MATHEMATICAL STRUCTURES IN LOGIC 2018
HOMEWORK 2

- Deadline: February 20 — at the **beginning** of the tutorial class.
- In exceptional cases homework can be submitted electronically (in a single pdf-file!) to Saul Fernandez (`saul.fdez.glez@gmail.com`).
- Grading is from 0 to 100 points.
- Discussion of problems is allowed, but each student should submit a homework they themselves have written.
- Good luck!

(1) (40pt) Do the following equations hold in any Heyting algebra? If yes, give a proof, if not, provide a counter-example.

(a) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$,

(b) $\neg\neg a \vee \neg a = 1$,

(c) $\neg\neg\neg a = \neg a$,

(d) $(a \to b) \vee (b \to a) = 1$.

Here $\neg a = a \to 0$.

(2) (20pt)

(a) Let B be a finite Boolean algebra and $At(B) = \{x \in B : x \text{ is an atom}\}$. Show that the map defined by

$$\eta(a) = \{x \in At(B) : x \leq a\}$$

is a lattice morphism from B to $\mathcal{P}(At(B))$. That is, show that the following holds for every $a, b \in B$:

(i) $\eta(a \wedge b) = \eta(a) \cap \eta(b)$,

(ii) $\eta(a \vee b) = \eta(a) \cup \eta(b)$.

(b) Let X be an infinite set. Show that every finite Boolean algebra B is isomorphic to a subalgebra of $\mathcal{P}(X)$. That is, show that there is an injective Boolean algebra homomorphism $h : B \to \mathcal{P}(X)$. (A bit tricky. Hint: Use the representation of finite Boolean algebras.)

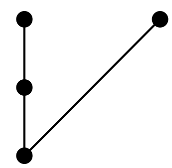
(3) (20pt) Let L be a lattice. We say that a non-zero element $a \in L$ is *join prime* if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. (Check exercise sheet 1 for the definition of join irreducible elements.)

(a) Show that in a distributive lattice the join irreducible elements coincide with the join prime elements.

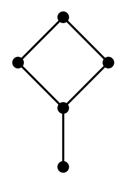
(b) Give an example of a lattice having a join irreducible element which is not a join prime element.

(4) (20pt)

(a) Draw the Heyting algebra of all up-sets of the poset drawn below.



(b) Let A be the Heyting algebra drawn below.



Find an embedding (injective HA homomorphism) $\iota\colon A \hookrightarrow \prod_{i\in I} A_i$ of HAs such that for each $i \in I$ the algebra A_i is a linear HA and $\pi_i \circ \iota\colon A \to A_i$ is surjective, where π_i is the i'th projection.

You can think of a finite (also infinite) product of Heyting algebras $A_1, \dots, A_n$ as follows. Take $A = A_1 \times \cdots \times A_n$ and define $\leq$ on A as follows:

$$(a_1, \dots a_n) \leq (b_1, \dots, b_n) \text{ iff } a_i \leq_i b_i \text{ for each } i = 1, \dots, n.$$

Then A is a HA and it is $\prod_{i=1}^{n} A_i$.