Expressibility in the Lambda Calculus with μ

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Motivation

motivation:

- $\boldsymbol{\lambda}_{\text{letrec}}$ as an abstraction & the core of functional languages
- infinite λ-terms ~ unfolding semantics of functional programs
- optimizing program transformations on $\boldsymbol{\lambda}_{\text{letrec}}$

question:

- which infinite λ-terms are $\boldsymbol{\lambda}_{\text{letrec}}$-expressible, i.e. can be obtained from $\boldsymbol{\lambda}_{\text{letrec}}$-terms by infinite unfolding?

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we restrict to:

- $\boldsymbol{\lambda}_{\boldsymbol{\mu}}$ instead of $\boldsymbol{\lambda}_{\mathsf{letrec}}$

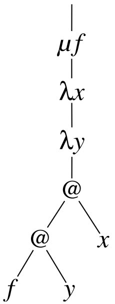
we develop:

- 2 rewriting characterizations of $\boldsymbol{\lambda}_{\boldsymbol{\mu}}$-expressible infinite λ-terms

Which infinite λ-terms are expressible finitely in $\boldsymbol{\lambda}_{\mu}$?

Example

$\mu f.\lambda x.\lambda y.f\,y\,x$



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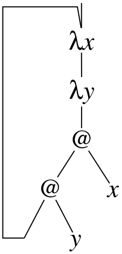

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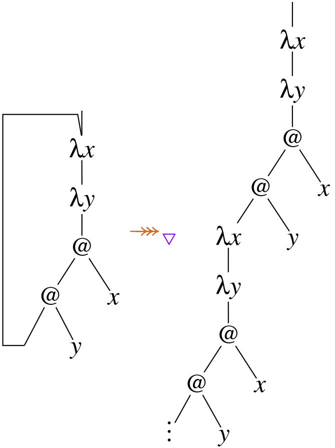

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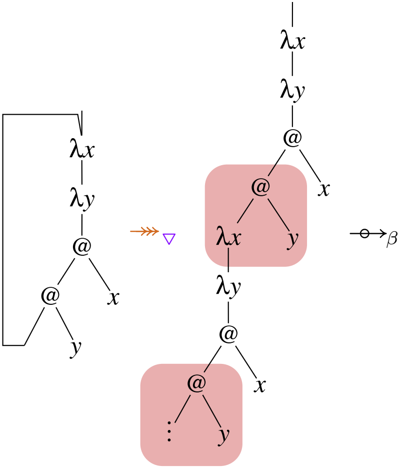

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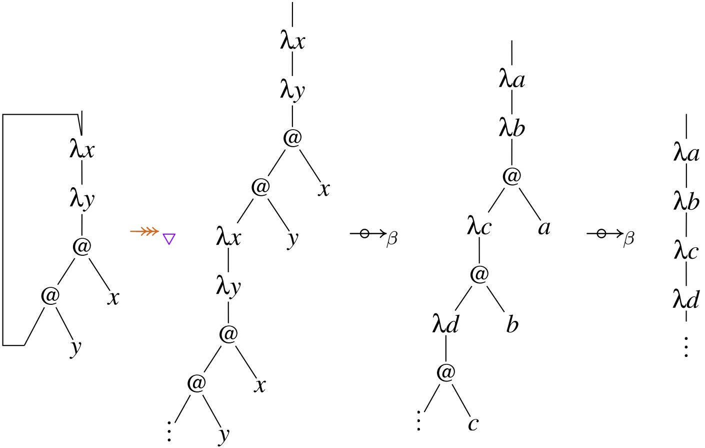

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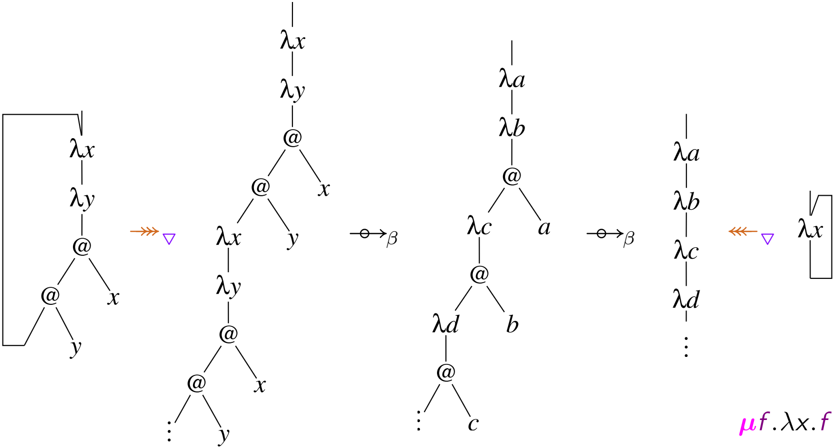
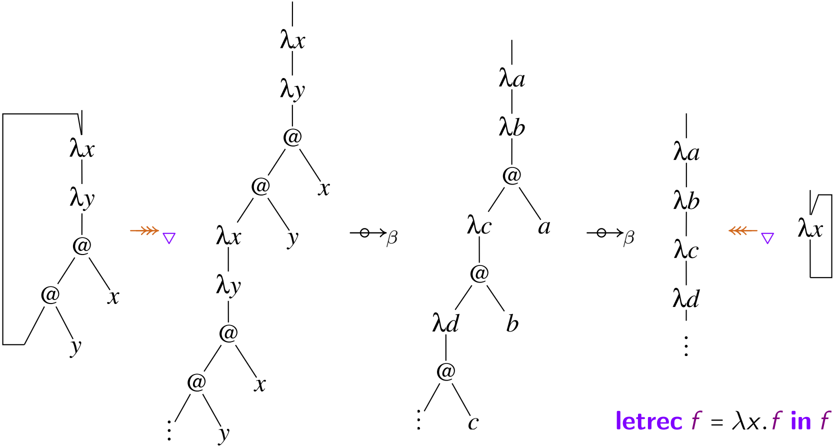


Which infinite λ-terms are expressible finitely in $\boldsymbol{\lambda}_{\mathsf{letrec}}$?

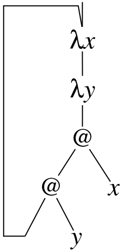
Example

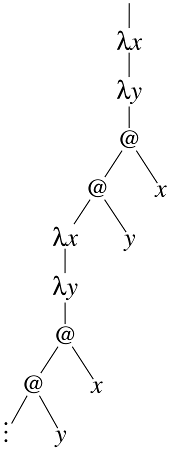
letrec $f = \lambda xy.\, f\; y\; x$ **in** f $\quad \twoheadrightarrow_{\mathsf{letrec}} \quad \lambda xy.(\lambda xy.(\lambda xy.(\ldots)\; y\; x)\; y\; x)\; y\; x$



$\boldsymbol{\lambda}_{\boldsymbol{\mu}}$-expressible 'regular' $\boldsymbol{\lambda}^{\infty}$-term

$\mu f.\lambda xy.f\ y\ x$

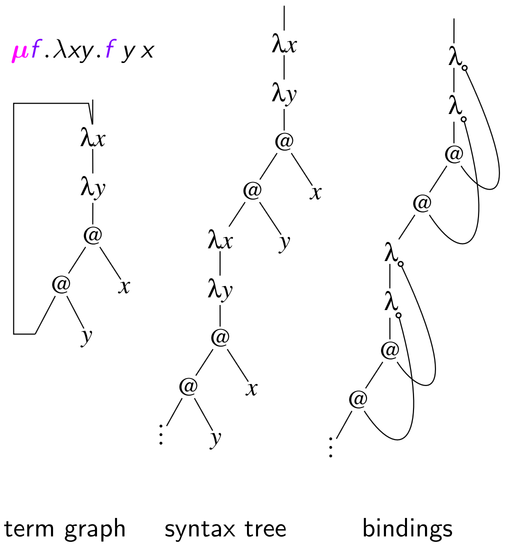




term graph syntax tree

$\boldsymbol{\lambda_\mu}$-expressible 'regular' $\boldsymbol{\lambda^\infty}$-term

$\mu f.\lambda xy.f\ y\ x$



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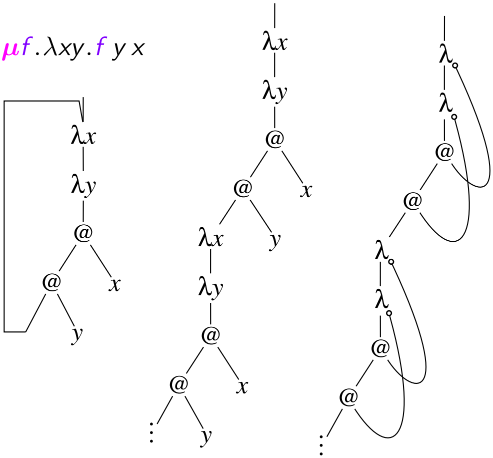
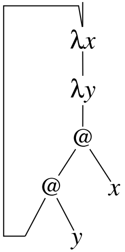


term graph   syntax tree   bindings

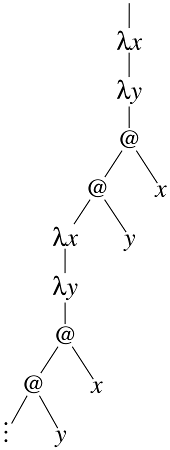
finite
entanglement

$\boldsymbol{\lambda_{\mu}}$-expressible 'regular' $\boldsymbol{\lambda^{\infty}}$-term

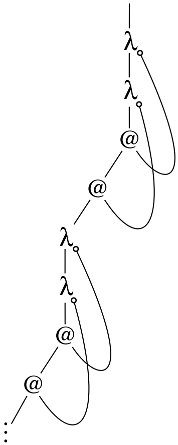
$\mu f.\lambda xy.f\ y\ x$



term graph

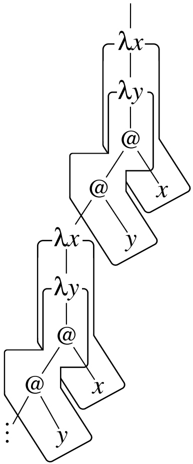


syntax tree



bindings

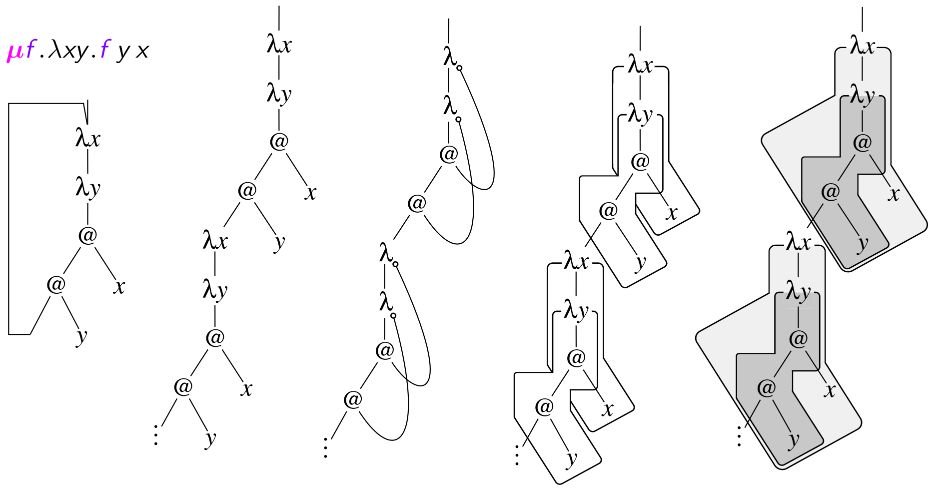
finite entanglement



scopes

$\boldsymbol{\lambda}_{\mu}$-expressible 'regular' $\boldsymbol{\lambda}^{\infty}$-term

