MATHEMATICAL STRUCTURES IN LOGIC 2018
HOMEWORK 3

- Deadline: February 27 — at the **beginning** of the tutorial class.
- In exceptional cases homework can be submitted electronically (in a single pdf-file!) to Saul Fernandez (`saul.fdez.glez@gmail.com`).
- Grading is from 0 to 100 points.
- Discussion of problems is allowed, but each student should submit a homework they themselves have written.
- Good luck!

(1) (20pt) Let $(X, \leq)$ be a pre-ordered set (i.e., $\leq$ is reflexive and transitive). Let (X, τ) be the corresponding Alexandroff topology (where τ is the set of all up-sets of X).

(a) Define $\preceq$ on X by $x \preceq y$ iff ($x \in U$ implies $y \in U$ for each $U \in \tau$). Show that $x \leq y$ iff $x \preceq y$.

(b) Recall that a topological space is called T_0 if for each $x \neq y$ there exists an open set U such that ($x \in U$ and $y \notin U$) or ($y \in U$ and $x \notin U$). What is a necessary and sufficient condition on $\leq$ so that (X, τ) is a T_0-space. Justify your solution.

(2) (20pt) Let A be a Heyting algebra and $\mathrm{Rg}(A) = \{\neg\neg a : a \in A\}$. Then $\mathrm{Rg}(A)$ is a Boolean algebra, where $a \dot{\vee} b = \neg\neg(a \vee b)$. Show that

(a) $\mathrm{Rg}(A) = \{a = \neg\neg a : a \in A\}$.

(b) Show that $\neg\neg : A \to \mathrm{Rg}(A)$ is a $\vee$-homomorphism. (In fact, it also a $\to$ and $\wedge$-homomorphism, but you do not have to show that.)

(3) (20pt) Let L be a lattice and $A \subseteq L$ a non-empty set. Show that

$$[A) = \uparrow\{a_1 \wedge \ldots \wedge a_n : n \in \mathbb{N}, a_1, \ldots, a_n \in A\}$$

is a filter, and moreover it is contained in any filter F of L which contains A.

(4) (20pt) Let A be a Boolean algebra. A filter F of the form $\uparrow a$ for some $a \in A$ is called a *principal filter*. Let $\mathrm{FinCofin}(\mathbb{N})$ be the Boolean algebra of all finite and cofinite subsets of $\mathbb{N}$.

(a) Characterize all principal ultrafilters in $\mathrm{FinCofin}(\mathbb{N})$.

(b) Show that there is a unique (!) non-principle ultrafilter in $\mathrm{FinCofin}(\mathbb{N})$.

(5) (20pt) Let A be a finite Boolean algebra. We order the set of all filters of A by inclusion. Show that A has a least non-unital filter iff A is isomorphic to a two-element Boolean algebra. Note that a least non-unital filter is a filter $F \subseteq A$ such that $F \neq \{1\}$ and for each filter $F' \neq \{1\}$ we have $F \subseteq F'$.