Derivability and Admissibility of Inference Rules in Abstract Hilbert Systems

Clemens Grabmayer

Department of Computer Science, Vrije Universiteit Amsterdam,
de Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands,
clemens@cs.vu.nl,
WWW home page: http://www.cs.vu.nl/~clemens

Abstract. We give an overview of results, presented in the form of a poster at CSL03/KGC, about a general study of the notions of rule derivability and admissibility in Hilbert-style proof systems. The basis of our investigation consists in the concept of "abstract Hilbert system", a framework for Hilbert-style proof systems in which it is abstracted from the syntax of formulas and the operational content of rules. We adapt known definitions of rule derivability and admissibility to abstract Hilbert systems, propose two variant notions of rule derivability, s-derivability and m-derivability, and investigate how these four notions are related. Furthermore, we consider relations that compare abstract Hilbert systems with respect to rule admissibility or with respect to one of the three notions of rule derivability, and study their interrelations. Finally, we report of a theorem that describes a correspondence between abstract notions of rule elimination and the notions of rule admissibility and derivability.

This paper intends to give a short overview of results presented in the form of a poster with the same title at the 8th Kurt Gödel Colloquium that was jointly held with the conference CSL 2003 in Vienna, Austria, August 25–30, 2003. As it was the case for the poster, also this overview follows closely the report [2] to which we refer the reader for the proofs, for more details and for related results.

1 Introduction

The notions of derivability and admissibility of inference rules are usually studied in the context of concrete systems of formal logic. A rule R is generally called 'derivable' (or 'derived') in a formal system $\mathcal{S}$ if all applications (instances) of R can, in some sense, be mimicked by appropriate derivations in $\mathcal{S}$. And a rule R is called 'admissible' in a formal system $\mathcal{S}$ if the class of theorems of $\mathcal{S}$ is closed under applications of R. Our aim is to collect a number of basic results about these notions that are applicable to all Hilbert-style systems of simplest kind.

By this we mean systems, sometimes just called 'axiom systems', in which each rule application α within a derivation $\mathcal{D}$ is the inference of a single conclusion from a finite sequence of premises; a rule application α does not depend on the presence or absence of assumptions in subderivations of $\mathcal{D}$ leading to α. There is a correspondence (cf. [2, Appendix E]) between systems of this nature, endowed with consequence relations as considered below, and 'pure', single-conclusioned 'Hilbert-style systems for consequence' in the characterization of Hilbert systems in sequent-style formalization due to Avron in [1]. For studying general properties of the notions of rule derivability and admissibility, we introduce an abstract framework for Hilbert systems of the kind outlined above.

2 Abstract Hilbert Systems

For a formal concept embracing all Hilbert systems of the kind sketched in the Introduction, we abstract away, by analogy with the notion of 'abstract rewriting systems' (cf. [6,5]), from the syntax of the formula language, and consequently also from specific ways in which rules can be defined syntactically. In an "abstract Hilbert system" a rule is a set of applications (inference steps) that is endowed with a premise and a conclusion function, which respectively assign a finite sequence of premises and a conclusion to every application. Every such system consists of a set of formulas, a set of axioms and a set of rules. In the coming definitions we denote, for all sets X, by $\mathcal{P}_{\mathrm{f}}(X)$, $\mathcal{M}_{\mathrm{f}}(X)$, and $Seqs_{\mathrm{f}}(X)$ the sets of *finite sets*, *finite multisets*, and *finite sequences over* X, respectively.

Definition 1. (An abstract notion of rule). *Let Fo be a set. An* AHS-rule R on Fo *(later just called a* rule R on Fo*) is a triple* $\langle Apps, \mathsf{prem}, \mathsf{concl}\rangle$ *where*

- *Apps is the set of* applications of R,
- $\mathsf{prem} : Apps \to Seqs_{\mathrm{f}}(Fo)$ *is the* premise function *of* R,
- $\mathsf{concl} : Apps \to Fo$ *is the* conclusion function *of* R.