$\mu f . \lambda x y . f\, y\, x$



term graph

syntax tree

bindings
finite
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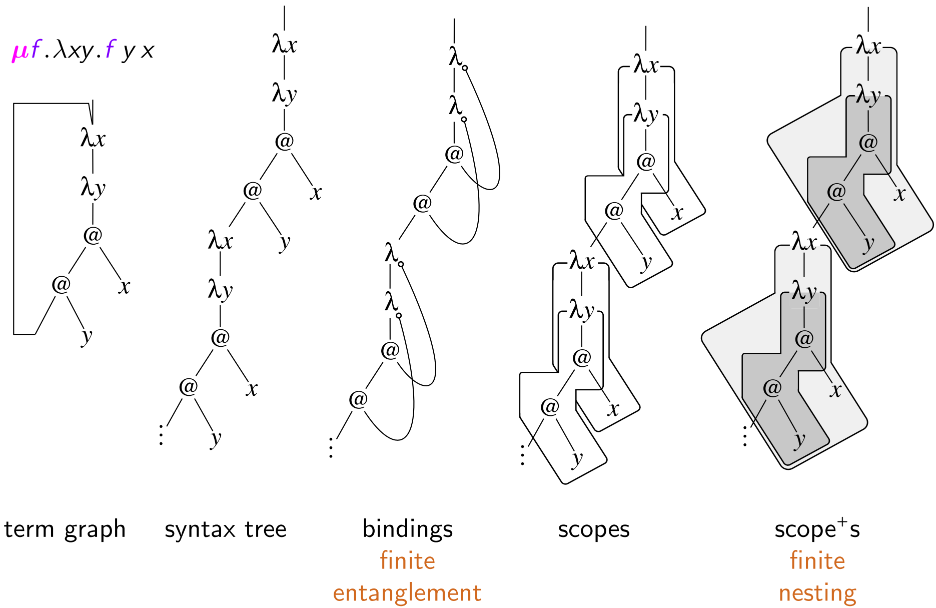
scopes

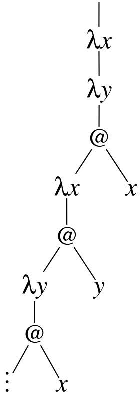
scope^{+}s

$\boldsymbol{\lambda}_{\mu}$-expressible 'regular' $\boldsymbol{\lambda}^{\infty}$-term

$\mu f.\lambda xy.f\ y\ x$

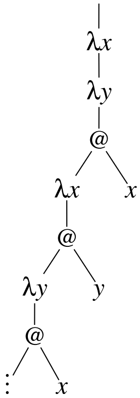


Not $\boldsymbol{\lambda}_{\mu}$-expressible 'regular' $\boldsymbol{\lambda}^{\infty}$-term

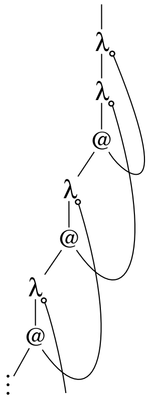


syntax tree

Not $\boldsymbol{\lambda}_{\mu}$-expressible ‘regular’ $\boldsymbol{\lambda}^{\infty}$-term



syntax tree



bindings

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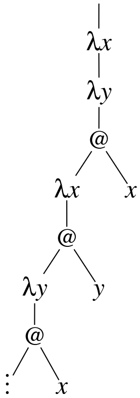

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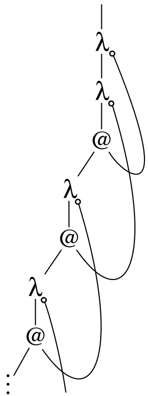
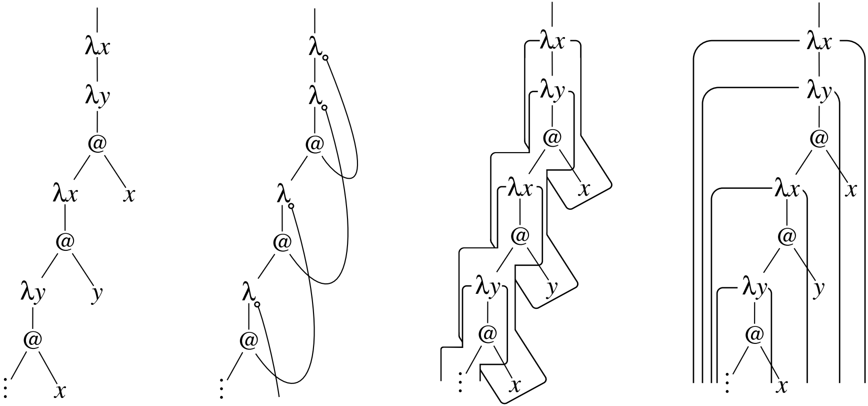


bindings

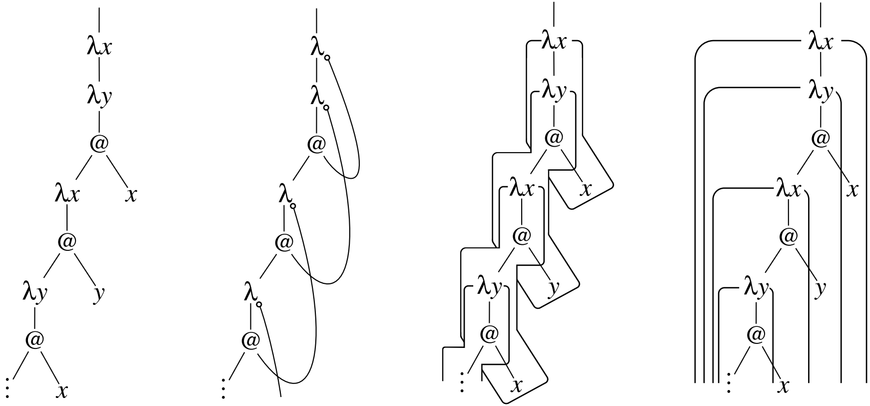
infinitely entangled

Not $\boldsymbol{\lambda}_{\mu}$-expressible ‘regular’ $\boldsymbol{\lambda}^{\infty}$-term



syntax tree — bindings (infinitely entangled) — scopes — scope^{+}s

Not $\boldsymbol{\lambda}_{\mu}$-expressible ‘regular’ $\boldsymbol{\lambda}^{\infty}$-term



syntax tree

bindings
infinitely entangled

scopes

scope^{+}s
infinite nesting

Concepts and results

We introduce:

- generalizations to λ^∞-terms of first-order concept of regularity
  - strong regularity
  - regularity
- proof systems for regularity and strong regularity

use:

- binding–capturing chains for λ^∞-terms [Melliés, van Oostrom]

Concepts and results

We introduce:

- generalizations to $\boldsymbol{\lambda}^{\infty}$-terms of first-order concept of regularity
  - strong regularity
  - regularity
- proof systems for regularity and strong regularity

use:

- binding–capturing chains for $\boldsymbol{\lambda}^{\infty}$-terms [Melliés, van Oostrom]

and show:

Results

- $\boldsymbol{\lambda}_{\mu}$-expressibility = strong regularity
- strong regularity =

  regularity + a binding–capturing chains property

Deconstructing/observing $\boldsymbol{\lambda}^\infty$-terms

$$()\,\lambda x.\lambda y.x\,x\,y$$

Deconstructing/observing $\boldsymbol{\lambda}^\infty$-terms

$$() \, \lambda x.\lambda y.x\,x\,y \to_\lambda$$
$$(\lambda x) \, \lambda y.x\,x\,y$$

$$(\lambda x_1 \ldots x_n) \, \lambda x_{n+1}.\, T_0 \;\; \to_\lambda \;\; (\lambda x_1 \ldots x_{n+1}) \, T_0$$

Deconstructing/observing $\boldsymbol{\lambda}^\infty$-terms

$$
\begin{aligned}
&(\,)\,\lambda x.\lambda y.x\,x\,y \to_\lambda \\
&(\lambda x)\,\lambda y.x\,x\,y \to_\lambda \\
&(\lambda xy)\,x\,x\,y
\end{aligned}
$$

$$(\lambda x_1 \ldots x_n)\,\lambda x_{n+1}.\,T_0 \;\to_\lambda\; (\lambda x_1 \ldots x_{n+1})\,T_0$$

Deconstructing/observing $\boldsymbol{\lambda}^\infty$-terms

$$
\begin{aligned}
&()\,\lambda x.\lambda y.x\,x\,y \to_\lambda \\
&(\lambda x)\,\lambda y.x\,x\,y \to_\lambda \\
&(\lambda xy)\,x\,x\,y \to_{@_0} \\
&(\lambda xy)\,x\,x
\end{aligned}
$$

$$
\begin{aligned}
(\lambda x_1 \dots x_n)\, T_0\ T_1 &\to_{@_i} (\lambda x_1 \dots x_n)\, T_i \qquad (i \in \{0,1\}) \\
(\lambda x_1 \dots x_n)\, \lambda x_{n+1}.\, T_0 &\to_\lambda (\lambda x_1 \dots x_{n+1})\, T_0
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&(\lambda xy)\,x\,x\,y \to_{@_0} \\
&(\lambda xy)\,x\,x \to_{\mathsf{S}} \\
&(\lambda x)\,x\,x
\end{aligned}
$$

$$
\begin{aligned}
(\lambda x_1 \ldots x_n)\,T_0\;T_1 &\to_{@_i} (\lambda x_1 \ldots x_n)\,T_i && (i \in \{0,1\}) \\
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(\lambda x_1 \ldots x_n x_{n+1})\,T_0 &\to_{\mathsf{S}} (\lambda x_1 \ldots x_n)\,T_0 && (\text{if } \lambda x_{n+1} \text{ is vacuous})
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$$

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&(\lambda xy)\,x\,x \to_{\mathsf{S}}\\
&(\lambda x)\,x\,x \to_{@_0}\\
&(\lambda x)\,x
\end{aligned}
$$

$\to_{\mathrm{reg}^+}$-generated subterms of $\lambda x.\lambda y.x\,x\,y$ w.r.t. rewrite relation $\to_{\mathrm{reg}^+}$:

$$
\begin{aligned}
(\lambda x_1 \dots x_n)\,T_0\,T_1 &\to_{@_i} (\lambda x_1 \dots x_n)\,T_i && (i \in \{0,1\})\\
(\lambda x_1 \dots x_n)\,\lambda x_{n+1}.\,T_0 &\to_{\lambda} (\lambda x_1 \dots x_{n+1})\,T_0\\
(\lambda x_1 \dots x_n x_{n+1})\,T_0 &\to_{\mathsf{S}} (\lambda x_1 \dots x_n)\,T_0 && (\text{if } \lambda x_{n+1} \text{ is vacuous})
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&(\lambda xy)\,x\,x \to_{\mathsf{S}}\\
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(\lambda x_1 \ldots x_n)\,\lambda x_{n+1}.\,T_0 &\to_{\lambda} (\lambda x_1 \ldots x_{n+1})\,T_0 &&\\
(\lambda x_1 \ldots x_n x_{n+1})\,T_0 &\to_{\mathsf{S}} (\lambda x_1 \ldots x_n)\,T_0 && (\text{if } \lambda x_{n+1} \text{ is vacuous})
\end{aligned}
$$

formalized as a CRS, e.g. rule:

$$\mathrm{pre}_n([x_1 \ldots x_n]\mathrm{abs}([x_{n+1}]Z(\vec{x}))) \to \mathrm{pre}_{n+1}([x_1 \ldots x_{n+1}]Z(\vec{x}))$$

