Unique Normal Forms in Infinitary Weakly Orthogonal Term Rewriting

Jörg Endrullis[⋆] Clemens Grabmayer[△] Dimitri Hendriks[⋆]
Jan Willem Klop[⋆] Vincent van Oostrom[△]

[△]) Universiteit Utrecht
[⋆]) Vrije Universiteit Amsterdam

RTA 2010, Edinburgh, UK
July 11–13, 2010

Weakly orthogonal vs. orthogonal

Weakly orthogonal (first-/higher-order) rewrite systems:

- definition: ‘harmless’ weakening of orthogonality
- for finitary TRSs: most ‘nice’ properties of orthogonal systems are preserved
- but: new concepts, and non-trivial adaptations are needed

In this paper we:

- investigate infinitary weakly orthogonal rewrite systems
- show that uniqueness of infinitary normal forms fails in contrast to orthogonal systems
- explain how this failure can be repaired

Weakly orthogonal vs. orthogonal

Weakly orthogonal (first-/higher-order) rewrite systems:

- definition: ‘harmless’ weakening of orthogonality
- for finitary TRSs: most ‘nice’ properties of orthogonal systems are preserved
- but: new concepts, and non-trivial adaptations are needed

In this paper we:

- investigate infinitary weakly orthogonal rewrite systems
- show that uniqueness of infinitary normal forms fails in contrast to orthogonal systems
- explain how this failure can be repaired

Overview

- Definitions: weakly orthogonal, UN^{∞}
- Counterexample to UN^{∞} for weakly orthogonal TRSs
- Counterexample to UN^{∞} for $\lambda^{\infty}\beta\eta$
- Restoring infinitary confluence
- Diamond and triangle properties for developments

Weakly orthogonal

Weakly orthogonal (first-/higher-order) systems:

- left-linear
- all critical pairs are trivial.

Examples.

- Successor/Predecessor TRS:

$$\mathsf{P}(\mathsf{S}(x)) \to x \qquad \mathsf{S}(\mathsf{P}(x)) \to x$$

with critical pairs:

$$\mathsf{S}(x) \leftarrow \underline{\mathsf{S}}(\underline{\mathsf{P}}(\overline{\mathsf{S}}(x))) \to \mathsf{S}(x) \qquad \mathsf{P}(x) \leftarrow \underline{\mathsf{P}}(\underline{\mathsf{S}}(\overline{\mathsf{P}}(x))) \to \mathsf{P}(x)$$

- Parallel-Or TRS ('almost orthogonal'):

$$\mathsf{por}(\mathsf{true}, x) \to \mathsf{true}$$
$$\mathsf{por}(x, \mathsf{true}) \to \mathsf{true}$$
$$\mathsf{por}(\mathsf{false}, \mathsf{false}) \to \mathsf{false}$$

Weakly orthogonal

Weakly orthogonal (first-/higher-order) systems:

- left-linear
- all critical pairs are trivial.

Examples.

- Successor/Predecessor TRS:

$$\mathsf{P}(\mathsf{S}(x)) \to x \qquad \mathsf{S}(\mathsf{P}(x)) \to x$$

with critical pairs:

$$\mathsf{S}(x) \leftarrow \underline{\mathsf{S}}(\overline{\underline{\mathsf{P}}}(\overline{\mathsf{S}}(x))) \to \mathsf{S}(x) \qquad \mathsf{P}(x) \leftarrow \underline{\mathsf{P}}(\overline{\underline{\mathsf{S}}}(\overline{\mathsf{P}}(x))) \to \mathsf{P}(x)$$

- Parallel-Or TRS ('almost orthogonal'):

$$\mathsf{por}(\mathsf{true}, x) \to \mathsf{true}$$
$$\mathsf{por}(x, \mathsf{true}) \to \mathsf{true}$$
$$\mathsf{por}(\mathsf{false}, \mathsf{false}) \to \mathsf{false}$$

Weakly orthogonal

Weakly orthogonal (first-/higher-order) systems:

- left-linear
- all critical pairs are trivial.

Examples.

- Successor/Predecessor TRS:

$$\mathsf{P}(\mathsf{S}(x)) \to x \qquad \mathsf{S}(\mathsf{P}(x)) \to x$$

with critical pairs:

$$\mathsf{S}(x) \leftarrow \underline{\mathsf{S}}(\overline{\underline{\mathsf{P}}}(\overline{\mathsf{S}}(x))) \to \mathsf{S}(x) \qquad \mathsf{P}(x) \leftarrow \underline{\mathsf{P}}(\overline{\underline{\mathsf{S}}}(\overline{\mathsf{P}}(x))) \to \mathsf{P}(x)$$

- Parallel-Or TRS ('almost orthogonal'):

$$\mathsf{por}(\mathsf{true}, x) \to \mathsf{true}$$
$$\mathsf{por}(x, \mathsf{true}) \to \mathsf{true}$$
$$\mathsf{por}(\mathsf{false}, \mathsf{false}) \to \mathsf{false}$$

Weakly orthogonal

Weakly orthogonal (first-/higher-order) systems:

- left-linear
- all critical pairs are trivial.

Examples.

- Successor/Predecessor TRS:

$$\mathsf{P}(\mathsf{S}(x)) \to x \qquad \mathsf{S}(\mathsf{P}(x)) \to x$$

with critical pairs:

$$\mathsf{S}(x) \leftarrow \underline{\mathsf{S}}(\overline{\underline{\mathsf{P}}}(\overline{\mathsf{S}}(x))) \to \mathsf{S}(x) \qquad \mathsf{P}(x) \leftarrow \underline{\mathsf{P}}(\overline{\underline{\mathsf{S}}}(\overline{\mathsf{P}}(x))) \to \mathsf{P}(x)$$

- Parallel-Or TRS ('almost orthogonal'):

$$\mathsf{por}(\mathsf{true}, x) \to \mathsf{true}$$
$$\mathsf{por}(x, \mathsf{true}) \to \mathsf{true}$$
$$\mathsf{por}(\mathsf{false}, \mathsf{false}) \to \mathsf{false}$$

Weakly orthogonal

Weakly orthogonal (first-/higher-order) systems:

- left-linear
- all critical pairs are trivial.

Examples.

- Successor/Predecessor TRS:

$$\mathsf{P}(\mathsf{S}(x)) \to x \qquad \mathsf{S}(\mathsf{P}(x)) \to x$$

with critical pairs:

$$\mathsf{S}(x) \leftarrow \underline{\mathsf{S}}(\underline{\overline{\mathsf{P}}}(\overline{\mathsf{S}}(x))) \to \mathsf{S}(x) \qquad \mathsf{P}(x) \leftarrow \underline{\mathsf{P}}(\underline{\overline{\mathsf{S}}}(\overline{\mathsf{P}}(x))) \to \mathsf{P}(x)$$

- Parallel-Or TRS ('almost orthogonal'):

$$\mathsf{por}(\mathsf{true}, x) \to \mathsf{true}$$
$$\mathsf{por}(x, \mathsf{true}) \to \mathsf{true}$$
$$\mathsf{por}(\mathsf{false}, \mathsf{false}) \to \mathsf{false}$$

Weakly orthogonal

Weakly orthogonal (first-/higher-order) systems:

- left-linear
- all critical pairs are trivial.

Examples.

- Successor/Predecessor TRS:

$$\mathsf{P}(\mathsf{S}(x)) \to x \qquad \mathsf{S}(\mathsf{P}(x)) \to x$$

with critical pairs:

$$\mathsf{S}(x) \leftarrow \underline{\mathsf{S}}(\overline{\underline{\mathsf{P}}}(\overline{\mathsf{S}}(x))) \to \mathsf{S}(x) \qquad \mathsf{P}(x) \leftarrow \underline{\mathsf{P}}(\overline{\underline{\mathsf{S}}}(\overline{\mathsf{P}}(x))) \to \mathsf{P}(x)$$

- Parallel-Or TRS ('almost orthogonal'):

$$\mathsf{por}(\mathsf{true}, x) \to \mathsf{true}$$
$$\mathsf{por}(x, \mathsf{true}) \to \mathsf{true}$$
$$\mathsf{por}(\mathsf{false}, \mathsf{false}) \to \mathsf{false}$$

Weakly orthogonal

Weakly orthogonal (first-/higher-order) systems:

- left-linear
- all critical pairs are trivial.

Examples.

- Successor/Predecessor TRS:

$$\mathsf{P}(\mathsf{S}(x)) \to x \qquad \mathsf{S}(\mathsf{P}(x)) \to x$$

with critical pairs:

$$\mathsf{S}(x) \leftarrow \underline{\mathsf{S}}(\overline{\underline{\mathsf{P}}}(\overline{\mathsf{S}}(x))) \to \mathsf{S}(x) \qquad \mathsf{P}(x) \leftarrow \underline{\mathsf{P}}(\overline{\underline{\mathsf{S}}}(\overline{\mathsf{P}}(x))) \to \mathsf{P}(x)$$

- Parallel-Or TRS (‘almost orthogonal’):

$$\mathsf{por}(\mathsf{true}, x) \to \mathsf{true}$$
$$\mathsf{por}(x, \mathsf{true}) \to \mathsf{true}$$
$$\mathsf{por}(\mathsf{false}, \mathsf{false}) \to \mathsf{false}$$

CR^{∞} en UN^{∞} (definitions). Situation in OTRSs

- CR^{∞}: $t_1 \twoheadleftarrow t \twoheadrightarrow t_2 \implies \exists s.\ t_1 \twoheadrightarrow s \twoheadleftarrow t_2$
- UN^{∞}: $t_1 \twoheadleftarrow t \twoheadrightarrow t_2 \ \wedge\ t_1, t_2$ normal forms $\implies t_1 = t_2$
- SN^{∞}: all infinite rewrite sequences are progressive (str. conv.)