Definition 2. (Abstract Hilbert systems). *An* abstract Hilbert system *(an AHS)* $\mathcal{S}$ *is a triple* $\langle Fo, Ax, \mathcal{R}\rangle$ *consisting of sets Fo, Ax and* $\mathcal{R}$ *such that*

- *the elements of Fo, Ax and* $\mathcal{R}$ *are respectively called the* formulas, axioms *and* rules *of* $\mathcal{S}$,
- $Ax \subseteq Fo$ *holds, i.e. all axioms of* $\mathcal{S}$ *are formulas of* $\mathcal{S}$*, and*
- *every rule* $R \in \mathcal{R}$ *is an AHS-rule on Fo.*

We denote by $\mathfrak{H}$ *the class of all AHS's. For referring to the sets of formulas, of axioms, and of rules belonging to an AHS* $\mathcal{S}$*, we will use the denotations* $Fo_{\mathcal{S}}$, $Ax_{\mathcal{S}}$*, and* $\mathcal{R}_{\mathcal{S}}$*, respectively.*

For some purposes, the variant concept of *abstract Hilbert system with names* (n-AHS) is useful, where additionally a name function is present that assigns names to axioms and to rules. For this concept, and the straightforward inductive definition of *derivations in* an AHS (or n-AHS), which may start from unproven assumptions, we refer to [2]. For all AHS's $\mathcal{S}$, we denote by $Der(\mathcal{S})$ the *set of derivations* in $\mathcal{S}$ (or, of $\mathcal{S}$), and for all derivations $\mathcal{D} \in Der(\mathcal{S})$, we denote by

- $\mathsf{assm}(\mathcal{D}) \in \mathcal{M}_{\mathrm{f}}(Fo_{\mathcal{S}})$ the *multiset of assumptions* of $\mathcal{D}$, and by

– $\mathsf{concl}(\mathcal{D}) \in Fo_{\mathcal{S}}$ the *conclusion* of $\mathcal{D}$.

Based on these notions, we introduce three consequence relations for every AHS, and three mimicking relations between derivations in (possibly different) AHS's. Here and later we denote by $\mathrm{set}(\cdot)$ the operation that assigns to every finite multiset or sequence the *set* of all occurring elements.

Definition 3 (Three consequence relations on an AHS). *Let $\mathcal{S}$ be an AHS with set Fo of formulas. We define the* consequence relations $\vdash_{\mathcal{S}}$, $\vdash_{\mathcal{S}}^{(\mathrm{s})}$ *and* $\vdash_{\mathcal{S}}^{(\mathrm{m})}$ *on $\mathcal{S}$, where* $\vdash_{\mathcal{S}}, \vdash_{\mathcal{S}}^{(\mathrm{s})} \subseteq \mathcal{P}_{\mathrm{f}}(Fo) \times Fo$ *and* $\vdash_{\mathcal{S}}^{(\mathrm{m})} \subseteq \mathcal{M}_{\mathrm{f}}(Fo) \times Fo$, *by stipulating*

$$\begin{aligned}
\Sigma \vdash_{\mathcal{S}} A &\iff (\exists\, \mathcal{D} \in Der(\mathcal{S}))\, \big[\, \mathrm{set}(\mathsf{assm}(\mathcal{D})) \subseteq \Sigma \;\;\&\;\; \mathsf{concl}(\mathcal{D}) = A \,\big]\,, \\
\Sigma \vdash_{\mathcal{S}}^{(\mathrm{s})} A &\iff (\exists\, \mathcal{D} \in Der(\mathcal{S}))\, \big[\, \mathrm{set}(\mathsf{assm}(\mathcal{D})) = \Sigma \;\;\&\;\; \mathsf{concl}(\mathcal{D}) = A \,\big]\,, \\
\Gamma \vdash_{\mathcal{S}}^{(\mathrm{m})} A &\iff (\exists\, \mathcal{D} \in Der(\mathcal{S}))\, \big[\, \mathsf{assm}(\mathcal{D}) = \Gamma \;\;\&\;\; \mathsf{concl}(\mathcal{D}) = A \,\big]\,,
\end{aligned}$$

for all formulas $A \in Fo$, finite <u>s</u>ets Σ *on Fo and finite* <u>m</u>ultisets Γ *on Fo.*

