Mathematical structures in logic
Exercise sheet 7

Esakia duality; Jónsson's Lemma; locally finite, finitely generated and approximable varieties; Kripke completeness

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1 Esakia duality

(a) Compute the dual Esakia space of the Rieger-Nishimura lattice. (*Hint:* This space is often referred to as the Rieger-Nishimura ladder).

(b) Determine the Boolean algebra of regular elements of the Rieger-Nishimura lattice. (Compare the result with exercise 1.g from the previous exercise sheet).

2 Jónsson's Lemma

(a) Show that the variety of all Heyting algebras is not finitely generated;

(b) Determine the collection of subdirectly irreducible Heyting algebras in the variety $\mathsf{V}(\mathbf{A})$ generated by $\mathbf{A}$ when $\mathbf{A}$ is

$$\mathbf{C}_n, \quad \mathbf{2}, \quad (\mathbf{2}\times\mathbf{2})\oplus\mathbf{1}, \quad \mathbf{1}\oplus(\mathbf{2}\times\mathbf{2}), \quad (\mathbf{2}\times\mathbf{3})\oplus 1,$$

where $\mathbf{C}_n$ is a chain of n-elements and $-\oplus\mathbf{1}$ and $\mathbf{1}\oplus-$ denotes the operations of adding a new top and bottom element respectively. (*Hint:* You might find Esakia duality helpful for this.)

(c) Use exercise 2b to determine the lattice of subvarieties of the variety of Heyting algebras $\mathsf{V}(\mathbf{A})$, for $\mathbf{A}$ as in Exercise 2b.

Let Ω be a Signature.

(d) Let $\mathbb{V}$ be a congruence distributive variety of Ω-algebras and let $\mathbf{A}, \mathbf{B} \in \mathbb{V}$ be a pair of non-isomorphic subdirectly irreducible Ω-algebras with $\mathbf{A}$ finite. Show that if $|A| \leq |B|$ then there exists an equation that holds in $\mathbf{A}$ but fails in $\mathbf{B}$.

(e) Let $\mathbb{V}$ be a variety of Ω-algebras. Show that the collection of subvarieties of $\mathbb{V}$ forms a (bounded) lattice.

(f) Show that if $\mathbb{V}_1, \mathbb{V}_2$ are subvarieties of some congruence distributive variety of Ω–algebras then $(\mathbb{V}_1 \vee \mathbb{V}_2)_{si} = (\mathbb{V}_1)_{si} \cup (\mathbb{V}_2)_{si}$. (*Hint:* use Łos's Theorem[1].) Does the corresponding statement hold true for arbitrary joins as well?

(g) Show that if $\mathbb{V}$ is a congruence distributive variety of Ω-algebras then the lattice $\Lambda(\mathbb{V})$ of subvarieties of $\mathbb{V}$ is a distributive lattice.

[1] Recall that Łos's Theorem entails that $\Pi_{i\in I}\mathbf{A}_i/\mathcal{U} \models t \approx s$ iff $\{i \in I \colon \mathbf{A}_i \models t \approx s\} \in \mathcal{U}$, for all sets of algebras $\{\mathbf{A}_i\}_{i\in I}$ and all ultrafilters $\mathcal{U}$ on I.

3 Finitely approximable varieties

(a) Let Ω be a signature and let $\mathcal{K}$ be a class of Ω-algebras. An Ω-algebra $\mathbf{A}$ is *residually finite in* $\mathcal{K}$ iff for all $x, y \in A$ with $x \neq y$ there exist a finite algebra $\mathbf{B} \in \mathcal{K}$ together with an onto Ω-homomorphism $h\colon \mathbf{A} \twoheadrightarrow \mathbf{B}$ such that $h(x) \neq h(y)$.

Let $\mathbb{V}$ be a variety of Ω-algebras and let $\mathbf{A} \in \mathbb{V}$. Show that the following are equivalent

(i) The algebra $\mathbf{A}$ is residually finite in $\mathbb{V}$;

(ii) The intersection of all congruence with finite index is trivial, i.e.,

$$\Delta_A = \bigcap \{\theta \in \mathrm{Con}(\mathbf{A})\colon |\mathbf{A}/\theta| < \aleph_0\};$$

(iii) The algebra $\mathbf{A}$ embeds into a product of finite $\mathbb{V}$-algebras.

(b) Show that a variety of Ω-algebras $\mathbb{V}$ is finitely approximable iff all finitely generated free $\mathbb{V}$-algebras are residually finite in $\mathbb{V}$.

(c) Describe the subdirectly irreducible Ω-algebras which are residually finite.

4 Locally finite and finitely generated varieties

Let Ω be any signature.

(a) Show that a variety of Ω-algebras is finitely generated iff it is generated by a single finite Ω-algebra.

(b) Let $\mathbb{V}$ be a variety of Ω-algebras. Show that the following are equivalent

(i) The variety $\mathbb{V}$ is locally finite;

(ii) All the finitely generated free $\mathbb{V}$-algebras are finite;

(iii) Every member of $\mathbb{V}$ is locally finite[2].

(c) Show that in general

$$\text{finitely generated} \implies \text{locally finite} \implies \text{finitely approximable}.$$

with both of the implications being strict. (*Hint:* Showing that the first implication is strict is a bit tricky. You might want to consider the variety $\mathbb{LC}$ of pre-linear Heyting algebras. Moreover, you may use that if $\mathbf{A} \cong \prod_{i \in I} \mathbf{A}_i$ for a set of algebras $\{\mathbf{A}_i\}_{i \in I}$ such that for each $n \in \omega$ there exist only finitely many non-isomorphic finitely generated subalgebras of the algebras $\{A_i\}_{i \in I}$, then $\mathbf{A}$ is locally finite.)