In orthogonal TRSs (well-known):

- $\mathrm{SN}^{\infty} \implies \mathrm{CR}^{\infty}$, and $\mathrm{CR}^{\infty} \implies \mathrm{UN}^{\infty}$.
- CR^{∞} fails (Kennaway).
  - But for non-collapsing TRSs: CR^{∞} holds.
- UN^{∞} holds (Kennaway/Klop).

CR^{∞} en UN^{∞} (definitions). Situation in OTRSs

- CR^{∞}: $t_1 \twoheadleftarrow\!\!\!\!\!\twoheadleftarrow t \twoheadrightarrow\!\!\!\!\!\twoheadrightarrow t_2 \implies \exists s.\ t_1 \twoheadrightarrow\!\!\!\!\!\twoheadrightarrow s \twoheadleftarrow\!\!\!\!\!\twoheadleftarrow t_2$
- UN^{∞}: $t_1 \twoheadleftarrow\!\!\!\!\!\twoheadleftarrow t \twoheadrightarrow\!\!\!\!\!\twoheadrightarrow t_2 \wedge t_1, t_2$ normal forms $\implies t_1 = t_2$
- SN^{∞}: all infinite rewrite sequences are progressive (str. conv.)

In orthogonal TRSs (well-known):

- $\mathrm{SN}^{\infty} \implies \mathrm{CR}^{\infty}$, and $\mathrm{CR}^{\infty} \implies \mathrm{UN}^{\infty}$.
- CR^{∞} fails (Kennaway).
  - But for non-collapsing TRSs: CR^{∞} holds.
- UN^{∞} holds (Kennaway/Klop).

CR^∞ en UN^∞ (definitions). Situation in OTRSs

- CR^∞: $t_1 \twoheadleftarrow t \twoheadrightarrow t_2 \implies \exists s.\ t_1 \twoheadrightarrow s \twoheadleftarrow t_2$
- UN^∞: $t_1 \twoheadleftarrow t \twoheadrightarrow t_2 \wedge t_1, t_2$ normal forms $\implies t_1 = t_2$
- SN^∞: all infinite rewrite sequences are progressive (str. conv.)

In orthogonal TRSs (well-known):

- $\mathrm{SN}^\infty \implies \mathrm{CR}^\infty$, and $\mathrm{CR}^\infty \implies \mathrm{UN}^\infty$.
- CR^∞ fails (Kennaway).
  - But for non-collapsing TRSs: CR^∞ holds.
- UN^∞ holds (Kennaway/Klop).

CR^{∞} en UN^{∞} (definitions). Situation in OTRSs

- CR^{∞}: $t_1 \twoheadleftarrow\!\!\!\twoheadleftarrow t \twoheadrightarrow\!\!\!\twoheadrightarrow t_2 \implies \exists s.\ t_1 \twoheadrightarrow\!\!\!\twoheadrightarrow s \twoheadleftarrow\!\!\!\twoheadleftarrow t_2$
- UN^{∞}: $t_1 \twoheadleftarrow\!\!\!\twoheadleftarrow t \twoheadrightarrow\!\!\!\twoheadrightarrow t_2 \wedge t_1, t_2$ normal forms $\implies t_1 = t_2$
- SN^{∞}: all infinite rewrite sequences are progressive (str. conv.)

In orthogonal TRSs (well-known):

- $\mathrm{SN}^{\infty} \implies \mathrm{CR}^{\infty}$, and $\mathrm{CR}^{\infty} \implies \mathrm{UN}^{\infty}$.
- CR^{∞} fails (Kennaway).
  - But for non-collapsing TRSs: CR^{∞} holds.
- UN^{∞} holds (Kennaway/Klop).

CR^{∞} en UN^{∞} (definitions). Situation in OTRSs

- CR^{∞}: $t_1 \twoheadleftarrow\!\!\!\!\leftarrow t \twoheadrightarrow\!\!\!\!\rightarrow t_2 \implies \exists s.\ t_1 \twoheadrightarrow\!\!\!\!\rightarrow s \twoheadleftarrow\!\!\!\!\leftarrow t_2$
- UN^{∞}: $t_1 \twoheadleftarrow\!\!\!\!\leftarrow t \twoheadrightarrow\!\!\!\!\rightarrow t_2 \ \wedge\ t_1, t_2$ normal forms $\implies t_1 = t_2$
- SN^{∞}: all infinite rewrite sequences are progressive (str. conv.)

In orthogonal TRSs (well-known):

- $\mathrm{SN}^{\infty} \implies \mathrm{CR}^{\infty}$, and $\mathrm{CR}^{\infty} \implies \mathrm{UN}^{\infty}$.
- CR^{∞} fails (Kennaway).
  - But for non-collapsing TRSs: CR^{∞} holds.
- UN^{∞} holds (Kennaway/Klop).

CR^{∞} en UN^{∞} (definitions). Situation in OTRSs

- CR^{∞}: $t_1 \twoheadleftarrow t \twoheadrightarrow t_2 \implies \exists s.\ t_1 \twoheadrightarrow s \twoheadleftarrow t_2$
- UN^{∞}: $t_1 \twoheadleftarrow t \twoheadrightarrow t_2 \,\wedge\, t_1, t_2$ normal forms $\implies t_1 = t_2$
- SN^{∞}: all infinite rewrite sequences are progressive (str. conv.)

In orthogonal TRSs (well-known):

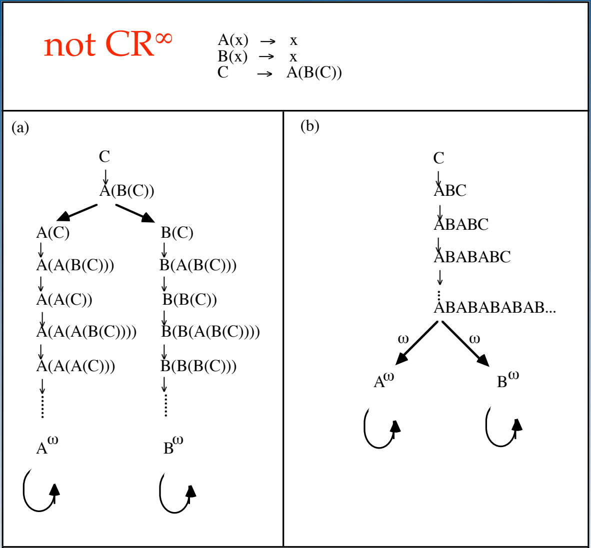
- $\mathrm{SN}^{\infty} \implies \mathrm{CR}^{\infty}$, and $\mathrm{CR}^{\infty} \implies \mathrm{UN}^{\infty}$.
- CR^{∞} fails (Kennaway).
  - But for non-collapsing TRSs: CR^{∞} holds.
- UN^{∞} holds (Kennaway/Klop).

CR^{∞} en UN^{∞} (definitions). Situation in OTRSs

- CR^{∞}: $t_1 \twoheadleftarrow\!\!\!\!\leftarrow t \twoheadrightarrow\!\!\!\!\rightarrow t_2 \implies \exists s.\ t_1 \twoheadrightarrow\!\!\!\!\rightarrow s \twoheadleftarrow\!\!\!\!\leftarrow t_2$
- UN^{∞}: $t_1 \twoheadleftarrow\!\!\!\!\leftarrow t \twoheadrightarrow\!\!\!\!\rightarrow t_2 \ \wedge\ t_1, t_2$ normal forms $\implies t_1 = t_2$
- SN^{∞}: all infinite rewrite sequences are progressive (str. conv.)

In orthogonal TRSs (well-known):

- $\mathrm{SN}^{\infty} \implies \mathrm{CR}^{\infty}$, and $\mathrm{CR}^{\infty} \implies \mathrm{UN}^{\infty}$.
- CR^{∞} fails (Kennaway).
  - But for non-collapsing TRSs: CR^{∞} holds.
- UN^{∞} holds (Kennaway/Klop).



Failure of infinitary confluence

Overview

1. Counterexample to UN^{∞} for weakly orthogonal iTRSs
2. Counterexample to UN^{∞} in $\lambda^{\infty}\beta\eta$
3. Restoring infinitary confluence
4. Diamond and triangle properties for developments
5. Summary

Counterexample: UN^{∞} fails weakly-ortho iTRS

In the Successor/Predecessor TRS:

$$\mathsf{P}(\mathsf{S}(x)) \to x \qquad\qquad \mathsf{S}(\mathsf{P}(x)) \to x$$

with the normal forms $\mathsf{S}^{\omega} = \mathsf{SSS}\ldots$ and $\mathsf{P}^{\omega} = \mathsf{PPP}\ldots$ we consider:

$$\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\mathsf{S}^4\,\mathsf{P}^5\,\mathsf{S}^6\ldots = \mathsf{P\,SS\,PPP\,SSSS\,PPPPP\,SSSSSS}\ldots$$

We find:

$$\begin{aligned}
\psi &= \mathsf{P\,SS\,PPP\,SSSS\,PPPPP\,SSSSSS}\ldots\\
&\to \mathsf{S\,PPP\,SSSS\,PPPPP\,SSSSSS}\ldots\\
&\to \mathsf{S\,PP\,SSS\,PPPPP\,SSSSSS}\ldots\\
&\to \mathsf{S\,P\,SS\,PPPPP\,SSSSSS}\ldots\\
&\to \mathsf{S\,S\,PPPPP\,SSSSSS}\ldots\\
&\twoheadrightarrow \mathsf{S\,S\,S\,S\,S\,S}\ldots = \mathsf{S}^{\omega}
\end{aligned}$$