Definition 4 (Three mimicking relations between AHS-derivations). *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be AHS's, and let $\mathcal{D}_1 \in Der(\mathcal{S}_1)$ and $\mathcal{D}_2 \in Der(\mathcal{S}_2)$ be derivations. We say that $\mathcal{D}_1$* mimics *$\mathcal{D}_2$ (denoted by $\mathcal{D}_1 \precsim \mathcal{D}_2$) if and only if*

$$\mathrm{set}(\mathsf{assm}(\mathcal{D}_1)) \subseteq \mathrm{set}(\mathsf{assm}(\mathcal{D}_2)) \;\;\&\;\; \mathsf{concl}(\mathcal{D}_1) = \mathsf{concl}(\mathcal{D}_2)\,.$$

Furthermore, we stipulate that $\mathcal{D}_1$ s-mimics *$\mathcal{D}_2$ (denoted by $\mathcal{D}_1 \simeq^{(\mathrm{s})} \mathcal{D}_2$), and that $\mathcal{D}_1$* m-mimics *$\mathcal{D}_2$ (denoted by $\mathcal{D}_1 \simeq^{(\mathrm{m})} \mathcal{D}_2$) if and only if respectively* (1) *and* (2) *hold:*

$$\mathrm{set}(\mathsf{assm}(\mathcal{D}_1)) = \mathrm{set}(\mathsf{assm}(\mathcal{D}_2)) \;\;\&\;\; \mathsf{concl}(\mathcal{D}_1) = \mathsf{concl}(\mathcal{D}_2)\,, \tag{1}$$

$$\mathsf{assm}(\mathcal{D}_1) = \mathsf{assm}(\mathcal{D}_2) \;\;\&\;\; \mathsf{concl}(\mathcal{D}_1) = \mathsf{concl}(\mathcal{D}_2)\,. \tag{2}$$

3 Rule Derivability and Admissibility in AHS's

Now we adapt well-known definitions (e.g. [3, p.70]) of the notions of rule derivability and admissibility to abstract Hilbert systems. For rule derivability, we base our definition on the three kinds of consequence relations from Definition 3 and accordingly formulate three versions. We denote by $\mathrm{mset}(\cdot)$ the operation that assigns, for all sets X, to every finite sequence σ over X *that* finite *multiset* over X in which every element of X occurs precisely as often as in σ.

Definition 5. (Rule admissibility and three versions of rule derivability in AHS's). *Let $\mathcal{S}$ be an AHS and let $R = \langle Apps, \mathsf{prem}, \mathsf{concl}\rangle$ be a rule on $Fo_{\mathcal{S}}$.*

(i) The rule R is admissible in $\mathcal{S}$ *if and only if it holds that*[1]

$$(\forall \alpha \in Apps)\, \big[\, (\forall A \in \mathrm{set}(\mathsf{prem}(\alpha)))\, [\, \vdash_{\mathcal{S}} A \,] \implies \;\vdash_{\mathcal{S}} \mathsf{concl}(\alpha) \,\big]\,.$$

[1] For every AHS $\mathcal{S}$ and for all formulas A of $\mathcal{S}$, we use the customary abbreviation $\vdash_{\mathcal{S}} A$ for $\emptyset \vdash_{\mathcal{S}} A$.

(ii) The rule R is derivable in $\mathcal{S}$ *if and only if*

$$(\forall \alpha \in \mathit{Apps}) \left[\operatorname{set}(\mathsf{prem}(\alpha)) \vdash_{\mathcal{S}} \mathsf{concl}(\alpha) \right]$$

holds. Similarly, we say that R is s-derivable *or that R is* m-derivable *if and only if, respectively, the assertions* (3) *and* (4) *hold:*

$$(\forall \alpha \in \mathit{Apps}) \left[\operatorname{set}(\mathsf{prem}(\alpha)) \vdash_{\mathcal{S}}^{(\mathrm{s})} \mathsf{concl}(\alpha) \right] , \tag{3}$$

$$(\forall \alpha \in \mathit{Apps}) \left[\operatorname{mset}(\mathsf{prem}(\alpha)) \vdash_{\mathcal{S}}^{(\mathrm{m})} \mathsf{concl}(\alpha) \right] . \tag{4}$$

For every AHS $\mathcal{S}$ and all formulas $A \in \mathit{Fo}_{\mathcal{S}}$, we call A *admissible* as well as *derivable*, *s-derivable*, and *m-derivable in* $\mathcal{S}$ if and only if $\vdash_{\mathcal{S}} A$ holds, i.e. iff A is a theorem of $\mathcal{S}$.