Deconstructing/observing $\boldsymbol{\lambda}^{\infty}$-terms

$$
\begin{array}{l}
()\,\lambda x.\lambda y.x\,x\,y \to_{\lambda} \\
(\lambda x)\,\lambda y.x\,x\,y \to_{\lambda} \\
(\lambda xy)\,x\,x\,y \to_{@_1} \\
(\lambda xy)\,y
\end{array}
\qquad
\begin{array}{l}
()\,\lambda x.\lambda y.x\,x\,y \to_{\lambda} \\
(\lambda x)\,\lambda y.x\,x\,y \to_{\lambda} \\
(\lambda xy)\,x\,x\,y \to_{@_0} \\
(\lambda xy)\,x\,x \to_{S} \\
(\lambda x)\,x\,x \to_{@_0} \\
(\lambda x)\,x
\end{array}
\qquad
\begin{array}{l}
()\,\lambda x.\lambda y.x\,x\,y \to_{\lambda} \\
(\lambda x)\,\lambda y.x\,x\,y \to_{\lambda} \\
(\lambda xy)\,x\,x\,y \to_{@_0} \\
(\lambda xy)\,x\,x \to_{S} \\
(\lambda x)\,x\,x \to_{@_1} \\
(\lambda x)\,x
\end{array}
$$

$\to_{\mathrm{reg}^+}$-generated subterms of $\lambda x.\lambda y.x\,x\,y$ w.r.t. rewrite relation $\to_{\mathrm{reg}^+}$:

$$
\begin{array}{rcll}
(\lambda x_1 \dots x_n)\,T_0\,T_1 & \to_{@_i} & (\lambda x_1 \dots x_n)\,T_i & (i \in \{0,1\}) \\
(\lambda x_1 \dots x_n)\,\lambda x_{n+1}.\,T_0 & \to_{\lambda} & (\lambda x_1 \dots x_{n+1})\,T_0 & \\
(\lambda x_1 \dots x_n x_{n+1})\,T_0 & \to_{S} & (\lambda x_1 \dots x_n)\,T_0 & (\text{if } \lambda x_{n+1} \text{ is vacuous})
\end{array}
$$

formalized as a CRS, e.g. rule:

$$
\mathrm{pre}_n([x_1 \dots x_n]\,\mathrm{abs}([x_{n+1}]\,Z(\vec{x}))) \to \mathrm{pre}_{n+1}([x_1 \dots x_{n+1}]\,Z(\vec{x}))
$$

Generated subterms

$$\begin{array}{l} ()\,\lambda x.\lambda y.x\,x\,y \to_{\lambda} \\ (\lambda x)\,\lambda y.x\,x\,y \to_{\lambda} \\ (\lambda xy)\,x\,x\,y \to_{@_1} \\ (\lambda xy)\,y \end{array}
\qquad
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$\to_{\mathrm{reg}^+}$-generated subterms of $\lambda x.\lambda y.x\,x\,y$ w.r.t. rewrite relation $\to_{\mathrm{reg}^+}$:

$$\begin{array}{rcll} (\lambda x_1 \ldots x_n)\,T_0\,T_1 & \to_{@_i} & (\lambda x_1 \ldots x_n)\,T_i & (i \in \{0,1\}) \\ (\lambda x_1 \ldots x_n)\,\lambda x_{n+1}.\,T_0 & \to_{\lambda} & (\lambda x_1 \ldots x_{n+1})\,T_0 & \\ (\lambda x_1 \ldots x_n x_{n+1})\,T_0 & \to_{S} & (\lambda x_1 \ldots x_n)\,T_0 & (\text{if } \lambda x_{n+1} \text{ is vacuous}) \end{array}$$

$\to_{\mathrm{reg}}$-generated subterms w.r.t. rewrite relation $\to_{\mathrm{reg}}$:

$$(\lambda x_1 \ldots x_i \ldots x_{n+1})\,T_0 \to_{\mathrm{del}} (\lambda x_1 \ldots x_{i-1} x_{i+1} \ldots x_{n+1})\,T_0 \quad (\text{if } \lambda x_i \text{ is vacuous})$$

Generated subterms

$() \lambda x.\lambda y.x\,x\,y \to_\lambda$
$(\lambda x)\,\lambda y.x\,x\,y \to_\lambda$
$(\lambda xy)\,x\,x\,y \to_{@_1}$
$(\lambda xy)\,y \to_{\mathrm{del}}$
$(\lambda y)\,y$

$() \lambda x.\lambda y.x\,x\,y \to_\lambda$
$(\lambda x)\,\lambda y.x\,x\,y \to_\lambda$
$(\lambda xy)\,x\,x\,y \to_{@_0}$
$(\lambda xy)\,x\,x \to_{\mathsf{S}}$
$(\lambda x)\,x\,x \to_{@_0}$
$(\lambda x)\,x$

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$(\lambda x)\,x$

$\to_{\mathrm{reg}^+}$-generated subterms of $\lambda x.\lambda y.x\,x\,y$ w.r.t. rewrite relation $\to_{\mathrm{reg}^+}$:

$$(\lambda x_1 \dots x_n)\,T_0\,T_1 \to_{@_i} (\lambda x_1 \dots x_n)\,T_i \qquad (i \in \{0,1\})$$
$$(\lambda x_1 \dots x_n)\,\lambda x_{n+1}.T_0 \to_\lambda (\lambda x_1 \dots x_{n+1})\,T_0$$
$$(\lambda x_1 \dots x_n x_{n+1})\,T_0 \to_{\mathsf{S}} (\lambda x_1 \dots x_n)\,T_0 \qquad (\text{if } \lambda x_{n+1} \text{ is vacuous})$$

$\to_{\mathrm{reg}}$-generated subterms w.r.t. rewrite relation $\to_{\mathrm{reg}}$:

$$(\lambda x_1 \dots x_i \dots x_{n+1})\,T_0 \to_{\mathrm{del}} (\lambda x_1 \dots x_{i-1} x_{i+1} \dots x_{n+1})\,T_0 \quad (\text{if } \lambda x_i \text{ is vacuous})$$

Regularity and strong regularity

An infinite first-order term t is regular if:

t has only finitely many subterms.

Definition

1. A $\boldsymbol{\lambda}^{\infty}$-term T is strongly regular if:

   () T has only finitely many $\rightarrow_{\text{reg}^{+}}$-generated subterms.

Regularity and strong regularity

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t has only finitely many subterms.

Definition

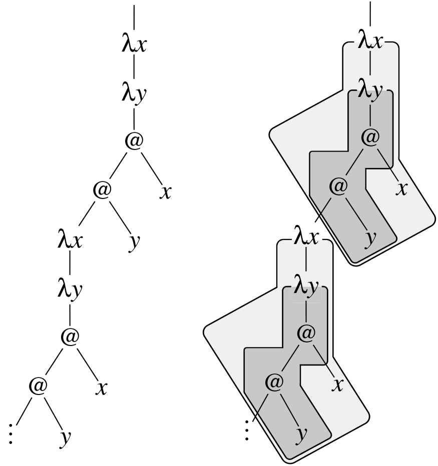
1. A $\boldsymbol{\lambda}^{\infty}$-term T is strongly regular if:

   () T has only finitely many $\rightarrow_{\text{reg}^{+}}$-generated subterms.

2. A $\boldsymbol{\lambda}^{\infty}$-term U is regular if:

   () U has only finitely many $\rightarrow_{\text{reg}}$-generated subterms.

strongly regular $\boldsymbol{\lambda}^{\infty}$-term

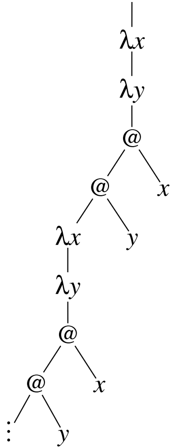


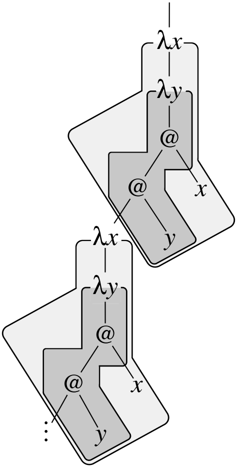
$() \, T \quad = \quad () \, \lambda xy.\, T \, y \, x$

$T = \lambda xy.\, T \, y \, x$

$\rightarrow_{\mathsf{reg}^+}$-generated subterms

strongly regular $\boldsymbol{\lambda}^{\infty}$-term



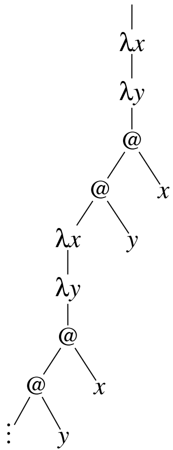


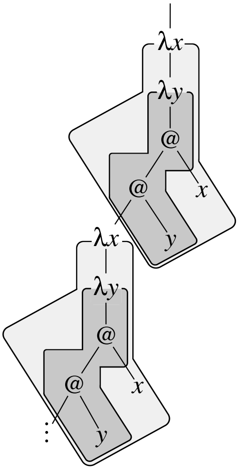
$$
\begin{aligned}
()\,T \quad &= \quad ()\,\lambda xy.\,T\;y\;x \\
&\to_{\lambda} \quad (\lambda x)\,\lambda y.\,T\;y\;x
\end{aligned}
$$

$T = \lambda xy.\,T\;y\;x$

$\to_{\text{reg}^{+}}$-generated subterms

strongly regular $\boldsymbol{\lambda}^{\infty}$-term



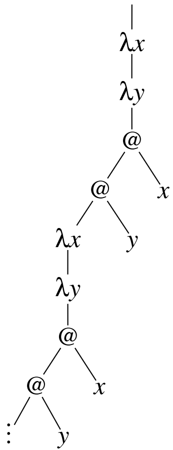


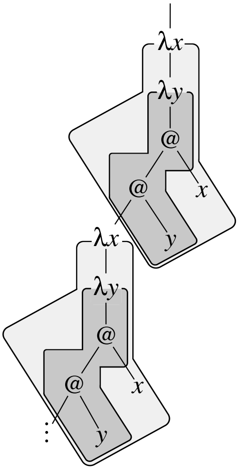
$$
\begin{aligned}
()\,T \quad &= \quad ()\,\lambda xy.\,T\;y\;x \\
&\to_{\lambda} \quad (\lambda x)\,\lambda y.\,T\;y\;x \\
&\to_{\lambda} \quad (\lambda xy)\,T\;y\;x
\end{aligned}
$$