And similarly:

Counterexample: UN^{∞} fails weakly-ortho iTRS

In the Successor/Predecessor TRS:

$$\mathsf{PS} \to x \qquad\qquad \mathsf{SP} \to x$$

with the normal forms $\mathsf{S}^{\omega} = \mathsf{SSS}\ldots$ and $\mathsf{P}^{\omega} = \mathsf{PPP}\ldots$ we consider:

$$\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\mathsf{S}^4\,\mathsf{P}^5\,\mathsf{S}^6 \ldots = \mathsf{P\;SS\;PPP\;SSSS\;PPPPP\;SSSSSS}\;\ldots$$

We find:

$$\begin{aligned}
\psi &= \mathsf{P\;SS\;PPP\;SSSS\;PPPPP\;SSSSSS}\;\ldots \\
&\to \mathsf{S\;PPP\;SSSS\;PPPPP\;SSSSSS}\;\ldots \\
&\to \mathsf{S\;PP\;SSS\;PPPPP\;SSSSSS}\;\ldots \\
&\to \mathsf{SP\;SS\;PPPPP\;SSSSSS}\;\ldots \\
&\to \mathsf{S\;S\;PPPPP\;SSSSSS}\;\ldots \\
&\twoheadrightarrow \mathsf{S\;S\;S\;S\;S\;S}\;\ldots = \mathsf{S}^{\omega}
\end{aligned}$$

And similarly:

Counterexample: UN^{∞} fails weakly-ortho iTRS

In the Successor/Predecessor TRS:

$$\mathsf{PS} \to x \qquad\qquad \mathsf{SP} \to x$$

with the normal forms $\mathsf{S}^{\omega} = \mathsf{SSS}\ldots$ and $\mathsf{P}^{\omega} = \mathsf{PPP}\ldots$ we consider:

$$\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\mathsf{S}^4\,\mathsf{P}^5\,\mathsf{S}^6 \ldots = \mathsf{P\;SS\;PPP\;SSSS\;PPPPP\;SSSSSS}\;\ldots$$

We find:

$$\begin{aligned}
\psi &= \mathsf{P\;SS\;PPP\;SSSS\;PPPPP\;SSSSSS}\;\ldots \\
&\to \mathsf{S\;PPP\;SSSS\;PPPPP\;SSSSSS}\;\ldots \\
&\to \mathsf{S\;PP\;SSS\;PPPPP\;SSSSSS}\;\ldots \\
&\to \mathsf{S\;P\;SS\;PPPPP\;SSSSSS}\;\ldots \\
&\to \mathsf{S\;S\;PPPPP\;SSSSSS}\;\ldots \\
&\twoheadrightarrow \mathsf{S\;S\;S\;S\;S\;S}\;\ldots = \mathsf{S}^{\omega}
\end{aligned}$$

And similarly:

Counterexample: UN^{∞} fails weakly-ortho iTRS

In the Successor/Predecessor TRS:

$$\mathsf{PS} \to x \qquad\qquad \mathsf{SP} \to x$$

with the normal forms $\mathsf{S}^{\omega} = \mathsf{SSS}\ldots$ and $\mathsf{P}^{\omega} = \mathsf{PPP}\ldots$ we consider:

$$\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\mathsf{S}^4\,\mathsf{P}^5\,\mathsf{S}^6\ldots = \mathsf{P}\;\mathsf{SS}\;\mathsf{PPP}\;\mathsf{SSSS}\;\mathsf{PPPPP}\;\mathsf{SSSSSS}\;\ldots$$

We find:

$$\psi = \mathsf{P}\;\mathsf{SS}\;\mathsf{PPP}\;\mathsf{SSSS}\;\mathsf{PPPPP}\;\mathsf{SSSSSS}\;\ldots$$
$$\to \mathsf{S}\;\mathsf{PPP}\;\mathsf{SSSS}\;\mathsf{PPPPP}\;\mathsf{SSSSSS}\;\ldots$$
$$\to \mathsf{S}\;\mathsf{PP}\;\mathsf{SSS}\;\mathsf{PPPPP}\;\mathsf{SSSSSS}\;\ldots$$
$$\to \mathsf{S}\mathsf{P}\;\mathsf{SS}\;\mathsf{PPPPP}\;\mathsf{SSSSSS}\;\ldots$$
$$\to \mathsf{S}\;\mathsf{S}\;\mathsf{PPPPP}\;\mathsf{SSSSSS}\;\ldots$$
$$\twoheadrightarrow \mathsf{S}\,\mathsf{S}\,\mathsf{S}\,\mathsf{S}\,\mathsf{S}\,\mathsf{S}\;\ldots = \mathsf{S}^{\omega}$$

And similarly:

Counterexample: UN^{∞} fails weakly-ortho iTRS

In the Successor/Predecessor TRS:

$$\mathsf{PS} \to x \qquad\qquad \mathsf{SP} \to x$$

with the normal forms $\mathsf{S}^{\omega} = \mathsf{SSS}\ldots$ and $\mathsf{P}^{\omega} = \mathsf{PPP}\ldots$ we consider:

$$\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\mathsf{S}^4\,\mathsf{P}^5\,\mathsf{S}^6\ldots = \mathsf{P\;SS\;PPP\;SSSS\;PPPPP\;SSSSSS}\;\ldots$$

We find:

$$\begin{aligned}
\psi &= \mathsf{P\;SS\;PPP\;SSSS\;PPPPP\;SSSSSS}\;\ldots\\
&\to \mathsf{S\;PPP\;SSSS\;PPPPP\;SSSSSS}\;\ldots\\
&\to \mathsf{S\;PP\;SSS\;PPPPP\;SSSSSS}\;\ldots\\
&\to \mathsf{SP\;SS\;PPPPP\;SSSSSS}\;\ldots\\
&\to \mathsf{S\;S\;PPPPP\;SSSSSS}\;\ldots\\
&\twoheadrightarrow \mathsf{S\;S\;S\;S\;S\;S}\;\ldots = \mathsf{S}^{\omega}
\end{aligned}$$

And similarly:

Counterexample: UN^{∞} fails weakly-ortho iTRS

In the Successor/Predecessor TRS:

$$\mathsf{PS} \to x \qquad\qquad \mathsf{SP} \to x$$

with the normal forms $\mathsf{S}^{\omega} = \mathsf{SSS}\ldots$ and $\mathsf{P}^{\omega} = \mathsf{PPP}\ldots$ we consider:

$$\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\mathsf{S}^4\,\mathsf{P}^5\,\mathsf{S}^6\ldots = \mathsf{P\,SS\,PPP\,SSSS\,PPPPP\,SSSSSS}\ldots$$

We find:

$$\begin{aligned}\psi &= \mathbf{P\,S}\mathsf{S\,PPP\,SSSS\,PPPPP\,SSSSSS}\ldots\\ &\to \mathsf{S\,PP}\mathbf{P\,S}\mathsf{SSS\,PPPPP\,SSSSSS}\ldots\\ &\to \mathsf{S\,PP\,SSS\,PPPPP\,SSSSSS}\ldots\\ &\to \mathsf{S}\mathbf{P\,S}\mathsf{S\,PPPPP\,SSSSSS}\ldots\\ &\to \mathsf{S\,S\,PPPP}\mathbf{P\,S}\mathsf{SSSSS}\ldots\\ &\twoheadrightarrow \mathsf{S\,S\,S\,S\,S\,S}\ldots = \mathsf{S}^{\omega}\end{aligned}$$

And similarly:

Counterexample: UN^{∞} fails weakly-ortho iTRS

In the Successor/Predecessor TRS:

$$\mathsf{PS} \to x \qquad\qquad \mathsf{SP} \to x$$

with the normal forms $\mathsf{S}^{\omega} = \mathsf{SSS}\ldots$ and $\mathsf{P}^{\omega} = \mathsf{PPP}\ldots$ we consider:

$$\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\mathsf{S}^4\,\mathsf{P}^5\,\mathsf{S}^6 \ldots = \mathsf{P\;SS\;PPP\;SSSS\;PPPPP\;SSSSSS}\ \ldots$$

We find:

$$\begin{aligned}
\psi &= \mathbf{P}\,\mathbf{S}\mathsf{S}\;\mathsf{PPP}\;\mathsf{SSSS}\;\mathsf{PPPPP}\;\mathsf{SSSSSS}\ \ldots\\
&\to \mathsf{S}\;\mathsf{PP}\mathbf{P}\;\mathbf{S}\mathsf{SSS}\;\mathsf{PPPPP}\;\mathsf{SSSSSS}\ \ldots\\
&\to \mathsf{S}\;\mathsf{P}\mathbf{P}\;\mathbf{S}\mathsf{SS}\;\mathsf{PPPPP}\;\mathsf{SSSSSS}\ \ldots\\
&\to \mathsf{SP}\;\mathsf{SS}\;\mathsf{PPPPP}\;\mathsf{SSSSSS}\ \ldots\\
&\to \mathsf{S}\;\mathsf{S}\;\mathsf{PPPP}\mathbf{P}\;\mathbf{S}\mathsf{SSSSS}\ \ldots\\
&\twoheadrightarrow \mathsf{S\;S\;S\;S\;S\;S}\ \ldots = \mathsf{S}^{\omega}
\end{aligned}$$

And similarly:

Counterexample: UN^{∞} fails weakly-ortho iTRS

In the Successor/Predecessor TRS:

$$\mathsf{PS} \to x \qquad\qquad \mathsf{SP} \to x$$

with the normal forms $\mathsf{S}^{\omega} = \mathsf{SSS}\ldots$ and $\mathsf{P}^{\omega} = \mathsf{PPP}\ldots$ we consider:

$$\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\mathsf{S}^4\,\mathsf{P}^5\,\mathsf{S}^6 \ldots = \mathsf{P\,SS\,PPP\,SSSS\,PPPPP\,SSSSSS}\ \ldots$$