There is an obvious connection between the three notions of rule derivability defined above and the three notions of mimicking derivation from Definition 4: A rule R is derivable (s-derivable, m-derivable) in an AHS $\mathcal{S}$ if and only if for every (one-step derivation corresponding to an) application of R there exists a mimicking (s-mimicking, m-mimicking) derivation in $\mathcal{S}$.

Theorem 1 below, which in its main part is only a reformulation applicable to AHS's of Lemma 6.14 in [3, p.70], gathers basic facts about the interrelations of these four notions. For its statement, we vary the terminology used in [4] for extensions of 'first-order theories': For all AHS's $\mathcal{S}$ and $\mathcal{S}'$, we call $\mathcal{S}'$ an *extension by enlargement of* $\mathcal{S}$ iff $\mathcal{S}'$ results from $\mathcal{S}$ by adding new formulas, axioms, and/or rules. And for all AHS's $\mathcal{S}$, rules R on $\mathit{Fo}_{\mathcal{S}}$ (and sets $\Sigma \in \mathcal{P}_{\mathrm{f}}(\mathit{Fo}_{\mathcal{S}})$), we denote by $\mathcal{S}{+}R$ (by $\mathcal{S}{+}\Sigma$) the extension by enlargement of $\mathcal{S}$ by adding the rule R (by adding the formulas of Σ as new axioms).

Theorem 1 (Reformulation of a lemma by Hindley, Seldin). *Let $\mathcal{S}$ be an AHS and let R be a rule on $\mathit{Fo}_{\mathcal{S}}$. Then the following statements holds:*

(i) R is admissible in $\mathcal{S}$ iff $\mathcal{S}{+}R$ does not possess more theorems than $\mathcal{S}$.
(ii) If R is derivable in $\mathcal{S}$, then R is also admissible in $\mathcal{S}$. The implication in the opposite direction does not hold in general.
(iii) If R is derivable in $\mathcal{S}$, then R is derivable in every ext. by enlargement of $\mathcal{S}$.
(iv) If R is m-derivable in $\mathcal{S}$, then R is also s-derivable in $\mathcal{S}$; and if R is s-derivable in $\mathcal{S}$, then R is also derivable in $\mathcal{S}$. But for neither of these two implications does the reverse implication hold in general.

The next theorem establishes a link between parts (ii) and (iii) of Theorem 1, and it gives a characterization of rule derivability in an AHS $\mathcal{S}$ in terms of rule admissibility in extensions by enlargement of $\mathcal{S}$.

Theorem 2. *Let $\mathcal{S}$ be an AHS with set Fo of formulas, and let R be a rule on Fo. Then the following three statements are equivalent:*

(i) R is derivable in $\mathcal{S}$.
(ii) R is admissible in every AHS $\mathcal{S}{+}\Sigma$ with $\Sigma \in \mathcal{P}_{\mathrm{f}}(\mathit{Fo})$ arbitrary.
(iii) R is admissible in every extension by enlargement of $\mathcal{S}$.

4 (Mutual) Inclusion Relations between AHS's

Inspired by another lemma of Hindley and Seldin, Lemma 6.16 on p. 71 in [3], we consider relations that compare AHS's with respect to the introduced notions of rule derivability and admissibility, and with respect to the three kinds of consequence relations on them. That is, we consider the question: What kind of relationships hold for all AHS's $\mathcal{S}_1$ and $\mathcal{S}_2$ between assertions like, for example, "every rule of $\mathcal{S}_1$ is derivable in $\mathcal{S}_2$, and vice versa", "$\mathcal{S}_1$ and $\mathcal{S}_2$ have the same m-derivable rules", and "$\mathcal{S}_1$ and $\mathcal{S}_2$ have the same theorems"? For this purpose, we introduce, in the three following definitions, a total of twelve "inclusion relations" between AHS's, and twelve "mutual inclusion relations" that are induced by respective inclusion relations.