$T = \lambda xy.\,T\;y\;x$

$\to_{\mathrm{reg}^{+}}$-generated subterms

strongly regular $\boldsymbol{\lambda}^{\infty}$-term



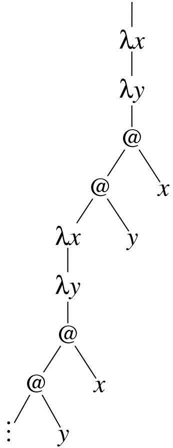


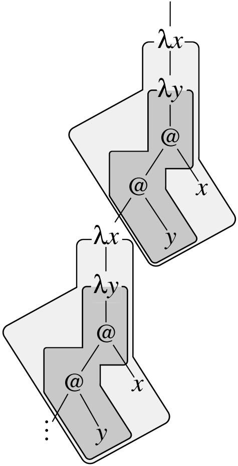
$$
\begin{array}{lcl}
()\,T & = & ()\,\lambda xy.\,T\,y\,x \\
 & \to_{\lambda} & (\lambda x)\,\lambda y.\,T\,y\,x \\
 & \to_{\lambda} & (\lambda xy)\,T\,y\,x \\
 & \to_{@_0} & (\lambda xy)\,T\,y
\end{array}
$$

$T = \lambda xy.\,T\,y\,x$

$\to_{\mathsf{reg}^{+}}$-generated subterms

strongly regular $\boldsymbol{\lambda}^{\infty}$-term



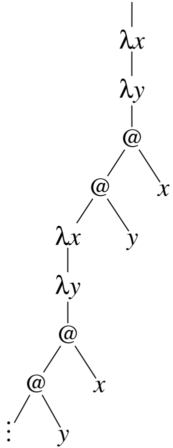


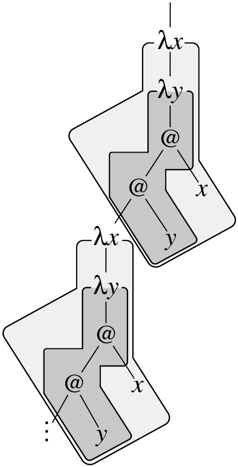
$$
\begin{aligned}
()\, T &= ()\, \lambda xy.\, T\, y\, x \\
&\to_{\lambda} (\lambda x)\, \lambda y.\, T\, y\, x \\
&\to_{\lambda} (\lambda xy)\, T\, y\, x \\
&\to_{@_0} (\lambda xy)\, T\, y \\
&\to_{@_0} (\lambda xy)\, T
\end{aligned}
$$

$T = \lambda xy.\, T\, y\, x$

$\to_{\text{reg}^+}$-generated subterms

strongly regular $\boldsymbol{\lambda}^{\infty}$-term



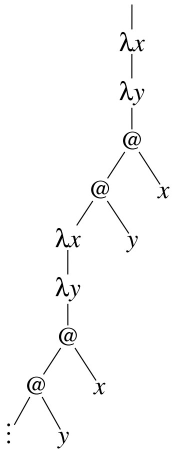


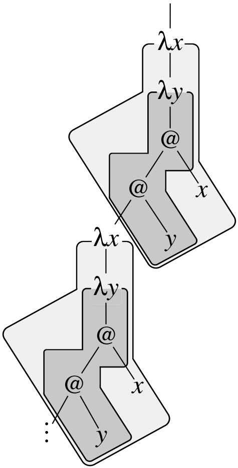
$$
\begin{aligned}
()\,T &= ()\,\lambda xy.\,T\,y\,x \\
&\to_{\lambda} (\lambda x)\,\lambda y.\,T\,y\,x \\
&\to_{\lambda} (\lambda xy)\,T\,y\,x \\
&\to_{@_0} (\lambda xy)\,T\,y \\
&\to_{@_0} (\lambda xy)\,T \\
&\to_{\mathsf{S}} (\lambda x)\,T
\end{aligned}
$$

$T = \lambda xy.\,T\,y\,x$

$\to_{\mathrm{reg}^+}$-generated subterms

strongly regular λ^∞-term



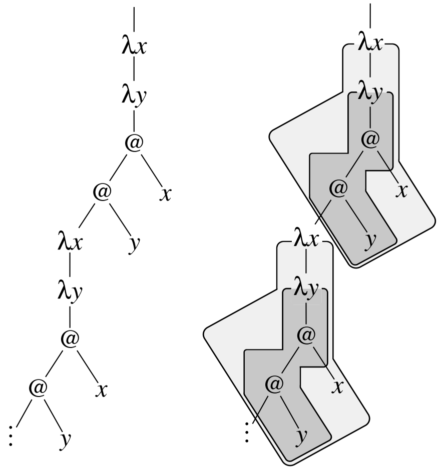


$$
\begin{aligned}
()\,T &= ()\,\lambda xy.\,T\;y\;x\\
&\to_\lambda (\lambda x)\,\lambda y.\,T\;y\;x\\
&\to_\lambda (\lambda xy)\,T\;y\;x\\
&\to_{@_0} (\lambda xy)\,T\;y\\
&\to_{@_0} (\lambda xy)\,T\\
&\to_{S} (\lambda x)\,T\\
&\to_{S} ()\,T
\end{aligned}
$$

$T = \lambda xy.\,T\;y\;x$

$\to_{reg^+}$-generated subterms

strongly regular λ^∞-term

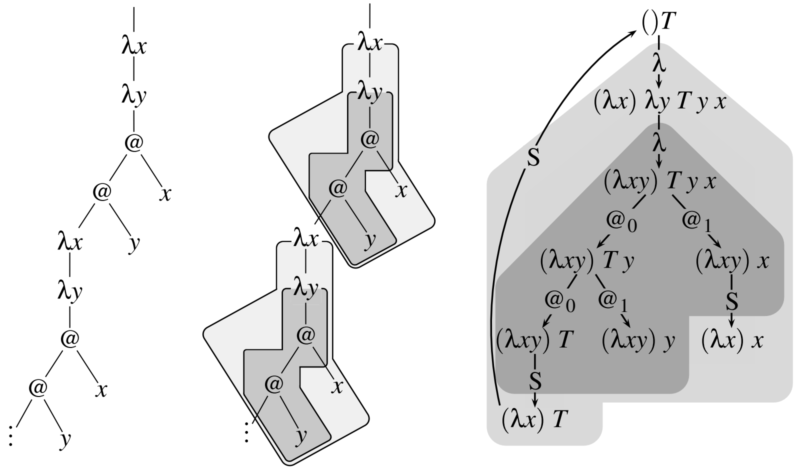


$$
\begin{aligned}
()\,T &= ()\,\lambda xy.\,T\,y\,x \\
&\to_\lambda (\lambda x)\,\lambda y.\,T\,y\,x \\
&\to_\lambda (\lambda xy)\,T\,y\,x \\
&\to_{@_0} (\lambda xy)\,T\,y \\
&\to_{@_0} (\lambda xy)\,T \\
&\to_S (\lambda x)\,T \\
&\to_S ()\,T \\
&\dots
\end{aligned}
$$

$T = \lambda xy.\,T\,y\,x$

$\to_{\mathrm{reg}^+}$-generated subterms

strongly regular $\boldsymbol{\lambda}^{\infty}$-term

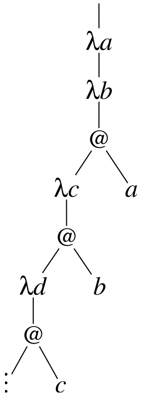


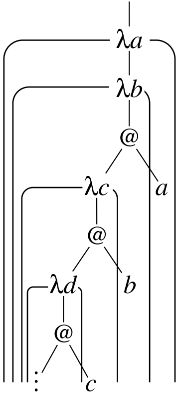
$T = \lambda xy.\, T\ y\ x$

finitely many $\rightarrow_{\text{reg}^+}$-generated subterms

$\Longrightarrow$ T is strongly regular

Not strongly regular $\boldsymbol{\lambda}^{\infty}$-term



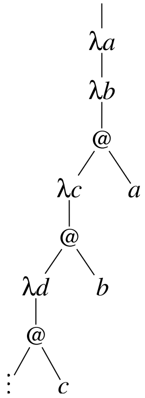


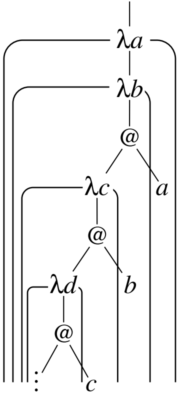
$U \quad = \quad ()\,\lambda a.\lambda b.(\ldots)\,a$

$\boldsymbol{\lambda}^{\infty}$-term U

$\rightarrow_{\text{reg}^+}$-generated subterms

Not strongly regular $\boldsymbol{\lambda}^{\infty}$-term



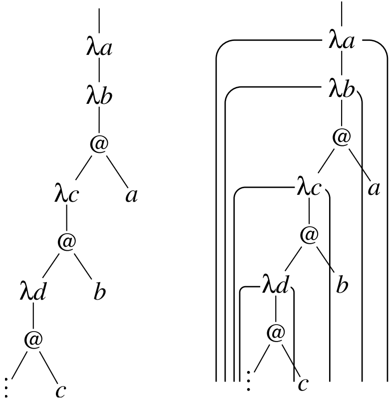


$$U \quad = \quad ()\,\lambda a.\lambda b.(\ldots)\,a$$
$$\rightarrow_{\lambda} \quad (\lambda a)\,\lambda b.(\lambda c.\ \ldots)\,a$$

$\boldsymbol{\lambda}^{\infty}$-term U

$\rightarrow_{\mathsf{reg}^+}$-generated subterms

Not strongly regular $\boldsymbol{\lambda}^\infty$-term

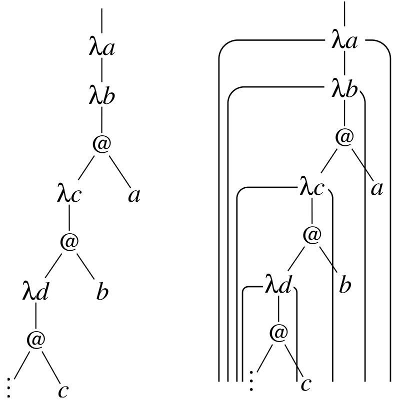


$$
\begin{aligned}
U \quad &= \quad ()\,\lambda a.\lambda b.(\ldots)\,a \\
&\to_\lambda \quad (\lambda a)\,\lambda b.(\lambda c.\ \ldots)\,a \\
&\to_\lambda \quad (\lambda ab)\,(\lambda c.(\ldots)\,b)\,a
\end{aligned}
$$

$\boldsymbol{\lambda}^\infty$-term U

$\to_{\mathsf{reg}^+}$-generated subterms

Not strongly regular $\boldsymbol{\lambda}^{\infty}$-term

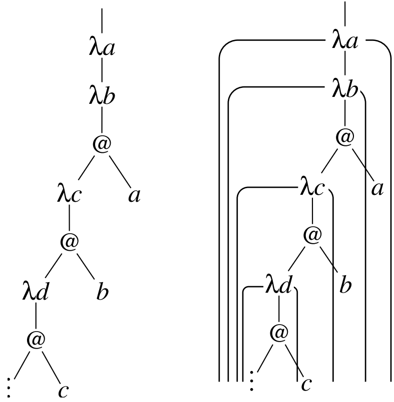


$\boldsymbol{\lambda}^{\infty}$-term U

$$
\begin{aligned}
U \quad &= \quad ()\,\lambda a.\lambda b.(\ldots)\,a \\
\to_{\lambda} \quad & \quad (\lambda a)\,\lambda b.(\lambda c.\ \ldots)\,a \\
\to_{\lambda} \quad & \quad (\lambda ab)\,(\lambda c.(\ldots)\,b)\,a \\
\to_{@_0} \quad & \quad (\lambda ab)\,\lambda c.(\lambda d.\ \ldots)\,b
\end{aligned}
$$