We find:

$$\begin{aligned}\psi &= \mathbf{P\,S}\mathsf{S\,PPP\,SSSS\,PPPPP\,SSSSSS}\ \ldots\\ &\to \mathsf{S\,PP}\mathbf{P\,S}\mathsf{SSS\,PPPPP\,SSSSSS}\ \ldots\\ &\to \mathsf{S\,P}\mathbf{P\,S}\mathsf{SS\,PPPPP\,SSSSSS}\ \ldots\\ &\to \mathsf{S}\mathbf{P\,S}\mathsf{S\,PPPPP\,SSSSSS}\ \ldots\\ &\to \mathsf{S\,S\,PPPPP\,SSSSSS}\ \ldots\\ &\twoheadrightarrow \mathsf{S\,S\,S\,S\,S\,S}\ \ldots = \mathsf{S}^{\omega}\end{aligned}$$

And similarly:

Counterexample: UN^{∞} fails weakly-ortho iTRS

In the Successor/Predecessor TRS:

$$\mathsf{PS} \to x \qquad\qquad \mathsf{SP} \to x$$

with the normal forms $\mathsf{S}^{\omega} = \mathsf{SSS}\ldots$ and $\mathsf{P}^{\omega} = \mathsf{PPP}\ldots$ we consider:

$$\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\mathsf{S}^4\,\mathsf{P}^5\,\mathsf{S}^6\ldots = \mathsf{P\,SS\,PPP\,SSSS\,PPPPP\,SSSSSS}\,\ldots$$

We find:

$$\begin{aligned}
\psi &= \mathbf{P}\,\mathbf{S}\mathsf{S}\,\mathsf{PPP}\,\mathsf{SSSS}\,\mathsf{PPPPP}\,\mathsf{SSSSSS}\,\ldots\\
&\to \mathsf{S}\,\mathsf{PP}\mathbf{P}\,\mathbf{S}\mathsf{SSS}\,\mathsf{PPPPP}\,\mathsf{SSSSSS}\,\ldots\\
&\to \mathsf{S}\,\mathsf{P}\mathbf{P}\,\mathbf{S}\mathsf{SS}\,\mathsf{PPPPP}\,\mathsf{SSSSSS}\,\ldots\\
&\to \mathsf{S}\mathbf{P}\,\mathbf{S}\mathsf{S}\,\mathsf{PPPPP}\,\mathsf{SSSSSS}\,\ldots\\
&\to \mathsf{S}\,\mathsf{S}\,\mathsf{PPPP}\mathbf{P}\,\mathbf{S}\mathsf{SSSSS}\,\ldots\\
&\twoheadrightarrow \mathsf{S\,S\,S\,S\,S\,S}\,\ldots = \mathsf{S}^{\omega}
\end{aligned}$$

And similarly:

Counterexample: UN^{∞} fails weakly-ortho iTRS

In the Successor/Predecessor TRS:

$$\mathsf{PS} \to x \qquad\qquad \mathsf{SP} \to x$$

with the normal forms $\mathsf{S}^{\omega} = \mathsf{SSS}\ldots$ and $\mathsf{P}^{\omega} = \mathsf{PPP}\ldots$ we consider:

$$\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\mathsf{S}^4\,\mathsf{P}^5\,\mathsf{S}^6\ldots = \mathsf{P\,SS\,PPP\,SSSS\,PPPPP\,SSSSSS}\ldots$$

We find:

$$\begin{aligned}
\psi &= \mathbf{P\,S}\mathsf{S\,PPP\,SSSS\,PPPPP\,SSSSSS}\ldots \\
&\to \mathsf{S\,PP}\mathbf{P\,S}\mathsf{SSS\,PPPPP\,SSSSSS}\ldots \\
&\to \mathsf{S\,P}\mathbf{P\,S}\mathsf{SS\,PPPPP\,SSSSSS}\ldots \\
&\to \mathsf{S}\mathbf{P\,S}\mathsf{S\,PPPPP\,SSSSSS}\ldots \\
&\to \mathsf{S\,S\,PPPP}\mathbf{P\,S}\mathsf{SSSSS}\ldots \\
&\twoheadrightarrow \mathsf{S\,S\,S\,S\,S\,S}\ldots = \mathsf{S}^{\omega}
\end{aligned}$$

And similarly:

$$\psi = \mathsf{P\,SS\,PPP\,SSSS\,PPPPP\,SSSSSS}\ldots$$

Counterexample: UN^{∞} fails weakly-ortho iTRS

In the Successor/Predecessor TRS:

$$\mathsf{PS} \to x \qquad\qquad \mathsf{SP} \to x$$

with the normal forms $\mathsf{S}^{\omega} = \mathsf{SSS}\ldots$ and $\mathsf{P}^{\omega} = \mathsf{PPP}\ldots$ we consider:

$$\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\mathsf{S}^4\,\mathsf{P}^5\,\mathsf{S}^6 \ldots = \mathsf{P}\,\mathsf{SS}\,\mathsf{PPP}\,\mathsf{SSSS}\,\mathsf{PPPPP}\,\mathsf{SSSSSS}\,\ldots$$

We find:

$$\begin{aligned}
\psi &= \mathbf{P}\,\mathbf{S}\mathsf{S}\,\mathsf{PPP}\,\mathsf{SSSS}\,\mathsf{PPPPP}\,\mathsf{SSSSSS}\,\ldots \\
&\to \mathsf{S}\,\mathsf{PP}\mathbf{P}\,\mathbf{S}\mathsf{SSS}\,\mathsf{PPPPP}\,\mathsf{SSSSSS}\,\ldots \\
&\to \mathsf{S}\,\mathsf{P}\mathbf{P}\,\mathbf{S}\mathsf{SS}\,\mathsf{PPPPP}\,\mathsf{SSSSSS}\,\ldots \\
&\to \mathsf{S}\mathbf{P}\,\mathbf{S}\mathsf{S}\,\mathsf{PPPPP}\,\mathsf{SSSSSS}\,\ldots \\
&\to \mathsf{S}\,\mathsf{S}\,\mathsf{PPPP}\mathbf{P}\,\mathbf{S}\mathsf{SSSSS}\,\ldots \\
&\twoheadrightarrow \mathsf{S}\,\mathsf{S}\,\mathsf{S}\,\mathsf{S}\,\mathsf{S}\,\mathsf{S}\,\ldots = \mathsf{S}^{\omega}
\end{aligned}$$

And similarly:

$$\psi \to \mathsf{P}\,\mathsf{S}\,\mathsf{PP}\,\mathsf{SSSS}\,\mathsf{PPPPP}\,\mathsf{SSSSSS}\,\ldots$$

Counterexample: UN^{∞} fails weakly-ortho iTRS

In the Successor/Predecessor TRS:

$$\mathsf{PS} \to x \qquad\qquad \mathsf{SP} \to x$$

with the normal forms $\mathsf{S}^{\omega} = \mathsf{SSS}\ldots$ and $\mathsf{P}^{\omega} = \mathsf{PPP}\ldots$ we consider:

$$\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\mathsf{S}^4\,\mathsf{P}^5\,\mathsf{S}^6\ldots = \mathsf{P\;SS\;PPP\;SSSS\;PPPPP\;SSSSSS}\;\ldots$$

We find:

$$\begin{aligned}
\psi &= \mathsf{P\;SS\;PPP\;SSSS\;PPPPP\;SSSSSS}\;\ldots \\
&\to \mathsf{S\;PPP\;SSSS\;PPPPP\;SSSSSS}\;\ldots \\
&\to \mathsf{S\;PP\;SSS\;PPPPP\;SSSSSS}\;\ldots \\
&\to \mathsf{SP\;SS\;PPPPP\;SSSSSS}\;\ldots \\
&\to \mathsf{S\;S\;PPPPP\;SSSSSS}\;\ldots \\
&\twoheadrightarrow \mathsf{S\;S\;S\;S\;S\;S}\;\ldots = \mathsf{S}^{\omega}
\end{aligned}$$

And similarly:

$$\psi \twoheadrightarrow \mathsf{P\;P\;SSSS\;PPPPP\;SSSSSS}\;\ldots$$

Counterexample: UN^{∞} fails weakly-ortho iTRS

In the Successor/Predecessor TRS:

$$\mathsf{PS} \to x \qquad\qquad \mathsf{SP} \to x$$

with the normal forms $\mathsf{S}^{\omega} = \mathsf{SSS}\dots$ and $\mathsf{P}^{\omega} = \mathsf{PPP}\dots$ we consider:

$$\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\mathsf{S}^4\,\mathsf{P}^5\,\mathsf{S}^6\dots = \mathsf{P\ SS\ PPP\ SSSS\ PPPPP\ SSSSSS}\ \dots$$

We find:

$$\begin{aligned}\psi &= \mathbf{P}\,\mathbf{S}\mathsf{S}\ \mathsf{PPP\ SSSS\ PPPPP\ SSSSSS}\ \dots\\ &\to \mathsf{S\ PP}\mathbf{P}\,\mathbf{S}\mathsf{SSS\ PPPPP\ SSSSSS}\ \dots\\ &\to \mathsf{S\ P}\mathbf{P}\,\mathbf{S}\mathsf{SS\ PPPPP\ SSSSSS}\ \dots\\ &\to \mathsf{S}\mathbf{P}\,\mathbf{S}\mathsf{S\ PPPPP\ SSSSSS}\ \dots\\ &\to \mathsf{S\ S\ PPPP}\mathbf{P}\,\mathbf{S}\mathsf{SSSSS}\ \dots\\ &\twoheadrightarrow \mathsf{S\ S\ S\ S\ S\ S}\ \dots = \mathsf{S}^{\omega}\end{aligned}$$