Definition 6 (Relations between abstract Hilbert systems (I)). *The* inclusion relation $\preceq_{th}$*, a binary relation on the class* $\mathfrak{H}$ *of all AHS's, is defined by stipulating, for all* $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$,

$$\mathcal{S}_1 \preceq_{th} \mathcal{S}_2 \iff Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2} \;\&\; (\forall A \in Fo_{\mathcal{S}_1})[\,(\vdash_{\mathcal{S}_1} A) \;\Rightarrow\; (\vdash_{\mathcal{S}_2} A)\,]\,.$$

And the inclusion relations $\preceq_{rth}$, $\preceq^{(\mathrm{s})}_{rth}$, *and* $\preceq^{(\mathrm{m})}_{rth}$ *are binary relations on* $\mathfrak{H}$ *that are defined, for all* $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$, *by stipulating respectively*

$$\mathcal{S}_1 \preceq_{rth} \mathcal{S}_2 \iff Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2} \;\&\; (\forall \Sigma \in \mathcal{P}_{\mathrm{f}}(Fo_{\mathcal{S}_1}), A \in Fo_{\mathcal{S}_1})\big[(\Sigma \vdash_{\mathcal{S}_1} A) \Rightarrow (\Sigma \vdash_{\mathcal{S}_2} A)\big],$$

$$\mathcal{S}_1 \preceq^{(\mathrm{s})}_{rth} \mathcal{S}_2 \iff Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2} \;\&\; (\forall \Sigma \in \mathcal{P}_{\mathrm{f}}(Fo_{\mathcal{S}_1}), A \in Fo_{\mathcal{S}_1})\big[(\Sigma \vdash^{(\mathrm{s})}_{\mathcal{S}_1} A) \Rightarrow (\Sigma \vdash^{(\mathrm{s})}_{\mathcal{S}_2} A)\big],$$

$$\mathcal{S}_1 \preceq^{(\mathrm{m})}_{rth} \mathcal{S}_2 \iff Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2} \;\&\; (\forall \Gamma \in \mathcal{M}_{\mathrm{f}}(Fo_{\mathcal{S}_1}), A \in Fo_{\mathcal{S}_1})\big[(\Gamma \vdash^{(\mathrm{m})}_{\mathcal{S}_1} A) \Rightarrow (\Gamma \vdash^{(\mathrm{m})}_{\mathcal{S}_2} A)\big].$$

These four inclusion relations induce *respective* mutual inclusion relations*:* $\preceq_{th}$ *induces the binary relation* $\sim_{th}$ *on* $\mathfrak{H}$ *that is defined, for all* $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$, *by*

$$\mathcal{S}_1 \sim_{th} \mathcal{S}_2 \iff \mathcal{S}_1 \preceq_{th} \mathcal{S}_2 \;\&\; \mathcal{S}_2 \preceq_{th} \mathcal{S}_1 \;; \tag{5}$$

if, for some $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$, $\mathcal{S}_1 \sim_{th} \mathcal{S}_2$ *holds, we say that* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are* (<u>th</u>eorem-) equivalent. *And the inclusion relations* $\preceq_{rth}$, $\preceq^{(\mathrm{s})}_{rth}$ *and* $\preceq^{(\mathrm{m})}_{rth}$ *induce respectively, by stipulations analogous to* (5)*, the* mutual inclusion relations $\sim_{rth}$, $\sim^{(\mathrm{s})}_{rth}$ *and* $\sim^{(\mathrm{m})}_{rth}$.

Definition 7 (Relations between Abstract Hilbert Systems (II)). *The* inclusion relation $\preceq_{adm}$*, a binary relation on the class* $\mathfrak{H}$*, is defined by stipulating, for all* $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$,

$$\begin{aligned}\mathcal{S}_1 \preceq_{adm} \mathcal{S}_2 \iff & Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2} \;\&\; (\forall A \in Fo_{\mathcal{S}_1})\big[A \textit{ is adm. in } \mathcal{S}_1 \Rightarrow A \textit{ is adm. in } \mathcal{S}_2\big] \\ & \&\; (\forall R \textit{ rule on } Fo_{\mathcal{S}_1})\big[R \textit{ is adm. in } \mathcal{S}_1 \Rightarrow R \textit{ is adm. in } \mathcal{S}_2\big]\,.\end{aligned}$$

