaved specification $\mathcal{E}$ with $sh(\mathcal{E}) = sh(e)$ such that $P(e)$, and also $P(f)$ is a solution of $\mathcal{E}$.

By the Lemma $sh(\mathcal{E}) \geq n$ follows, entailing $sh(e) \geq n$.

Hence the minimal star height of $f_n$ is $n$. 

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Star Height of Regular Expressions under Bisimulation
Lemma (Main Lemma)

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Let $n \in \mathbb{N} \setminus \{0\}$ arbitrary. It suffices to show that $sh(f_n) = n$.

Let $e \in RegExps(\{a\})$ such that $e \leftrightarrow_{P} f_n$. Then by the Ref.Def.Lemma there exists a well-behaved specification $\mathcal{E}$ with $sh(\mathcal{E}) = sh(e)$ such that $P(e)$, and also $P(f)$ is a solution of $\mathcal{E}$.

By the Lemma $sh(\mathcal{E}) \geq n$ follows, entailing $sh(e) \geq n$.

Hence the minimal star height of $f_n$ is $n$. \qed
Lemma (Main Lemma)

For every well-behaved specification $\mathcal{E}$ that has $P(f_n)$ as a solution, $sh(\mathcal{E}) \geq n$ holds.

(Alternative) Proof of the Theorem.

Let $n \in \mathbb{N} \setminus \{0\}$ arbitrary. It suffices to show that $sh(f_n) = n$.

Let $e \in \text{RegExps} \{a\}$ such that $e \leftrightarrow_{P} f_n$. Then by the Ref.Def.Lemma there exists a well-behaved specification $\mathcal{E}$ with $sh(\mathcal{E}) = sh(e)$ such that $P(e)$, and also $P(f)$ is a solution of $\mathcal{E}$.

By the Lemma $sh(\mathcal{E}) \geq n$ follows, entailing $sh(e) \geq n$.

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   - Alternative Proof of Milner’s Conjecture

4. Summary
We consider regular expressions under Milner’s process interpretation.

We use the correspondence with “well-behaved” specifications from [BC05] to show:

- The Star Height Problem for Regular Expressions under Milner’s Process Interpretation is solvable. (Using the reducibility lemma for well-behaved spec’s from [BCG05].)

- For regular expressions over a single-letter alphabet, minimal star height w.r.t. the process interpretation defines a proper hierarchy (Hirshfeld/Moller, 2000).
Questions for Further Research

1. How could a better algorithm for the star-height problem look like?

2. Is there a relationship with known decision algorithms for the (restricted) star-height problem for regular languages?

3. Are there appealing interpretations for (generalised) regular expressions (allowing \textit{complementation and intersection} operators) in process theory?

4. Is it possible, to find, for all $e \in \text{RegExp}$ an $e_{\text{min}} \in \text{RegExp}$ of minimal star height such that $e_{\text{min}} \Leftrightarrow_P e$ and $e_{\text{min}} = e$ \textit{is provable} in Milner’s adaptation for $P$ of Salomaa’s axiomatisation for $L$?
Example: $\text{Spec}(a(a^*b + c) + (c^* + a^*b)^* + a)$
Example: Canonical Solution of $\text{Spec}(a(a^*b + c) + \ldots)$
References

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