And similarly:

$$\psi \twoheadrightarrow \mathsf{P\ P\ SSS\ PPPP\ SSSSSS}\ \dots$$

Counterexample: UN^{∞} fails weakly-ortho iTRS

In the Successor/Predecessor TRS:

$$\mathsf{PS} \to x \qquad\qquad \mathsf{SP} \to x$$

with the normal forms $\mathsf{S}^{\omega} = \mathsf{SSS}\ldots$ and $\mathsf{P}^{\omega} = \mathsf{PPP}\ldots$ we consider:

$$\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\mathsf{S}^4\,\mathsf{P}^5\,\mathsf{S}^6\ldots = \mathsf{P\,SS\,PPP\,SSSS\,PPPPP\,SSSSSS}\ldots$$

We find:

$$\begin{aligned}
\psi &= \mathbf{P\,S}\mathsf{S\,PPP\,SSSS\,PPPPP\,SSSSSS}\ldots\\
&\to \mathsf{S\,PP}\mathbf{P\,S}\mathsf{SSS\,PPPPP\,SSSSSS}\ldots\\
&\to \mathsf{S\,P}\mathbf{P\,S}\mathsf{SS\,PPPPP\,SSSSSS}\ldots\\
&\to \mathsf{S}\mathbf{P\,S}\mathsf{S\,PPPPP\,SSSSSS}\ldots\\
&\to \mathsf{S\,S\,PPPP}\mathbf{P\,S}\mathsf{SSSSS}\ldots\\
&\twoheadrightarrow \mathsf{S\,S\,S\,S\,S\,S}\ldots = \mathsf{S}^{\omega}
\end{aligned}$$

And similarly:

$$\psi \twoheadrightarrow \mathsf{P\,P\,SS\,PPP\,SSSSSS}\ldots$$

Counterexample: UN^{∞} fails weakly-ortho iTRS

In the Successor/Predecessor TRS:

$$\mathsf{PS} \to x \qquad\qquad \mathsf{SP} \to x$$

with the normal forms $\mathsf{S}^{\omega} = \mathsf{SSS}\ldots$ and $\mathsf{P}^{\omega} = \mathsf{PPP}\ldots$ we consider:

$$\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\mathsf{S}^4\,\mathsf{P}^5\,\mathsf{S}^6\ldots = \mathsf{P\;SS\;PPP\;SSSS\;PPPPP\;SSSSSS}\;\ldots$$

We find:

$$\begin{aligned}
\psi &= \mathbf{P\,S}\mathsf{S\;PPP\;SSSS\;PPPPP\;SSSSSS}\;\ldots\\
&\to \mathsf{S\;PP}\mathbf{P\,S}\mathsf{SSS\;PPPPP\;SSSSSS}\;\ldots\\
&\to \mathsf{S\;P}\mathbf{P\,S}\mathsf{SS\;PPPPP\;SSSSSS}\;\ldots\\
&\to \mathsf{S}\mathbf{P\,S}\mathsf{S\;PPPPP\;SSSSSS}\;\ldots\\
&\to \mathsf{S\;S\;PPPP}\mathbf{P\,S}\mathsf{SSSSS}\;\ldots\\
&\twoheadrightarrow \mathsf{S\;S\;S\;S\;S\;S}\;\ldots = \mathsf{S}^{\omega}
\end{aligned}$$

And similarly:

$$\psi \twoheadrightarrow \mathsf{P\;P\;S\;PP\;SSSSSS}\;\ldots$$

Counterexample: UN^{∞} fails weakly-ortho iTRS

In the Successor/Predecessor TRS:

$$\mathsf{PS} \to x \qquad\qquad \mathsf{SP} \to x$$

with the normal forms $\mathsf{S}^{\omega} = \mathsf{SSS}\ldots$ and $\mathsf{P}^{\omega} = \mathsf{PPP}\ldots$ we consider:

$$\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\mathsf{S}^4\,\mathsf{P}^5\,\mathsf{S}^6\ldots = \mathsf{P\,SS\,PPP\,SSSS\,PPPPP\,SSSSSS}\ \ldots$$

We find:

$$\begin{aligned}
\psi &= \mathbf{P}\,\mathbf{S}\mathsf{S}\ \mathsf{PPP}\ \mathsf{SSSS}\ \mathsf{PPPPP}\ \mathsf{SSSSSS}\ \ldots\\
&\to \mathsf{S}\ \mathsf{PP}\mathbf{P}\ \mathbf{S}\mathsf{SSS}\ \mathsf{PPPPP}\ \mathsf{SSSSSS}\ \ldots\\
&\to \mathsf{S}\ \mathsf{P}\mathbf{P}\ \mathbf{S}\mathsf{SS}\ \mathsf{PPPPP}\ \mathsf{SSSSSS}\ \ldots\\
&\to \mathsf{S}\mathbf{P}\ \mathbf{S}\mathsf{S}\ \mathsf{PPPPP}\ \mathsf{SSSSSS}\ \ldots\\
&\to \mathsf{S}\ \mathsf{S}\ \mathsf{PPPP}\mathbf{P}\ \mathbf{S}\mathsf{SSSSS}\ \ldots\\
&\twoheadrightarrow \mathsf{S\,S\,S\,S\,S\,S}\ \ldots = \mathsf{S}^{\omega}
\end{aligned}$$

And similarly:

$$\psi \twoheadrightarrow \mathsf{P\,P\,P}\ \mathsf{SSSSSS}\ \ldots$$

Counterexample: UN^{∞} fails weakly-ortho iTRS

In the Successor/Predecessor TRS:

$$\mathsf{PS} \to x \qquad\qquad \mathsf{SP} \to x$$

with the normal forms $\mathsf{S}^{\omega} = \mathsf{SSS}\ldots$ and $\mathsf{P}^{\omega} = \mathsf{PPP}\ldots$ we consider:

$$\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\mathsf{S}^4\,\mathsf{P}^5\,\mathsf{S}^6\ldots = \mathsf{P\;SS\;PPP\;SSSS\;PPPPP\;SSSSSS}\;\ldots$$

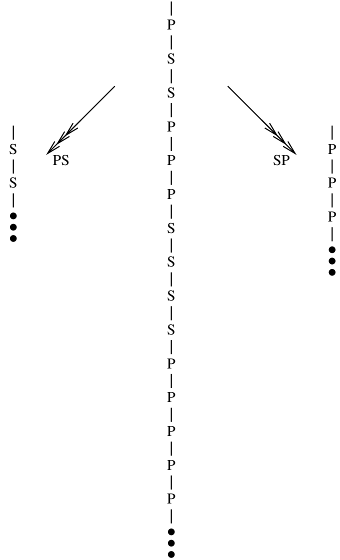
We find:

$$\begin{aligned}
\psi &= \mathbf{P}\;\mathbf{S}\mathsf{S}\;\mathsf{PPP}\;\mathsf{SSSS}\;\mathsf{PPPPP}\;\mathsf{SSSSSS}\;\ldots\\
&\to \mathsf{S}\;\mathsf{PP}\mathbf{P}\;\mathbf{S}\mathsf{SSS}\;\mathsf{PPPPP}\;\mathsf{SSSSSS}\;\ldots\\
&\to \mathsf{S}\;\mathsf{P}\mathbf{P}\;\mathbf{S}\mathsf{SS}\;\mathsf{PPPPP}\;\mathsf{SSSSSS}\;\ldots\\
&\to \mathsf{S}\mathbf{P}\;\mathbf{S}\mathsf{S}\;\mathsf{PPPPP}\;\mathsf{SSSSSS}\;\ldots\\
&\to \mathsf{S}\;\mathsf{S}\;\mathsf{PPPP}\mathbf{P}\;\mathbf{S}\mathsf{SSSSS}\;\ldots\\
&\twoheadrightarrow \mathsf{S\;S\;S\;S\;S\;S}\;\ldots = \mathsf{S}^{\omega}
\end{aligned}$$

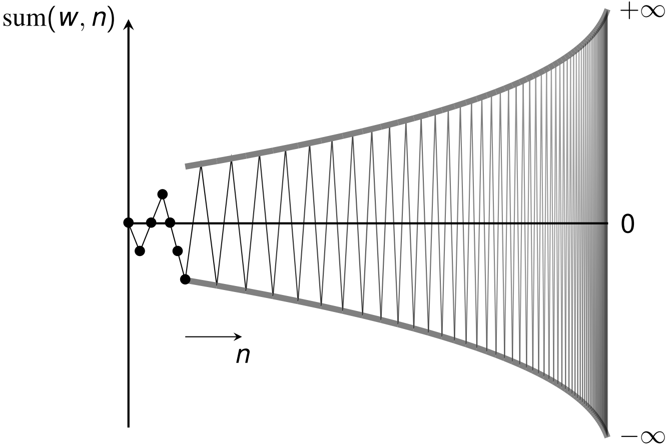
And similarly:

$$\psi \twoheadrightarrow \mathsf{P\;P\;P\;P\;P}\;\ldots = \mathsf{P}^{\omega}$$

Counterexample: UN^{∞} fails weakly-ortho iTRS



Counterexample: UN^{∞} fails weakly-ortho iTRS



Graph for the oscillating PS-word $\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\ \ldots$.