The inclusion relations $\preceq_{der}$, $\preceq^{(\mathrm{s})}_{der}$ *and* $\preceq^{(\mathrm{m})}_{der}$ *are defined analogously by using "derivable", "s-derivable" and "m-derivable" instead of "admissible". These four inclusion relations* induce*, by respective stipulations analogous to* (5)*, the four* mutual inclusion relations $\sim_{adm}$, $\sim_{der}$, $\sim^{(\mathrm{s})}_{der}$ *and* $\sim^{(\mathrm{m})}_{der}$.

Definition 8 (Relations between Abstract Hilbert Systems (III)). *We define the* inclusion relation $\preceq_{r/adm}$, *also a binary relation on the class* $\mathfrak{H}$, *by stipulating for all* $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$*:*

$$\mathcal{S}_1 \preceq_{r/adm} \mathcal{S}_2 \iff Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2} \;\&\; (\forall A \in Ax_{\mathcal{S}_1})\big[\, A \text{ is admissible in } \mathcal{S}_2 \,\big] \;\& \\ \&\; (\forall R \in \mathcal{R}_{\mathcal{S}_1})\big[\, R \text{ is admissible in } \mathcal{S}_2 \,\big] \,.$$

The inclusion relations $\preceq_{r/der}$, $\preceq^{(\mathrm{s})}_{r/der}$ *and* $\preceq^{(\mathrm{m})}_{r/der}$ *are defined analogously by using "derivable", "s-derivable" and "m-derivable" instead of "admissible". These four relations* induce*, by respective stipulations analogous to* (5)*, the* mutual inclusion relations $\sim_{r/adm}$, $\sim_{r/der}$, $\sim^{(\mathrm{s})}_{r/der}$ *and* $\sim^{(\mathrm{m})}_{r/der}$.

The following theorem is the outcome of a systematic examination of the relationship towards each other of the twelve inclusion relations, and of the twelve mutual inclusion relations.

Theorem 3 (Interrelations between (mutual) inclusion relations). *The following three statements hold about interrelations between the relations defined in Definitions 6–8:*

(i) *The implications and equivalences in the two* interrelations prisms *shown in Figure 1 hold, for all AHS's* $\mathcal{S}_1$ *and* $\mathcal{S}_2$*, between statements* $\mathcal{S}_1 \preceq \mathcal{S}_2$*, where* $\preceq$ *is an introduced* inclusion relation*, and respectively, between statements* $\mathcal{S}_1 \sim \mathcal{S}_2$*, where* $\sim$ *is an introduced* mutual inclusion relation*.*

(ii) *Each implication arrow in Figure 1 that is not inverted indicates that an implication in the opposite direction* does not hold in general*.*

(iii) *In the case of the interrelations prism for the* inclusion relations*, in general* no implication holds *between an assertion* $\mathcal{S}_1 \preceq_{r/adm} \mathcal{S}_2$ *and any of* $\mathcal{S}_1 \preceq_{r/der} \mathcal{S}_2$*,* $\mathcal{S}_1 \preceq^{(\mathrm{s})}_{r/der} \mathcal{S}_2$ *or* $\mathcal{S}_1 \preceq^{(\mathrm{m})}_{r/der} \mathcal{S}_2$*.*

5 Notions of Rule Elimination from Derivations in AHS's

On the poster, we also investigated the following question: What consequences does the fact that a rule R is admissible, derivable, s-derivable or m-derivable in an AHS $\mathcal{S}$ have for the possibility to eliminate applications of R from derivations in $\mathcal{S}+R$? For this purpose, we introduced four abstract notions of rule elimination with respect to derivations in AHS's, using the three mimicking relations of Definition 4. For example, we stipulated that, for a rule R of an AHS $\mathcal{S}$, *R-elimination holds for derivations in $\mathcal{S}$* if and only if for every derivation $\mathcal{D}$ in $\mathcal{S}$ there exists a derivation $\mathcal{D}'$ in $\mathcal{S}$ without R-applications that mimics $\mathcal{D}$.