5 Discrete duality and logic

Let $\mathfrak{G} = (W, \leq)$ be an intuitionistic Kripke frame, i.e., a poset. A *valuation* on $\mathfrak{G}$ is a function $V\colon \mathsf{Prop} \to \mathsf{Up}(W, \leq)$, where Prop is a collection of propositional letters. A *pointed intuitionistic Kripke model* is an intuitionistic Kripke frame $(W, \leq)$ together with a valuation on $(W, \leq)$ and an element $w \in W$. We define a relation $\Vdash$ (pronounced "forces")

[2] An algebra is called *locally finite* if all of its finitely generated subalgebras are finite.

between pointed intuitionistic Kripke models and formulas of propositional intuitionistic logic as follows:

$$\begin{array}{rll} (\mathfrak{G}, V, w) \Vdash \bot & & \text{never;} \\ (\mathfrak{G}, V, w) \Vdash p & \text{iff} & w \in V(p); \\ (\mathfrak{G}, V, w) \Vdash \phi \wedge \psi & \text{iff} & (\mathfrak{G}, V, w) \Vdash \phi \text{ and } (\mathfrak{G}, V, w) \Vdash \psi; \\ (\mathfrak{G}, V, w) \Vdash \phi \vee \psi & \text{iff} & (\mathfrak{G}, V, w) \Vdash \phi \text{ or } (\mathfrak{G}, V, w) \Vdash \psi; \\ (\mathfrak{G}, V, w) \Vdash \phi \to \psi & \text{iff} & \forall v \in W((w \leq v \text{ and } (\mathfrak{G}, V, v) \Vdash \phi)) \implies (\mathfrak{G}, V, v) \Vdash \psi). \end{array}$$

We write $(\mathfrak{G}, V) \Vdash \phi$ iff $(\mathfrak{G}, V, w) \Vdash \phi$ for all $w \in W$ and finally we write $\mathfrak{G} \Vdash \phi$ iff $(\mathfrak{G}, V) \Vdash \phi$ for all valuations V on $\mathfrak{G}$.

An intermediate logic L is *Kripke complete* iff there exists a class of intuitionistic Kripke frames $\mathcal{K}$ such that

$$\forall \phi (\phi \in L \iff \forall \mathfrak{G} \in \mathcal{K} (\mathfrak{G} \Vdash \phi)).$$

(a) Let $\mathfrak{G} = (W, \leq)$ be an intuitionistic Kripke frame and let ϕ be a formula in the language of propositional intuitionistic logic. Show that $\mathfrak{G} \Vdash \phi$ iff $\mathfrak{G}^+ \models \phi \approx 1$.

(b) Let L be an intermediate logic. Show that L is Kripke complete iff the corresponding variety of Heyting algebras $\mathbb{V}_L$ is generated by a collection of Heyting algebras of the form $\mathfrak{G}^+$. Such varieties are called *complete varieties.*

(c) Show that if $\mathbb{V}_L$ is variety of Heyting algebras which is finitely approximable then the corresponding intermediate logic L is Kripke complete.

(d) Let L be an intermediate logic. Show that L is Kripke complete iff the corresponding variety of Heyting algebras $\mathbb{V}_L$ is generated by a collection of complete and completely join-generated Heyting algebras. (*Hint:* Use Exercise 5d. from the previous exercise sheet.)

6 MacNeille completions

Let P be a poset and for $S \subseteq P$ let $L(S)$ and $U(S)$ be the set of all lower and upper bounds of S, respectively. A subset $S \subseteq P$ is called a *normal ideal* if $S = L(U(S))$. Recall that a lattice-completion $j \colon \mathbf{L} \hookrightarrow \mathbf{C}$ is join-regular (meet-regular) if j preserves arbitrary existing joins (meets). Moreover, we say that $\mathbf{L}$ is join-dense (meet-dense) in $\mathbf{C}$ if every element of C is a join (meet) of elements from L.

(a) Let $\mathbf{L}$ be a lattice. Show that the collection $\mathfrak{I}_N(\mathbf{L})$ of normal ideals of $\mathbf{L}$ is a complete lattice.

(b) Let $\mathbf{L}$ be a lattice. Show that $\iota \colon \mathbf{L} \to \mathfrak{I}_N(\mathbf{L})$ given by $a \mapsto {\downarrow} a$ is a regular lattice embedding, i.e., both join- and meet-regular;

(c) Let $\mathbf{L}$ be a lattice. Show that $\mathfrak{I}_N(\mathbf{L})$ is (isomorphic to) the MacNeille completion of $\mathbf{L}$;

(d) Let $\mathbf{L}$ be a lattice and let $j \colon \mathbf{L} \hookrightarrow \mathbf{C}$ be a completion. Show that j is join-regular (resp. meet-regular) if $\mathbf{L}$ is meet-dense (resp. join-dense) in $\mathbf{C}$.

(e) Compute the MacNeille completion of the Boolean algebra $\mathrm{FC}(\omega)$ of the finite and cofinite subsets of ω;

(f) Let $\mathbf{D}$ be a distributive lattice. Show that if $\mathbf{D}$ is a Heyting algebra then so is the MacNeille completion $\overline{\mathbf{D}}$ with Heyting implication given by;

$$x \to_{\overline{\mathbf{D}}} y = \bigwedge \{a \to_{\mathbf{D}} b \colon a \leq x, y \leq b\};$$

(g) Let $\mathbf{A}$ be a Heyting algebra and let $I \subseteq A$ be an ideal. Show that I is a normal ideal iff I is closed under all existing joins.