Overview

1. Counterexample to UN^{∞} for weakly orthogonal iTRSs
2. Counterexample to UN^{∞} in $\lambda^{\infty}\beta\eta$
3. Restoring infinitary confluence
4. Diamond and triangle properties for developments
5. Summary

$\lambda^\infty\beta\eta$

Terms of $\lambda^\infty\beta\eta$: the (potentially) infinite λ-terms in $Ter^\infty(\lambda)$

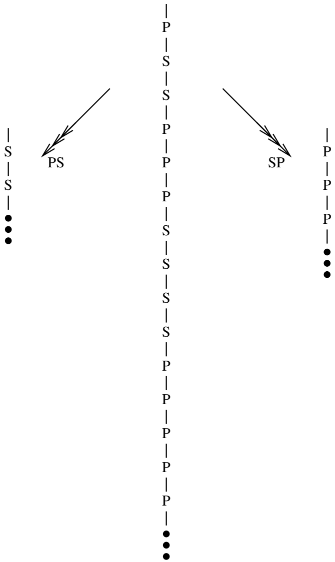
The rewrite rules of $\lambda^\infty\beta\eta$ are:

$$(\lambda x.M)N \xrightarrow{\beta} M[x{:=}N]$$
$$\lambda x.Mx \xrightarrow{\eta} M \qquad (x \text{ not free in } M)$$

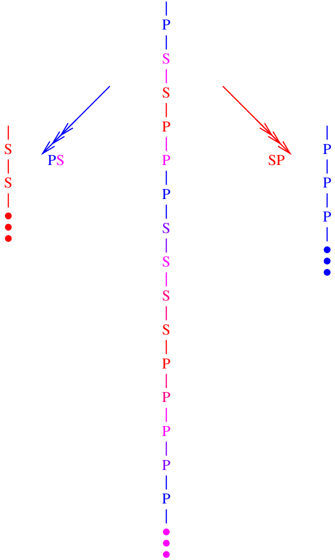
$\lambda^\infty\beta\eta$ is weakly orthogonal, since the critical pairs are trivial:

$$Mx \xleftarrow{\beta} (\lambda x.Mx)x \xrightarrow{\eta} Mx \qquad (x \text{ not free in } M)$$
$$\lambda x.M[y{:=}x] \xleftarrow{\beta} \lambda x.(\lambda y.M)x \xrightarrow{\eta} \lambda y.M \qquad (x \text{ not free in } \lambda y.M)$$

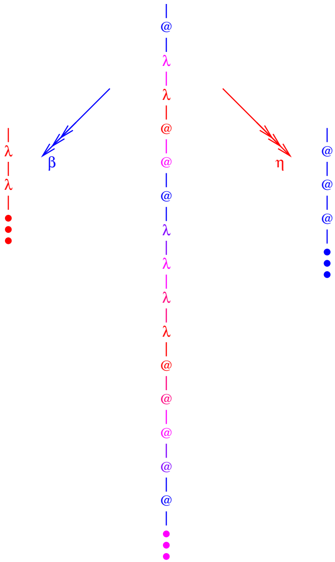
Translating the S-P-example to $\lambda^{\infty}\beta\eta$



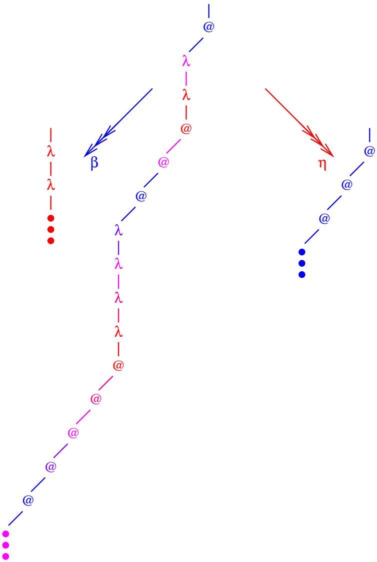
Translating the S-P-example to $\lambda^\infty\beta\eta$



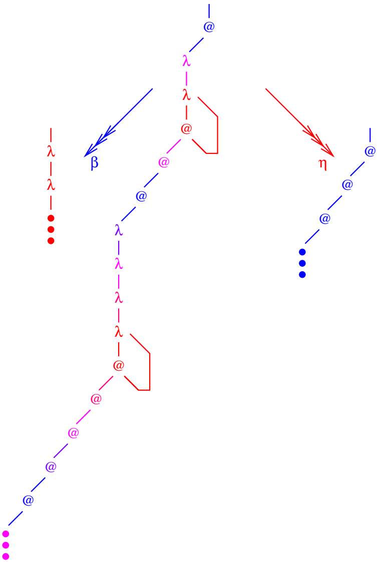
Translating the S-P-example to $\lambda^{\infty}\beta\eta$



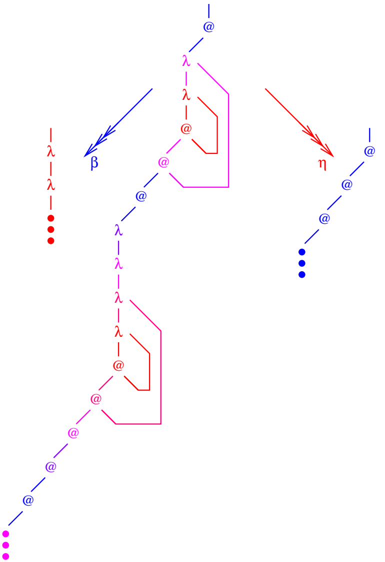
Translating the S-P-example to $\lambda^\infty\beta\eta$



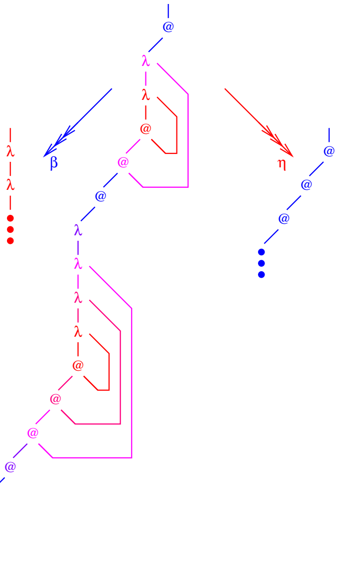
Translating the S-P-example to $\lambda^\infty\beta\eta$



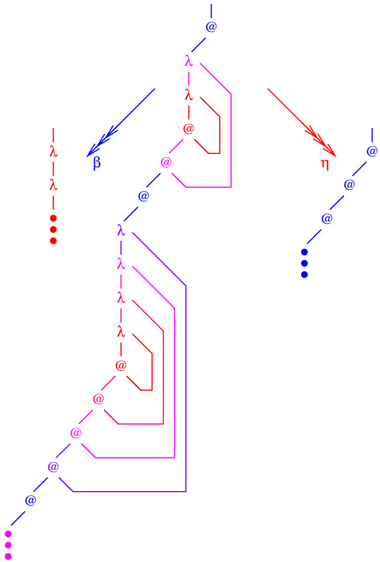
Translating the S-P-example to $\lambda^\infty\beta\eta$



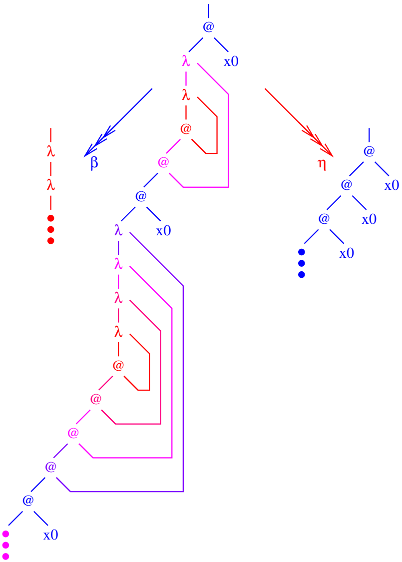
Translating the S-P-example to $\lambda^\infty\beta\eta$



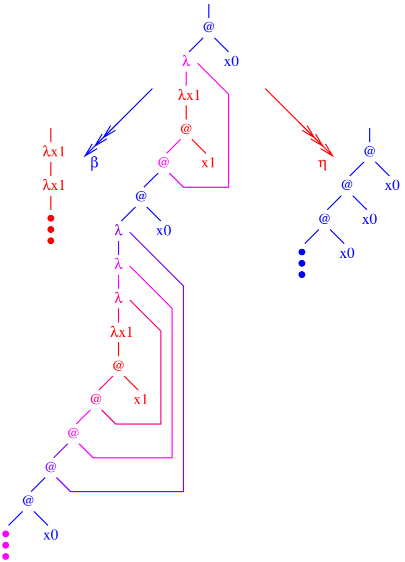
Translating the S-P-example to $\lambda^{\infty}\beta\eta$



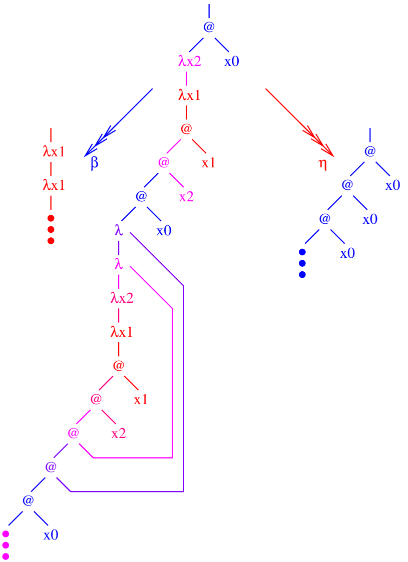
Translating the S-P-example to $\lambda^\infty\beta\eta$