And then we gave a theorem that asserts a direct correspondence of three of the four notions of rule elimination with respective notions of rule derivability and admissibility (in the fourth case only a weaker connection holds). E.g., the theorem states that, for a rule R of an AHS $\mathcal{S}$, R-elimination holds for derivations in $\mathcal{S}$ if and only if R is derivable in the AHS $\mathcal{S}-R$ that results from $\mathcal{S}$ by removing the rule R. – We refer the reader to the report-version [2] for the definition of the (three other) notions of rule elimination, for the mentioned theorem, and for a theorem concerning related notions of "strong rule elimination".

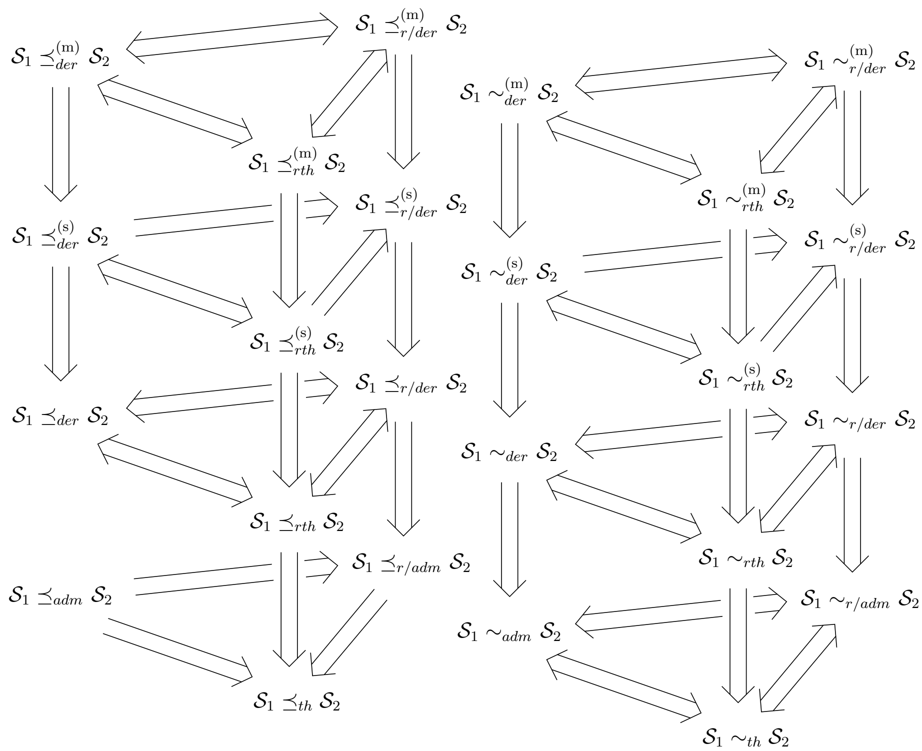

Figure 1. 'Interrelation prisms' between assertions involving the twelve inclusion relations (on the left-hand side), and involving the twelve mutual inclusion relations (on the right-hand side).

Acknowledgement. I want to thank Bas Luttik very much for his comments on drafts for the report-version [2] and for his often quite detailed suggestions. I was lead to defining "abstract Hilbert systems" by comments of Roel de Vrijer.

References

1. Avron, A.: Simple Consequence Relations. *Information and Computation* **92**:1 (1991) 105–139.
2. Grabmayer, C.: *Derivability and Admissibility of Inference Rules in Abstract Hilbert Systems*. Technical Report, Vrije Universiteit Amsterdam (2003), available at `http://www.cs.vu.nl/~clemens/dairahs.ps` .
3. Hindley, J.R., Seldin, J.P.: *Introduction to Combinators and Lambda-calculus*. Cambridge University Press (1986).
4. Shoenfield, J.R.: *Mathematical Logic*. Addison-Wesely (1967, 1973).
5. Terese: *Term Rewriting Systems*. Cambridge University Press (2003).
6. van Oostrom, V., de Vrijer, R.C.: Four Equivalent Equivalences of Reductions. *Artificial Intelligence Preprint Series*, Preprint nr. 035, Utrecht University (2002).