$\to_{\mathrm{reg}^+}$-generated subterms

Not strongly regular $\boldsymbol{\lambda}^\infty$-term

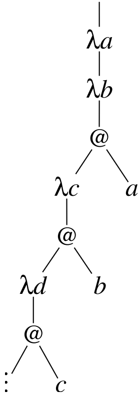


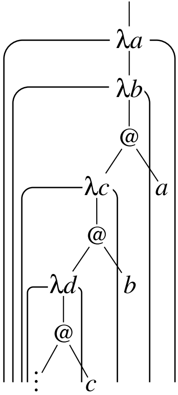
$$
\begin{aligned}
U \quad &= \quad ()\,\lambda a.\lambda b.(\ldots)\,a \\
\to_{\lambda} \quad & \quad (\lambda a)\,\lambda b.(\lambda c.\ \ldots)\,a \\
\to_{\lambda} \quad & \quad (\lambda ab)\,(\lambda c.(\ldots)\,b)\,a \\
\to_{@_0} \quad & \quad (\lambda ab)\,\lambda c.(\lambda d.\ \ldots)\,b \\
\to_{\lambda} \quad & \quad (\lambda abc)\,(\lambda d.(\ldots)\,c)\,b
\end{aligned}
$$

$\boldsymbol{\lambda}^\infty$-term U

$\to_{reg^+}$-generated subterms

Not strongly regular $\boldsymbol{\lambda}^\infty$-term



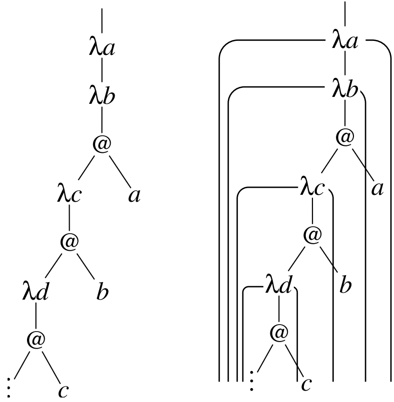


$$
\begin{aligned}
U \quad &= \quad ()\,\lambda a.\lambda b.(\ldots)\,a \\
&\rightarrow_{\lambda} \quad (\lambda a)\,\lambda b.(\lambda c.\ \ldots)\,a \\
&\rightarrow_{\lambda} \quad (\lambda ab)\,(\lambda c.(\ldots)\,b)\,a \\
&\rightarrow_{@_0} \quad (\lambda ab)\,\lambda c.(\lambda d.\ \ldots)\,b \\
&\rightarrow_{\lambda} \quad (\lambda abc)\,(\lambda d.(\ldots)\,c)\,b \\
&\rightarrow_{@_0} \quad (\lambda abc)\,\lambda d.(\lambda e.\ \ldots)\,c
\end{aligned}
$$

$\boldsymbol{\lambda}^\infty$-term U

$\rightarrow_{\text{reg}^+}$-generated subterms

Not strongly regular $\boldsymbol{\lambda}^{\infty}$-term

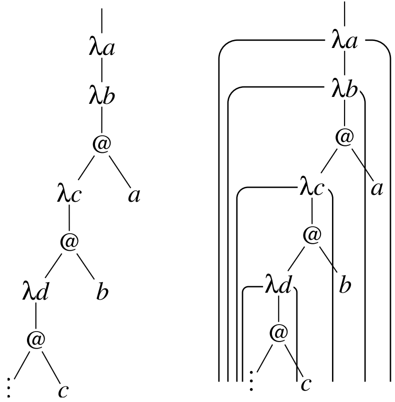


$\boldsymbol{\lambda}^{\infty}$-term U

$$
\begin{array}{rcl}
U & = & ()\,\lambda a.\lambda b.(\ldots)\,a \\
 & \to_{\lambda} & (\lambda a)\,\lambda b.(\lambda c.\;\ldots)\,a \\
 & \to_{\lambda} & (\lambda ab)\,(\lambda c.(\ldots)\,b)\,a \\
 & \to_{@_0} & (\lambda ab)\,\lambda c.(\lambda d.\;\ldots)\,b \\
 & \to_{\lambda} & (\lambda abc)\,(\lambda d.(\ldots)\,c)\,b \\
 & \to_{@_0} & (\lambda abc)\,\lambda d.(\lambda e.\;\ldots)\,c \\
 & \to_{\lambda} & (\lambda abcd)\,(\lambda e.(\ldots)\,d)\,c
\end{array}
$$

$\to_{\mathrm{reg}^{+}}$-generated subterms

Not strongly regular $\boldsymbol{\lambda}^\infty$-term



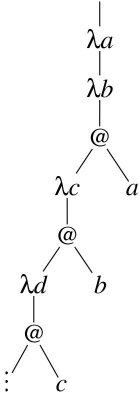
$$
\begin{array}{lll}
U & = & ()\,\lambda a.\lambda b.(\ldots)\,a \\
 & \to_\lambda & (\lambda a)\,\lambda b.(\lambda c.\ \ldots)\,a \\
 & \to_\lambda & (\lambda ab)\,(\lambda c.(\ldots)\,b)\,a \\
 & \to_{@_0} & (\lambda ab)\,\lambda c.(\lambda d.\ \ldots)\,b \\
 & \to_\lambda & (\lambda abc)\,(\lambda d.(\ldots)\,c)\,b \\
 & \to_{@_0} & (\lambda abc)\,\lambda d.(\lambda e.\ \ldots)\,c \\
 & \to_\lambda & (\lambda abcd)\,(\lambda e.(\ldots)\,d)\,c \\
 & \to_{@_0} & (\lambda abcd)\,\lambda e.(\lambda f.\ \ldots)\,d \\
 & \ldots &
\end{array}
$$

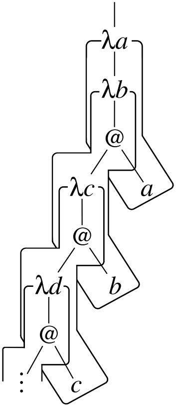
$\boldsymbol{\lambda}^\infty$-term U

infinitely many $\to_{\text{reg}^+}$-generated subterms

$\Longrightarrow$ U is not strongly regular

Regular $\boldsymbol{\lambda^\infty}$-term



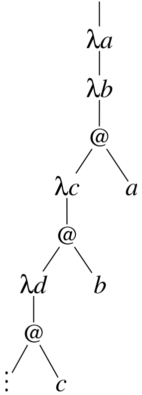


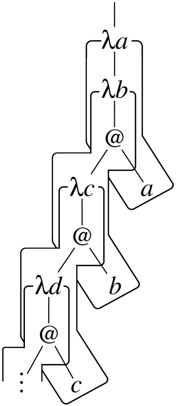
$U \quad = \quad ()\,\lambda a.\lambda b.(\ldots)\,a$

$\boldsymbol{\lambda^\infty}$-term U

$\rightarrow_{\text{reg}}$-generated subterms

Regular λ^∞-term



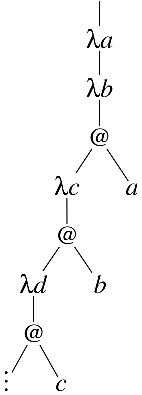


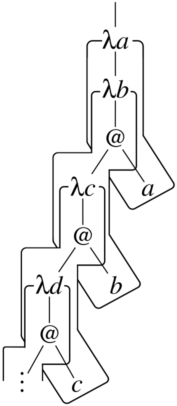
$$\begin{aligned} U &= & ()\,\lambda a.\lambda b.(\ldots)\,a \\ &\to_\lambda & (\lambda a)\,\lambda b.(\lambda c.\ \ldots)\,a \end{aligned}$$

λ^∞-term U

$\to_{\text{reg}}$-generated subterms

Regular λ^∞-term



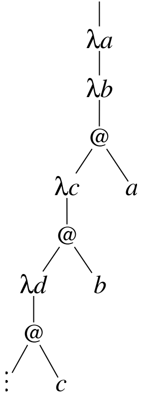


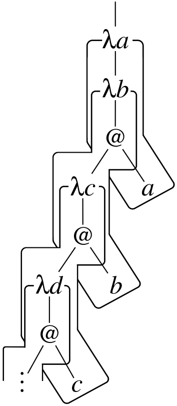
$$
\begin{aligned}
U &= && ()\,\lambda a.\lambda b.(\ldots)\,a \\
&\to_\lambda && (\lambda a)\,\lambda b.(\lambda c.\ \ldots)\,a \\
&\to_\lambda && (\lambda ab)\,(\lambda c.(\ldots)\,b)\,a
\end{aligned}
$$

λ^∞-term U

$\to_{\text{reg}}$-generated subterms

Regular $\boldsymbol{\lambda}^\infty$-term



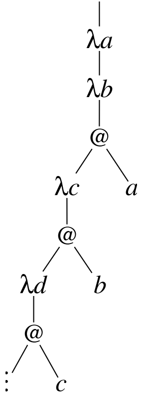


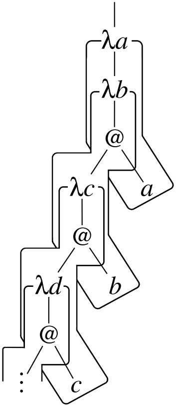
$$
\begin{aligned}
U \quad &= \quad (\,)\,\lambda a.\lambda b.(\ldots)\; a \\
&\rightarrow_{\lambda} \quad (\lambda a)\,\lambda b.(\lambda c.\;\ldots)\; a \\
&\rightarrow_{\lambda} \quad (\lambda ab)\,(\lambda c.(\ldots)\; b)\; a \\
&\rightarrow_{@_0} \quad (\lambda ab)\,\lambda c.(\lambda d.\;\ldots)\; b
\end{aligned}
$$

$\boldsymbol{\lambda}^\infty$-term U

$\rightarrow_{\mathrm{reg}}$-generated subterms

Regular $\boldsymbol{\lambda^\infty}$-term



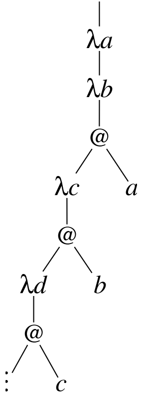


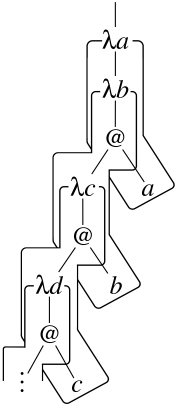
$$\begin{array}{lll}
U & = & ()\,\lambda a.\lambda b.(\ldots)\,a \\
 & \rightarrow_\lambda & (\lambda a)\,\lambda b.(\lambda c.\ \ldots)\,a \\
 & \rightarrow_\lambda & (\lambda ab)\,(\lambda c.(\ldots)\,b)\,a \\
 & \rightarrow_{@_0} & (\lambda ab)\,\lambda c.(\lambda d.\ \ldots)\,b \\
 & \rightarrow_{\mathsf{del}} & (\lambda b)\,\lambda c.(\lambda d.\ \ldots)\,b
\end{array}$$