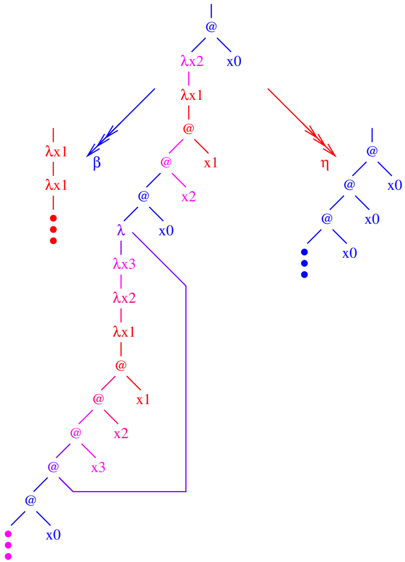
Translating the S-P-example to $\lambda^\infty\beta\eta$



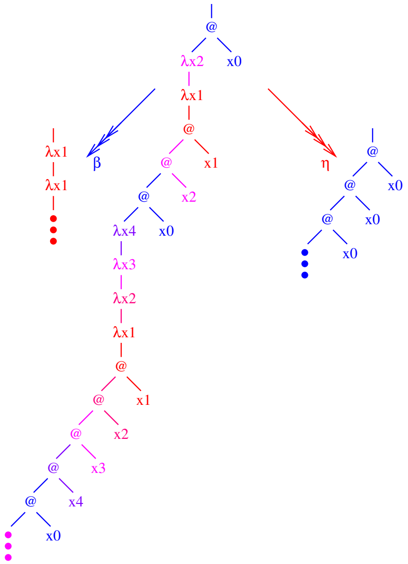
Translating the S-P-example to $\lambda^\infty\beta\eta$



Translating the S-P-example to $\lambda^{\infty}\beta\eta$



Counterexample: UN^{∞} fails in $\lambda^{\infty}\beta\eta$



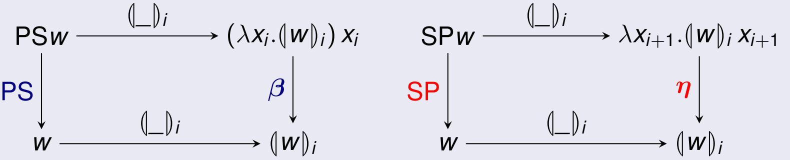
Translating the S-P-example to $\lambda^\infty\beta\eta$

$(\!|\_|\!) : \{\mathsf{P},\mathsf{S}\}^\omega \to Ter^\infty(\lambda)$ defined by:

- $(\!|w|\!) = (\!|w|\!)_0$;
- for all $w \in \{\mathsf{P},\mathsf{S}\}^\omega$, and $i \in \mathbb{Z}$:

$$(\!|\mathsf{P}w|\!)_i = (\!|w|\!)_{i-1}\, x_i \qquad\qquad (\!|\mathsf{S}w|\!)_i = \lambda x_{i+1}.(\!|w|\!)_{i+1}$$

Lemma



Contrast with $\lambda^{\infty}\beta$

We saw for $\lambda^{\infty}\beta\eta$:

- UN^{∞} **fails**
- Consequently: CR^{∞} **fails**

However for $\lambda^{\infty}\beta$ it holds:

- CR^{∞} **fails**
- But: UN^{∞} **holds!**

  Due to this, $\lambda^{\infty}\beta$ is important for the model theory of λ-calculus: for several models equality is captured by $\lambda^{\infty}\beta$-convertibility:

  - Böhm Trees
  - Lévy–Longo Trees
  - Berarducci Trees

Contrast with $\lambda^{\infty}\beta$

We saw for $\lambda^{\infty}\beta\eta$:

- UN^{∞} **fails**
- Consequently: CR^{∞} **fails**

However for $\lambda^{\infty}\beta$ it holds:

- CR^{∞} **fails**
- But: UN^{∞} **holds!**

  Due to this, $\lambda^{\infty}\beta$ is important for the model theory of λ-calculus: for several models equality is captured by $\lambda^{\infty}\beta$-convertibility:

  - Böhm Trees
  - Lévy–Longo Trees
  - Berarducci Trees

Contrast with $\lambda^\infty\beta$

We saw for $\lambda^\infty\beta\eta$:

- UN^∞ **fails**
- Consequently: CR^∞ **fails**

However for $\lambda^\infty\beta$ it holds:

- CR^∞ **fails**
- But: UN^∞ **holds!**

  Due to this, $\lambda^\infty\beta$ is important for the model theory of λ-calculus: for several models equality is captured by $\lambda^\infty\beta$-convertibility:

  - Böhm Trees
  - Lévy–Longo Trees
  - Berarducci Trees

Overview

Restoring infinitary confluence (preview)

Theorem

Weakly orthogonal TRSs without collapsing rules are inf. confluent.

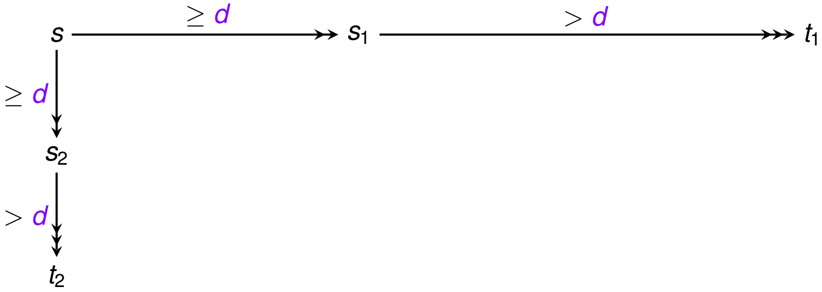
Proof.

Restoring infinitary confluence (preview)

Theorem

Weakly orthogonal TRSs without collapsing rules are inf. confluent.

Proof.

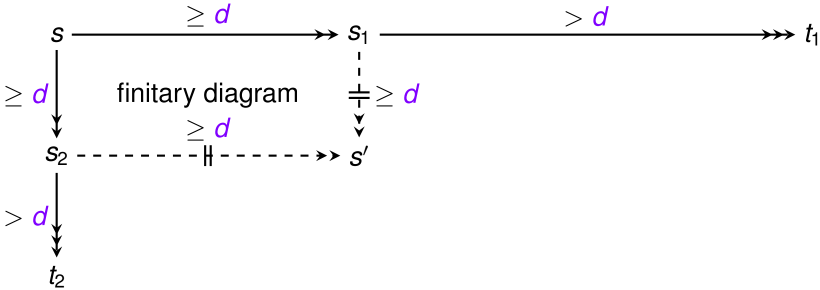


Restoring infinitary confluence (preview)

Theorem

Weakly orthogonal TRSs without collapsing rules are inf. confluent.

Proof.

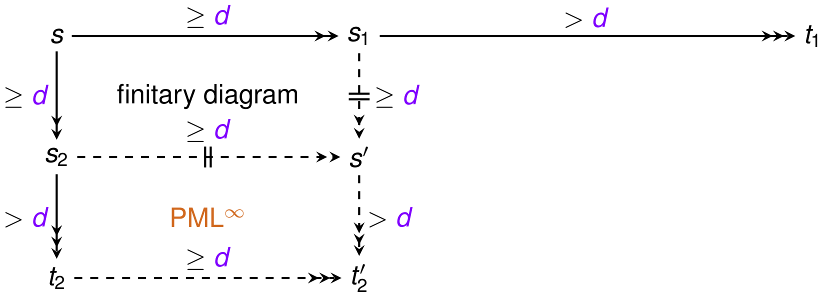


Restoring infinitary confluence (preview)

Theorem

Weakly orthogonal TRSs without collapsing rules are inf. confluent.

Proof.

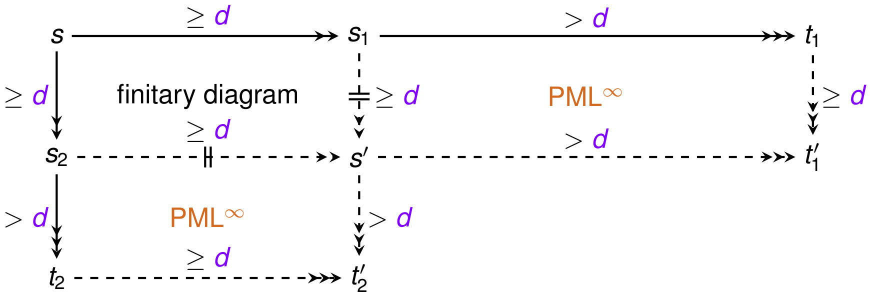


Restoring infinitary confluence (preview)

Theorem

Weakly orthogonal TRSs without collapsing rules are inf. confluent.

Proof.

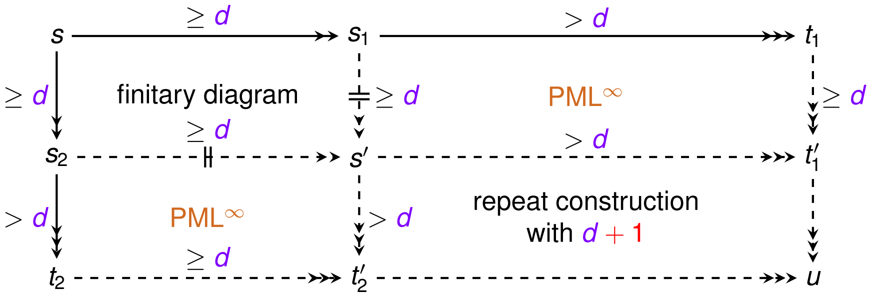


Restoring infinitary confluence (preview)

Theorem

Weakly orthogonal TRSs without collapsing rules are inf. confluent.

Proof.



