Derivability and Admissibility of Inference Rules in Abstract Hilbert Systems

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The notions of derivability and admissibility of inference rules are usually studied in the context of concrete systems of formal logic. A new rule R is generally called 'derivable' (or 'derived') in a formal system $\mathcal{S}$ if all applications (instances) of R can, in some sense, be modeled or mimicked by appropriate derivations in $\mathcal{S}$. And a rule R is understood to be 'admissible' in a formal system $\mathcal{S}$ if the class of theorems of $\mathcal{S}$ is closed under applications of R. The aim of this paper is to collect a number of basic results about these notions that are applicable to all Hilbert-style systems of the simplest kind. By this we mean systems, sometimes just called 'axiom systems', in which each rule application α within a derivation $\mathcal{D}$ is the inference of a single conclusion from a finite sequence of premises; each such rule application α does furthermore not depend on the presence or absence of assumptions in subderivations of $\mathcal{D}$ leading to α.

We introduce a general framework for such Hilbert-style proof systems by analogy with the notion of 'abstract rewriting systems' (cf. [2]). Thereby we abstract away from the syntax of the formula language and consequently also from the way in which rules can be defined syntactically. In an *abstract Hilbert system* (AHS) a rule is a set of applications (inference steps) that is endowed with a premise and a conclusion function, which respectively assign a finite sequence of premises and a conclusion to every application. An AHS consists of a set of formulas, a set of axioms and a set of rules. In the variant concept of *abstract Hilbert system with names* (n-AHS) additionally a name function is present that assigns names to axioms and to rules.

Next we adapt the notions of rule derivability and admissibility to AHS's and n-AHS's. In the case of rule derivability, we use three formalizations of the term 'mimicking derivation' for a rule application (mimicking , s-mimicking and m-mimicking derivation) leading to three variant notions: A rule R is *derivable* (*s-derivable*, *m-derivable*) in an AHS or n-AHS $\mathcal{S}$ if and only if for every application of R there exists a mimicking (s-mimicking, m-mimicking) derivation in $\mathcal{S}$. Starting out from an easy lemma in [1], we collect basic facts about the interrelations of these four notions. Also, we give characterizations of rule derivability and admissibility in terms of the respective other notion.

Inspired by another lemma in [1], we furthermore consider relations that compare AHS's (or analogously n-AHS's) with respect to the introduced notions of rule derivability and admissibility, and with respect to three consequence relations on them. That is, we consider the question: What kind of relationships hold for all AHS's $\mathcal{S}_1$ and $\mathcal{S}_2$ between assertions like, for example, 'every rule of $\mathcal{S}_1$ is derivable in $\mathcal{S}_2$, and vice versa', '$\mathcal{S}_1$ and $\mathcal{S}_2$ have the same m-derivable rules', and '$\mathcal{S}_1$ and $\mathcal{S}_2$ have the same theorems'? For this purpose, we introduce twelve *inclusion relations* between AHS's, and twelve *mutual inclusion relations* that are induced by respective inclusion relations. Then we carry out a systematic examination of the relationship between these relations and gather our findings: We prove two theorems that describe the logical implications and equivalences that hold in general, and that do not hold in general between statements that compare two AHS's with respect to the twelve inclusion relations, and respectively with respect to the twelve mutual inclusion relations. As a means to formulate these theorems, as well as to visualize them, we use pictures of 'interrelation prisms' that contain such statements.

In the last part, we investigate the question: What consequences does the fact that a rule R is admissible, derivable, s-derivable or m-derivable in an abstract Hilbert system $\mathcal{S}$ have for the possibility to eliminate applications of R from derivations in $\mathcal{S}$? We start by introducing four abstract notions of rule elimination with respect to arbitrary AHS's or n-AHS's. For this, we give three different formalizations of the term 'mimicking derivation' for a derivation before stipulating: For a rule R in an AHS or n-AHS $\mathcal{S}$, *R-elimination holds for derivations in $\mathcal{S}$* if and only if for every derivation in $\mathcal{S}$ there exists a 'mimicking derivation' that does not contain R-applications. We then show a direct correspondence of three of the four concepts of rule elimination with respective notions of rule derivability and admissibility (in the fourth case only a weaker connection holds).

Finally we prove that if a rule R is derivable in an n-AHS $\mathcal{S}$ then R-elimination for derivations in $\mathcal{S}$ can be performed effectively: For a considered derivation $\mathcal{D}$ in $\mathcal{S}$, pick an arbitrary application of R in $\mathcal{D}$ and replace it by a mimicking derivation; carry out such *mimicking steps* repeatedly until no further applications of R are present. We show correctness and termination for this nondeterministic procedure. And we find that in n-AHS's a similar result holds also for m-derivable rules.

References

[1] Hindley, J.R., Seldin, J.P.: *Introduction to Combinators and Lambda-calculus*, Cambridge University Press, 1986.

[2] Terese: *Term Rewriting Systems*, Cambridge Texts in Theoretical Computer Science 55, Cambridge University Press, 2003.