$\boldsymbol{\lambda^\infty}$-term U

$\rightarrow_{\mathsf{reg}}$-generated subterms

Regular $\boldsymbol{\lambda}^{\infty}$-term



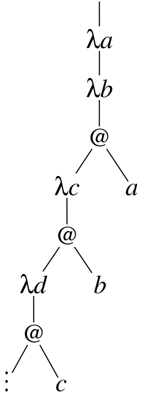


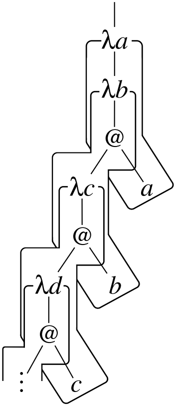
$$
\begin{array}{lll}
U & = & ()\,\lambda a.\lambda b.(\ldots)\,a \\
 & \to_{\lambda} & (\lambda a)\,\lambda b.(\lambda c.\ \ldots)\,a \\
 & \to_{\lambda} & (\lambda ab)\,(\lambda c.(\ldots)\,b)\,a \\
 & \to_{@_0} & (\lambda a b)\,\lambda c.(\lambda d.\ \ldots)\,b \\
 & \to_{\mathsf{del}} & (\lambda b)\,\lambda c.(\lambda d.\ \ldots)\,b \\
 & \to_{\lambda} & (\lambda bc)\,(\lambda d.(\ldots)\,c)\,b
\end{array}
$$

$\boldsymbol{\lambda}^{\infty}$-term U

$\to_{\mathsf{reg}}$-generated subterms

Regular $\boldsymbol{\lambda}^\infty$-term



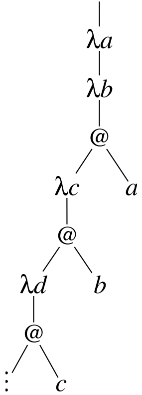


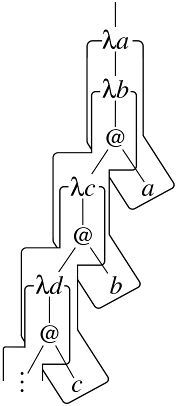
$$
\begin{aligned}
U \quad &= \quad (\,)\,\lambda a.\lambda b.(\ldots)\,a\\
&\to_{\lambda} \quad (\lambda a)\,\lambda b.(\lambda c.\ \ldots)\,a\\
&\to_{\lambda} \quad (\lambda ab)\,(\lambda c.(\ldots)\,b)\,a\\
&\to_{@_0} \quad (\lambda ab)\,\lambda c.(\lambda d.\ \ldots)\,b\\
&\to_{\mathsf{del}} \quad (\lambda b)\,\lambda c.(\lambda d.\ \ldots)\,b\\
&\to_{\lambda} \quad (\lambda bc)\,(\lambda d.(\ldots)\,c)\,b\\
&\to_{@_0} \quad (\lambda bc)\,\lambda d.(\lambda d.\ \ldots)\,c
\end{aligned}
$$

$\boldsymbol{\lambda}^\infty$-term U

$\to_{\mathsf{reg}}$-generated subterms

Regular $\boldsymbol{\lambda}^{\infty}$-term



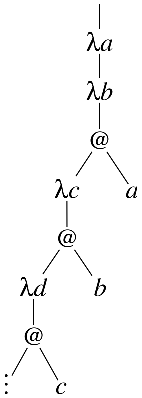


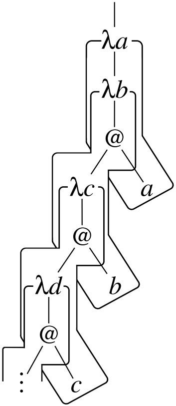
$$
\begin{array}{rll}
U & = & ()\,\lambda a.\lambda b.(\ldots)\,a \\
 & \to_{\lambda} & (\lambda a)\,\lambda b.(\lambda c.\ \ldots)\,a \\
 & \to_{\lambda} & (\lambda ab)\,(\lambda c.(\ldots)\,b)\,a \\
 & \to_{@_0} & (\lambda a b)\,\lambda c.(\lambda d.\ \ldots)\,b \\
 & \to_{\mathsf{del}} & (\lambda b)\,\lambda c.(\lambda d.\ \ldots)\,b \\
 & \to_{\lambda} & (\lambda bc)\,(\lambda d.(\ldots)\,c)\,b \\
 & \to_{@_0} & (\lambda bc)\,\lambda d.(\lambda d.\ \ldots)\,c \\
 & \to_{\mathsf{del}} & (\lambda c)\,\lambda d.(\lambda e.\ \ldots)\,d
\end{array}
$$

$\boldsymbol{\lambda}^{\infty}$-term U

$\to_{\mathsf{reg}}$-generated subterms

Regular λ^∞-term



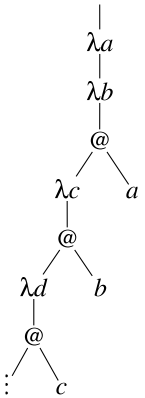


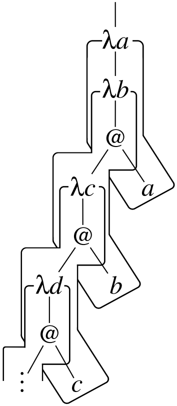
$$
\begin{array}{lll}
U & = & ()\,\lambda a.\lambda b.(\ldots)\,a \\
 & \to_\lambda & (\lambda a)\,\lambda b.(\lambda c.\ \ldots)\,a \\
 & \to_\lambda & (\lambda ab)\,(\lambda c.(\ldots)\,b)\,a \\
 & \to_{@_0} & (\lambda ab)\,\lambda c.(\lambda d.\ \ldots)\,b \\
 & \to_{\mathsf{del}} & (\lambda b)\,\lambda c.(\lambda d.\ \ldots)\,b \\
 & \to_\lambda & (\lambda bc)\,(\lambda d.(\ldots)\,c)\,b \\
 & \to_{@_0} & (\lambda bc)\,\lambda d.(\lambda d.\ \ldots)\,c \\
 & \to_{\mathsf{del}} & (\lambda c)\,\lambda d.(\lambda e.\ \ldots)\,d \\
 & \to_\lambda & (\lambda cd)\,(\lambda e.(\ldots)\,d)\,c
\end{array}
$$

λ^∞-term U

$\to_{\mathsf{reg}}$-generated subterms

Regular $\boldsymbol{\lambda}^\infty$-term



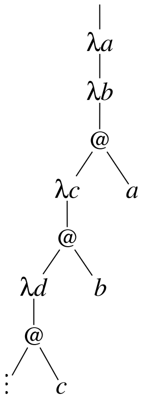


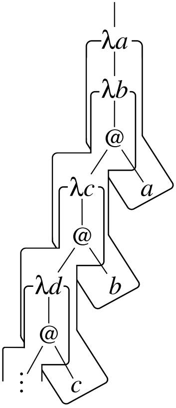
$$
\begin{aligned}
U \quad &= \quad ()\,\lambda a.\lambda b.(\ldots)\,a \\
&\to_\lambda \quad (\lambda a)\,\lambda b.(\lambda c.\ \ldots)\,a \\
&\to_\lambda \quad (\lambda ab)\,(\lambda c.(\ldots)\,b)\,a \\
&\to_{@_0} \quad (\lambda ab)\,\lambda c.(\lambda d.\ \ldots)\,b \\
&\to_{\mathsf{del}} \quad (\lambda b)\,\lambda c.(\lambda d.\ \ldots)\,b \\
&\to_\lambda \quad (\lambda bc)\,(\lambda d.(\ldots)\,c)\,b \\
&\to_{@_0} \quad (\lambda bc)\,\lambda d.(\lambda d.\ \ldots)\,c \\
&\to_{\mathsf{del}} \quad (\lambda c)\,\lambda d.(\lambda e.\ \ldots)\,d \\
&\to_\lambda \quad (\lambda cd)\,(\lambda e.(\ldots)\,d)\,c \\
&\to_{@_0} \quad (\lambda cd)\,\lambda e.(\lambda f.\ \ldots)\,d
\end{aligned}
$$

$\boldsymbol{\lambda}^\infty$-term U

$\to_{\mathsf{reg}}$-generated subterms

Regular $\boldsymbol{\lambda}^{\infty}$-term



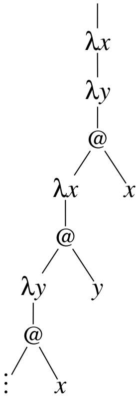


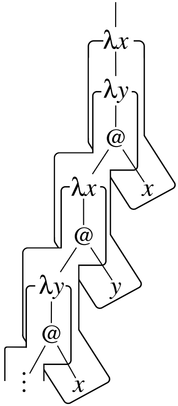
$$
\begin{aligned}
U \quad &= && ()\,\lambda a.\lambda b.(\ldots)\,a\\
&\to_{\lambda} && (\lambda a)\,\lambda b.(\lambda c.\ \ldots)\,a\\
&\to_{\lambda} && (\lambda ab)\,(\lambda c.(\ldots)\,b)\,a\\
&\to_{@_0} && (\lambda ab)\,\lambda c.(\lambda d.\ \ldots)\,b\\
&\to_{\mathsf{del}} && (\lambda b)\,\lambda c.(\lambda d.\ \ldots)\,b\\
&\to_{\lambda} && (\lambda bc)\,(\lambda d.(\ldots)\,c)\,b\\
&\to_{@_0} && (\lambda bc)\,\lambda d.(\lambda d.\ \ldots)\,c\\
&\to_{\mathsf{del}} && (\lambda c)\,\lambda d.(\lambda e.\ \ldots)\,d\\
&\to_{\lambda} && (\lambda cd)\,(\lambda e.(\ldots)\,d)\,c\\
&\to_{@_0} && (\lambda cd)\,\lambda e.(\lambda f.\ \ldots)\,d\\
&\to_{\mathsf{del}} && (\lambda d)\,\lambda e.(\lambda f.\ \ldots)\,d\\
& \ldots
\end{aligned}
$$

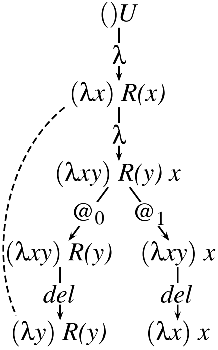
$\boldsymbol{\lambda}^{\infty}$-term U

$\to_{\mathsf{reg}}$-generated subterms

Regular $\boldsymbol{\lambda}^\infty$-term







$\boldsymbol{\lambda}^\infty$-term U

$\{U = \lambda xy.R(y)\,x,$
$R(z) = \lambda u.R(u)\,z\}$

finitely many $\rightarrow_{\text{reg}}$-generated subterms

$\Longrightarrow$ T is regular

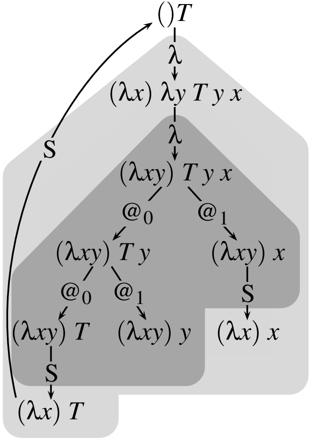
Strongly regular $\Rightarrow$ regular

Proposition

- Every strongly regular $\boldsymbol{\lambda}^{\boldsymbol{\infty}}$-term is also regular.
- Finite λ-terms are both regular and strongly regular.