Orthogonalization (of parallel steps)

Proposition

For parallel steps $\phi : s \mathrel{-\!\!\!+\!\!\!+\!\!\!\rightarrow} t_1$ and $\psi : s \mathrel{-\!\!\!+\!\!\!+\!\!\!\rightarrow} t_2$ in a w-o TRS there exists orthogonal steps ϕ' and ψ' such that $\phi' : s \mathrel{-\!\!\!+\!\!\!+\!\!\!\rightarrow} t_1$ and $\psi' : s \mathrel{-\!\!\!+\!\!\!+\!\!\!\rightarrow} t_2$ (the pair $\langle \phi', \psi' \rangle$ is an orthogonalization of ϕ and ψ).

Proof.

In case of overlaps, we replace the outer redex with the inner one

(by weak orthogonality overlapping redexes have the same effect).

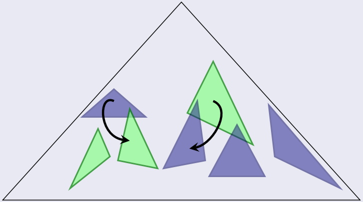
Orthogonalization (of parallel steps)

Proposition

For parallel steps $\phi : s \twoheadrightarrow\!\!\!\!\!\!\| \;\; t_1$ and $\psi : s \twoheadrightarrow\!\!\!\!\!\!\| \;\; t_2$ in a w-o TRS there exists orthogonal steps ϕ' and ψ' such that $\phi' : s \twoheadrightarrow\!\!\!\!\!\!\| \;\; t_1$ and $\psi' : s \twoheadrightarrow\!\!\!\!\!\!\| \;\; t_2$ (the pair $\langle \phi', \psi' \rangle$ is an orthogonalization of ϕ and ψ).

Proof.

In case of overlaps, we replace the outer redex with the inner one



(by weak orthogonality overlapping redexes have the same effect).

Orthogonalization (of parallel steps)

Proposition

For parallel steps $\phi : s \mathrel{-\!\!\!\parallel\!\!\!\to} t_1$ and $\psi : s \mathrel{-\!\!\!\parallel\!\!\!\to} t_2$ in a w-o TRS there exists orthogonal steps ϕ' and ψ' such that $\phi' : s \mathrel{-\!\!\!\parallel\!\!\!\to} t_1$ and $\psi' : s \mathrel{-\!\!\!\parallel\!\!\!\to} t_2$ (the pair $\langle \phi', \psi' \rangle$ is an orthogonalization of ϕ and ψ).

Proof.

In case of overlaps, we replace the outer redex with the inner one

(by weak orthogonality overlapping redexes have the same effect).

Infinitary Parallel Moves Lemma PML$^{\infty}$

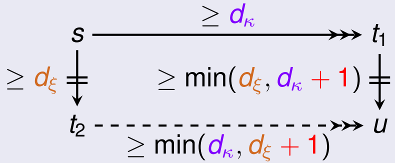
Using additionally a:

- refined compression lemma (preservation of min. depth of steps)

we show:

Lemma

Let R be a non-collapsing weakly orthogonal TRS. Then:

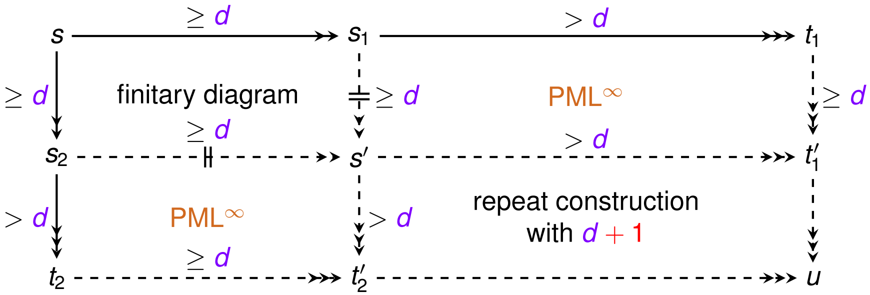


Restoring infinitary confluence

Theorem

Weakly orthogonal TRSs without collapsing rules are inf. confluent.

Proof.



Overview

1. Counterexample to UN^{∞} for weakly orthogonal iTRSs
2. Counterexample to UN^{∞} in $\lambda^{\infty}\beta\eta$
3. Restoring infinitary confluence
4. Diamond and triangle properties for developments
5. Summary

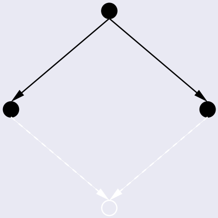
Diamond and triangle properties for developments

Definition

A binary relation $\rightarrow$ on A has:

- the diamond property if: $\leftarrow \cdot \rightarrow \;\subseteq\; \rightarrow \cdot \leftarrow$;
- the triangle property if:

  $\forall a \in A.\ \exists a^{\bullet} \in A.\ a \rightarrow a^{\bullet} \wedge (\forall b \in A.\ a \rightarrow b \Rightarrow \qquad).$



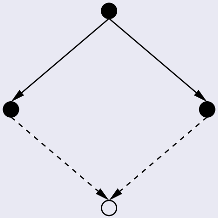
Diamond and triangle properties for developments

Definition

A binary relation $\rightarrow$ on A has:

- the diamond property if: $\leftarrow \cdot \rightarrow \;\subseteq\; \rightarrow \cdot \leftarrow$;
- the triangle property if:

  $\forall a \in A.\ \exists a^{\bullet} \in A.\ a \rightarrow a^{\bullet} \wedge (\forall b \in A.\ a \rightarrow b \Rightarrow \qquad)$.



Diamond and triangle properties for developments

Definition

A binary relation $\to$ on A has:

- the diamond property if: $\leftarrow \cdot \to \;\subseteq\; \to \cdot \leftarrow$;
- the triangle property if:

$$\forall a \in A.\ \exists a^{\bullet} \in A.\ a \to a^{\bullet} \wedge (\forall b \in A.\ a \to b \Rightarrow b \to a^{\bullet}).$$

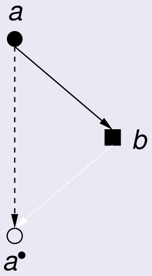
Diamond and triangle properties for developments

Definition

A binary relation $\to$ on A has:

- the diamond property if: $\leftarrow \cdot \to \;\subseteq\; \to \cdot \leftarrow$;
- the triangle property if:

$$\forall a \in A.\, \exists a^{\bullet} \in A.\, a \to a^{\bullet} \wedge (\forall b \in A.\, a \to b \;\Rightarrow\; b \to a^{\bullet}).$$



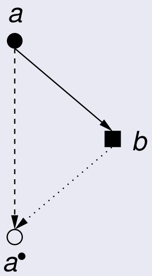
Diamond and triangle properties for developments

Definition

A binary relation $\to$ on A has:

- the diamond property if: $\leftarrow \cdot \to \;\subseteq\; \to \cdot \leftarrow$;
- the triangle property if:

$$\forall a \in A.\, \exists a^{\bullet} \in A.\; a \to a^{\bullet} \;\wedge\; (\forall b \in A.\; a \to b \;\Rightarrow\; b \to a^{\bullet})\,.$$



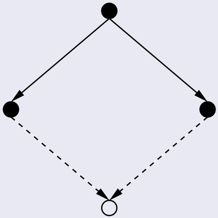
Diamond and triangle properties for developments

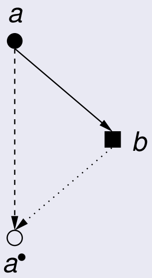
Definition

A binary relation $\to$ on A has:

- the diamond property if: $\leftarrow \cdot \to \;\subseteq\; \to \cdot \leftarrow$;
- the triangle property if:

$$\forall a \in A.\, \exists a^{\bullet} \in A.\, a \to a^{\bullet} \wedge (\forall b \in A.\, a \to b \;\Rightarrow\; b \to a^{\bullet})\,.$$





Diamond and triangle properties for developments

Theorem

For every weakly orthogonal TRS without collapsing rules, for infinitary developments there hold:

1. *the diamond property;*
2. *the triangle property.*

Our proof proceeds by:

- refining an earlier cluster analysis (I-clusters and Y-clusters) from the finite case;
- a top-down orthogonalization algorithm.

Diamond and triangle properties for developments

Theorem

For every weakly orthogonal TRS without collapsing rules, for infinitary developments there hold:

1. *the diamond property;*
2. *the triangle property.*

Our proof proceeds by:

- refining an earlier cluster analysis (I-clusters and Y-clusters) from the finite case;
- a top-down orthogonalization algorithm.

Overview

Summary

- Counterexample to $\mathrm{UN}^{\infty}/\mathrm{CR}^{\infty}$ for weakly orthogonal TRSs
- By translation: counterexample to $\mathrm{UN}^{\infty}/\mathrm{CR}^{\infty}$ for $\lambda^{\infty}\beta\eta$
- Restoring CR^{∞} (hence UN^{∞}) for non-collapsing w-o TRSs
- Diamond and triangle properties for developments in non-collapsing w-o TRSs

Summary

		finitary				*infinitary*			
		PML	CR	UN	NF	PML^{∞}	CR^{∞}	UN^{∞}	NF^{∞}
first-order	OTRS	yes	yes	yes	yes	yes	no	yes	yes
	WOTRS	yes	yes	yes	yes	**yes**	no	**no**	**no**
	nc-WOTRS	yes	yes	yes	yes	**yes**	**yes**	**yes**	**yes**
	1c-WOTRS	yes	yes	yes	yes	**yes**	no	**?**	**?**
higher-order	$\lambda\beta$	yes	yes	yes	yes	no	no	yes	yes
	fe-OCRS	yes	yes	yes	yes	no	no	yes	yes
	$\lambda\beta\eta$	yes	yes	yes	yes	no	no	**no**	**no**
	WOCRS	yes	yes	yes	yes	no	no	**no**	**no**