Mathematical structures in logic
Exercise class 7

March 20, 2018

1. (a) Compute the dual Esakia space of the Rieger-Nishimura lattice. (This space is often referred to as the *Rieger-Nishimura ladder*).

   (b) Determine the Boolean algebra of regular elements of the Rieger-Nishimura lattice.

2. Let V be a variety of Heyting algebras generated by a finite set K of finite algebras. Show that V is a finitely generated variety, i.e., it is generated by a single finite algebra.

3. Let (X, R) be a modal space. Recall that, for $U \subseteq X$, we define

$$\Diamond_R U := \{x \in X : \exists y (Rxy \,\&\, y \in U)\}.$$

   Show that the formula $p \to \Diamond p$ is valid on the modal algebra $(\mathsf{Clop}(X), \Diamond_R)$ iff R is reflexive.

   (*Hint:* Use Esakia's lemma, which states: if $\{U_i\}_{i \in I}$ is a downwards-directed[1] family of closed sets, then $\Diamond_R(\bigcap_{i\in I} U_i) = \bigcap_{i \in I}(\Diamond_R U_i)$.)

4. Take $(X, \leq)$ to be the set of non-positive integers with the usual order. Consider the **S4**-algebra $(\mathcal{P}(X), \Box_R)$. Show that the subalgebra generated by $U = \{0, -2, -4, -6, ...\}$ is infinite. Deduce that the variety of **S4**-algebras is not locally finite.

5. Let $f : (P, \leq) \to (P', \leq')$ be an order preserving function between partial orders. Show that the following are equivalent

   (a) The function f is a p-morphism;

   (b) $f^{-1}(\downarrow U') = \downarrow f^{-1}(U')$, for all $U' \subseteq P'$;

   (c) $f^{-1}(\downarrow p') = \downarrow f^{-1}(\{p'\})$, for all $p' \in P'$;

   (d) $\uparrow f(p) = f[\uparrow p]$ for all $p \in P$.

   (e) the function f is open as a continuous map between the induced Alexandroff spaces $(P, \tau_{\leq})$ and $(P', \tau_{\leq'})$.

[1] For every U_j, U_k there exists $U_i \subseteq U_j \cap U_k$