Proving strong regularity

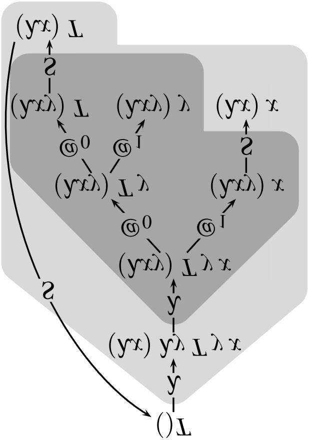
$T = \lambda xy.\, T\, x\, y$



$\rightarrow_{reg^+}$-reduction graph

Proving strong regularity

$T = \lambda xy.\, T\, x\, y$



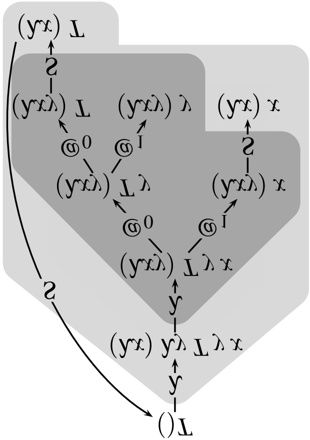
$\rightarrow_{\mathrm{reg}^+}$-reduction graph

Proving strong regularity

$T = \lambda xy.\, T\, x\, y$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{((\,)\,T)^{I}}{(\lambda x)\,T}\,\mathsf{S}}{(\lambda xy)\,T}\,\mathsf{S} \qquad \dfrac{}{(\lambda xy)\,y}\,0}{(\lambda xy)\,T\,y}\,@ \qquad \dfrac{\dfrac{}{(\lambda x)\,x}\,0}{(\lambda xy)\,x}\,\mathsf{S}}{(\lambda xy)\,T\,y\,x}\,@}{(\lambda x)\,\lambda y.\,T\,y\,x}\,\lambda}{(\,)\,\lambda xy.\,T\,y\,x}\,\lambda}{(\,)\,T}\,\mathsf{FIX}, I$$

proof in $\mathbf{Reg}_0^+$



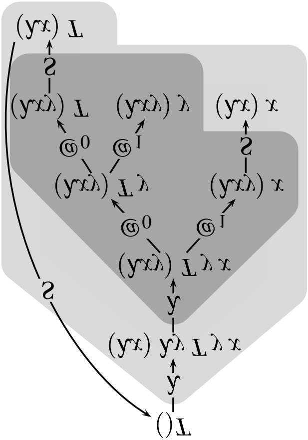
$\rightarrow_{\mathrm{reg}^+}$-reduction graph

Proving strong regularity

$T = \lambda xy.\, T\, x\, y$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{((\,)\,T)^{I}}{(\lambda x)\,T}\,\mathsf{S}}{(\lambda xy)\,T}\,\mathsf{S} \quad \dfrac{}{(\lambda xy)\,y}\,0}{(\lambda xy)\,T\,y}\,@ \quad \dfrac{\dfrac{}{(\lambda x)\,x}\,0}{(\lambda xy)\,x}\,\mathsf{S}}{(\lambda xy)\,T\,y\,x}\,@}{(\lambda x)\,\lambda y.\,T\,y\,x}\,\lambda}{(\,)\,\lambda xy.\,T\,y\,x}\,\lambda}{(\,)\,T}\,\mathsf{FIX}, I$$

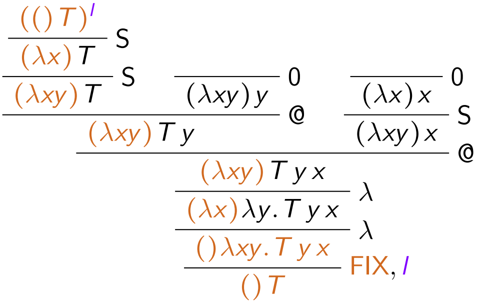
proof in $\mathbf{Reg}_0^+$



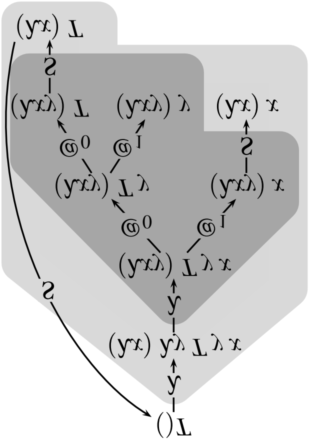
$\rightarrow_{\mathrm{reg}^+}$-reduction graph

Proving strong regularity

$T = \lambda xy.\, T\, x\, y$



proof in $\mathbf{Reg}_0^+$



$\rightarrow_{\mathrm{reg}^+}$-reduction graph

$\boldsymbol{\lambda}_\mu$-expressibility

μ-unfolding rule

$$\boldsymbol{\mu} f.M(f) \to M(\boldsymbol{\mu} f.M(f)) \qquad\qquad \mathsf{mu}([x]Z(x)) \to Z(\mathsf{mu}([x]Z(x)))$$

is orthogonal CRS with confluent rewrite relation $\to_\mu$.

$$\boldsymbol{\mu} f.\lambda x.\lambda y.f\,y\,x \;\to_\mu\; \lambda x.\lambda y.(\boldsymbol{\mu} f.\lambda x.\lambda y.f\,y\,x)\,y\,x$$

$$\ldots$$

$\boldsymbol{\lambda}_{\mu}$-expressibility

μ-unfolding rule

$$\boldsymbol{\mu} f . M(f) \to M(\boldsymbol{\mu} f . M(f)) \qquad \mathsf{mu}([x]\, Z(x)) \to Z(\mathsf{mu}([x]\, Z(x)))$$

is orthogonal CRS with confluent rewrite relation $\to_{\mu}$.

$$\boldsymbol{\mu} f . \lambda x . \lambda y . f\, y\, x \;\to_{\mu}\; \lambda x . \lambda y . (\boldsymbol{\mu} f . \lambda x . \lambda y . f\, y\, x)\, y\, x \;\to_{\mu} \ldots$$

$$\ldots \twoheadrightarrow_{\mu} \lambda x y . (\lambda x y . (\lambda x y . (\ldots)\, y\, x)\, y\, x)\, y\, x$$

$$\boldsymbol{\mu} g . g \;\to_{\mu}\; \boldsymbol{\mu} g . g \;\to_{\mu}\; \ldots$$

$\boldsymbol{\lambda}_\mu$-expressibility

μ-unfolding rule

$$\mu f.M(f) \to M(\mu f.M(f)) \qquad \mathsf{mu}([x]Z(x)) \to Z(\mathsf{mu}([x]Z(x)))$$

is orthogonal CRS with confluent rewrite relation $\to_\mu$.

$$\mu f.\lambda x.\lambda y.f\,y\,x \to_\mu \lambda x.\lambda y.(\mu f.\lambda x.\lambda y.f\,y\,x)\,y\,x \to_\mu \dots$$
$$\dots \twoheadrightarrow_\mu \lambda xy.(\lambda xy.(\lambda xy.(\dots)\,y\,x)\,y\,x)\,y\,x$$

$$\mu g.g \to_\mu \mu g.g \to_\mu \dots$$

Definition

Let M a $\boldsymbol{\lambda}_\mu$-term, T a $\boldsymbol{\lambda}^\infty$-term.

$$M \text{ expresses } T \iff M \twoheadrightarrow_\mu T$$

$\boldsymbol{\lambda}_\mu$-expressibility

μ-unfolding rule

$$\mu f.M(f) \to M(\mu f.M(f)) \qquad \mathsf{mu}([x]Z(x)) \to Z(\mathsf{mu}([x]Z(x)))$$

is orthogonal CRS with confluent rewrite relation $\to_\mu$.

$$\mu f.\lambda x.\lambda y.f\,y\,x \to_\mu \lambda x.\lambda y.(\mu f.\lambda x.\lambda y.f\,y\,x)\,y\,x \to_\mu \dots$$
$$\dots \twoheadrightarrow_\mu \lambda xy.(\lambda xy.(\lambda xy.(\dots)\,y\,x)\,y\,x)\,y\,x$$

$$\mu g.g \to_\mu \mu g.g \to_\mu \dots$$

Definition

Let M a $\boldsymbol{\lambda}_\mu$-term, T a $\boldsymbol{\lambda}^\infty$-term.

$$M \text{ expresses } T \iff M \twoheadrightarrow_\mu T$$

Proposition

1. the $\boldsymbol{\lambda}^\infty$-term T expressed by a $\boldsymbol{\lambda}_\mu$-term M is unique if it exists
2. outermost strategy for $\to_\mu$ is inf. normalizing [Ketema, Simonsen]

$\boldsymbol{\lambda}_\mu$-expressibility

Theorem ($\boldsymbol{\lambda}_\mu$-expressibility)

An $\boldsymbol{\lambda}^{\boldsymbol{\infty}}$-term is $\boldsymbol{\lambda}_\mu$-expressible if and only if it is strongly regular.

$\boldsymbol{\lambda}_\mu$-expressibility

Theorem ($\boldsymbol{\lambda}_\mu$-expressibility)

An $\boldsymbol{\lambda}^{\infty}$-term is $\boldsymbol{\lambda}_\mu$-expressible if and only if it is strongly regular.

Proof.

By a sequence of, mainly, proof-theoretic transformations.

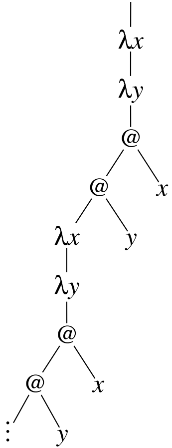
"$\Leftarrow$" ('λ_μ-term-extraction' direction)

For a strongly regular $\boldsymbol{\lambda}^{\infty}$-term T obtain:

1.
2.
3.
4.

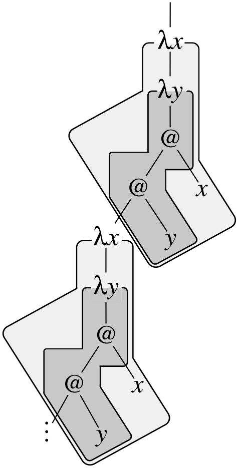
□

$\boldsymbol{\lambda}_{\mu}$-term extraction: step 1



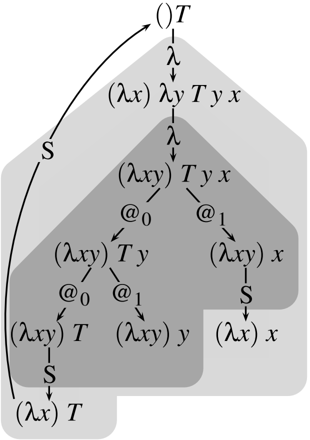
strongly regular λ-term

$\boldsymbol{\lambda}_\mu$-term extraction: step 1



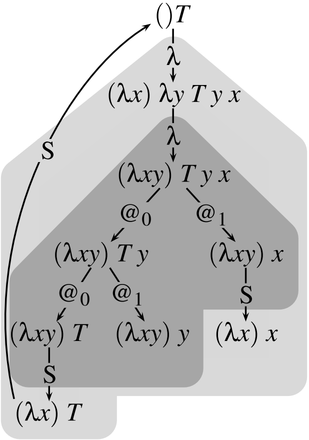
strongly regular λ-term with scope$^+$s

$\boldsymbol{\lambda}_{\mu}$-term extraction: step 1



$\rightarrow_{\text{reg}^+}$-reduction graph

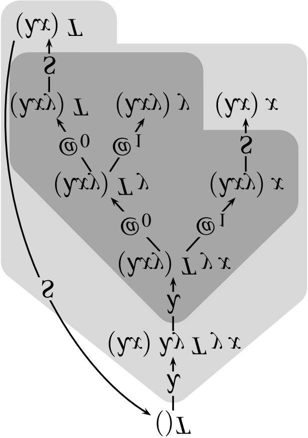
$\boldsymbol{\lambda}_\mu$-term extraction: step 1



$\rightarrow_{\mathrm{reg}^+}$-reduction graph

(only vertical sharing: expressible by a μ-term [Blom, 2001])

$\boldsymbol{\lambda}_\mu$-term extraction: step 2



$\rightarrow_{\text{reg}^+}$-reduction graph

(only vertical sharing: expressible by a μ-term [Blom, 2001])

$\boldsymbol{\lambda}_\mu$-term extraction: step 2

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{((\,)\,T)^{I}}{(\lambda x)\,T}\;\mathsf{S}
      }{(\lambda xy)\,T}\;\mathsf{S}
      \qquad
      \dfrac{}{(\lambda xy)\,y}\;\mathsf{0}
    }{(\lambda xy)\,T\,y}\;@
    \qquad
    \dfrac{
      \dfrac{}{(\lambda x)\,x}\;\mathsf{0}
    }{(\lambda xy)\,x}\;\mathsf{S}
  }{
    \dfrac{
      \dfrac{(\lambda xy)\,T\,y\,x}{(\lambda x)\,\lambda y.\,T\,y\,x}\;\lambda
    }{(\,)\,\lambda xy.\,T\,y\,x}\;\lambda
  }\;@
}{(\,)\,T}\;\mathsf{FIX},I
$$

closed derivation in $\mathbf{Reg}_{\mathbf{0}}^{+}$

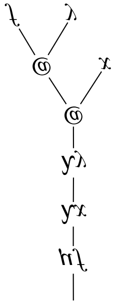
$\boldsymbol{\lambda}_\mu$-term extraction: step 3

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{(()\ \mathsf{c}_l : T)^{l}}{(\lambda x)\ \mathsf{c}_l : T}\ \mathsf{S}
      }{(\lambda xy)\ \mathsf{c}_l : T}\ \mathsf{S}
      \qquad
      \dfrac{}{(\lambda xy)\ y : y}\ 0
    }{(\lambda xy)\ \mathsf{c}_l\ y : T\ y}\ \mathtt{@}
    \qquad
    \dfrac{
      \dfrac{}{(\lambda x)\ x : x}\ 0
    }{(\lambda xy)\ x : x}\ \mathsf{S}
  }{
    \dfrac{
      \dfrac{(\lambda xy)\ \mathsf{c}_l\ y\ x : T\ y\ x}{(\lambda x)\ \lambda y.\mathsf{c}_l\ y\ x : T\ y\ x}\ \lambda
    }{()\ \lambda xy.\mathsf{c}_l\ y\ x : \lambda xy.\,T\ y\ x}\ \lambda
  }\ \mathtt{@}
}{()\ \boldsymbol{\mu} f.\lambda xy.f\ y\ x : T}\ \mathsf{FIX}, l
$$

$\boldsymbol{\lambda}_\mu$-term-annotated derivation in **Expr**

$\boldsymbol{\lambda}_\mu$-term extraction: step 3

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{((\,)\ c_l : T)^{l}}{(\lambda x)\ c_l : T}\,\mathsf{S}}{(\lambda xy)\ c_l : T}\,\mathsf{S} \qquad \dfrac{}{(\lambda xy)\ y : y}\,0}{(\lambda xy)\ c_l\ y : T\ y}\,@ \qquad \dfrac{\dfrac{}{(\lambda x)\ x : x}\,0}{(\lambda xy)\ x : x}\,\mathsf{S}}{(\lambda xy)\ c_l\ y\ x : T\ y\ x}\,@}{(\lambda x)\ \lambda y.c_l\ y\ x : T\ y\ x}\,\lambda}{(\,)\ \lambda xy.c_l\ y\ x : \lambda xy.\,T\ y\ x}\,\lambda}{(\,)\ \boldsymbol{\mu} f.\lambda xy.f\ y\ x : T}\,\mathsf{FIX}, l
$$

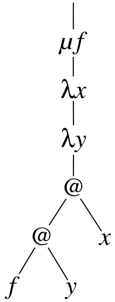


$\boldsymbol{\lambda}_\mu$-term-annotated derivation in **Expr**

$\boldsymbol{\lambda}_\mu$-term extraction: step 3

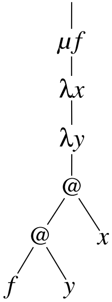
$$\dfrac{\dfrac{\dfrac{((\,)\ \mathsf{c}_l : T)^{l}}{(\lambda x)\ \mathsf{c}_l : T}\ \mathsf{S}}{(\lambda xy)\ \mathsf{c}_l : T}\ \mathsf{S} \qquad \dfrac{}{(\lambda xy)\ y : y}\ 0}{(\lambda xy)\ \mathsf{c}_l\, y : T\, y}\ @ \qquad \dfrac{\dfrac{}{(\lambda x)\ x : x}\ 0}{(\lambda xy)\ x : x}\ \mathsf{S}$$

$$\dfrac{(\lambda xy)\ \mathsf{c}_l\, y : T\, y \qquad (\lambda xy)\ x : x}{\dfrac{(\lambda xy)\ \mathsf{c}_l\, y\, x : T\, y\, x}{\dfrac{(\lambda x)\ \lambda y.\mathsf{c}_l\, y\, x : T\, y\, x}{\dfrac{(\,)\ \lambda xy.\mathsf{c}_l\, y\, x : \lambda xy.\, T\, y\, x}{(\,)\ \boldsymbol{\mu} f.\lambda xy.f\, y\, x : T}\ \mathsf{FIX}, l}\ \lambda}\ \lambda}\ @$$



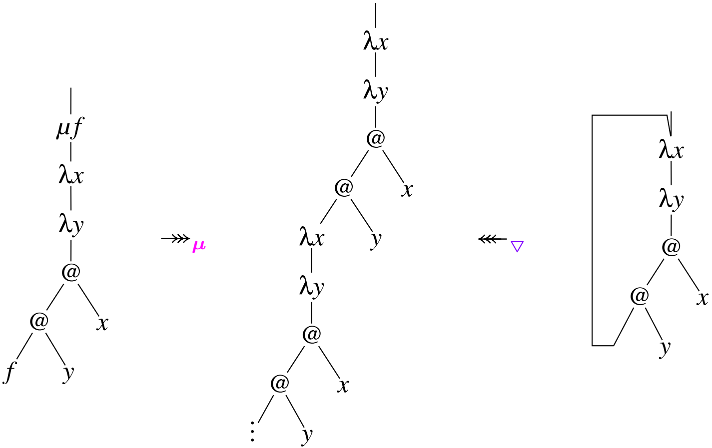
$\boldsymbol{\lambda}_\mu$-term-annotated derivation in **Expr**

$\boldsymbol{\lambda}_\mu$-term extraction: step 4



$\boldsymbol{\lambda}_\mu$-term

$\boldsymbol{\lambda}_{\mu}$-term extraction: step 4



$\boldsymbol{\lambda}_\mu$-expressibility

Theorem ($\boldsymbol{\lambda}_\mu$-expressibility)

An $\boldsymbol{\lambda}^\infty$-term is $\boldsymbol{\lambda}_\mu$-expressible if and only if it is strongly regular.

Proof.

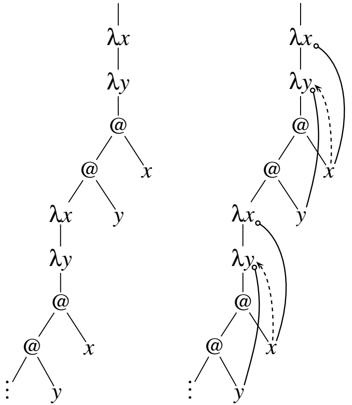
"$\Leftarrow$" ('λ_μ-term-extraction' direction)

For a strongly regular $\boldsymbol{\lambda}^\infty$-term T obtain:

1. finite $\rightarrow_{\text{reg}^+}$-reduction graph G of T
2. derivation $\mathcal{D}$ of $()\,T$ in $\mathbf{Reg}_0^+$
   (only vertical sharing $\Rightarrow$ expressible by μ-term [Blom, 2001])
3. the expressing $\boldsymbol{\lambda}_\mu$-term M from $\mathcal{D}$ by annotation
4. then $M \twoheadrightarrow_\mu T$ can be shown

□

Binding–capturing chains



Definition (Melliés, van Oostrom)

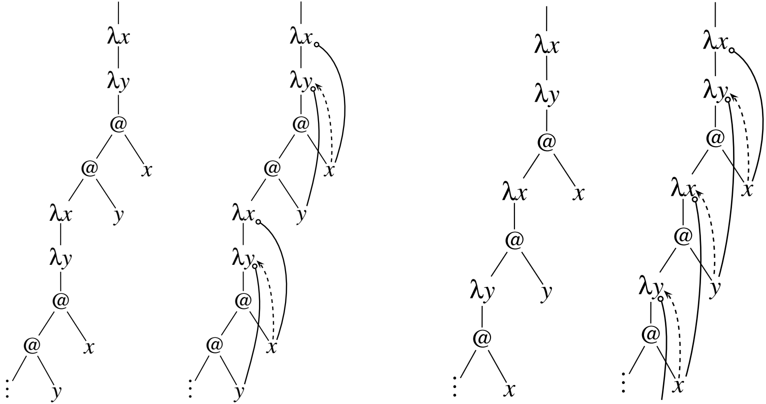
For positions p, q, r, s:

$$p \multimap q \;:\Longleftrightarrow\; \text{binder at } p \text{ binds variable occurrence at position } q$$

$$r \dashrightarrow s \;:\Longleftrightarrow\; \text{variable occurrence at } r \text{ is captured by binding at } s$$

Binding–capturing chains: $p_0 \multimap p_1 \dashrightarrow p_2 \multimap p_3 \dashrightarrow p_4 \multimap \ldots$

Binding–capturing chains



Definition (Melliés, van Oostrom)

For positions p, q, r, s:

$p \multimap q$ $:\Longleftrightarrow$ binder at p binds variable occurrence at position q

$r \dashrightarrow s$ $:\Longleftrightarrow$ variable occurrence at r is captured by binding at s

Binding–capturing chains: $p_0 \multimap p_1 \dashrightarrow p_2 \multimap p_3 \dashrightarrow p_4 \multimap \ldots$

Main theorem (extended)

Theorem (binding–capturing chains)

For all regular $\boldsymbol{\lambda}^{\infty}$-term T:

T *is strongly regular* $\iff$ T *has only finite binding–capturing chains.*

Theorem ($\boldsymbol{\lambda}_{\mu}$-expressibility, extended)

For all $\boldsymbol{\lambda}^{\infty}$-terms T the following are equivalent:

(i) T *is $\boldsymbol{\lambda}_{\mu}$-expressible.*

(ii) T *is strongly regular.*

(iii) T *is regular, and it only contains finite binding–capturing chains.*

Generalization and perspectives

- generalization to $\boldsymbol{\lambda}_{\text{letrec}}$
  - arxiv report Expressibility in the Lambda-Calculus with Letrec
    - same structure of proof
    - more technicalities: unfolding letrec-expressions
  - IWC 2013 talks Friday:
    - Confluent unfolding in λ_{letrec}
    - Confluent Let-floating
- practical relevance
  - recognize limits of optimization transformations
  - test efficiently for unfolding equivalence
  - implement maximal sharing for $\boldsymbol{\lambda}_{\text{letrec}}$-terms
- expressing regular $\boldsymbol{\lambda}^{\infty}$-terms
  - by higher-order rewrite rules
  - Chomsky hierarchy of finitely expressible $\boldsymbol{\lambda}^{\infty}$-terms?