Mathematical structures in logic
Exercise sheet 6

Priestley and Esakia duality, Discrete duality and some *logical* applications

April 24, 2017

1 Priestley and Esakia duality

(a) Let $(X, \tau, \leq)$ be a Priestley space show that

(i) the set $\uparrow x$ is closed for each $x \in X$;

(ii) the sets $\uparrow F$ and $\downarrow F$ are closed for each closed subset F of (X, τ).

(b) Let $\mathbf{D}$ be a bounded distributive lattice and let $X_D = (X, \tau, \leq)$ be its dual Priestley space. Show that for every clopen upset U of $X_\mathbf{D}$ there exists $a \in D$ such that $U = \phi_+(a)$, where $\phi_+(a)$ is the set of prime filters on $\mathbf{D}$ containing a (*Hint:* You will most likely have to use compactness twice, first for a cover of U^c and then for a cover of U.)

(c) Let $\mathbf{D}$ be a bounded distributive lattice. Show that the lattice $\mathrm{Con}(\mathbf{D})$ of congruences on $\mathbf{D}$ is anti-isomorphic to the lattice of closed subsets of the dual Priestley space $X_\mathbf{D}$. Deduce that the variety of bounded distributive lattices is congruence distributive.

(d) Let $\mathbf{A}$ be a Heyting algebra show. Show that the lattice $\mathrm{Con}(\mathbf{D})$ of Heyting algebra congruences on $\mathbf{A}$ is anti-isomorphic to the lattice of closed upsets of the dual Esakia space $X_\mathbf{A}$. Deduce that the variety of Heyting algebras is congruence distributive. (*Hint:* This exercise might be a bit difficult. Show that the closed upset are precisely the intersections of clopen upsets.)

(e) Show that a non-trivial Heyting algebra is subdirectly irreducible iff it has a second greatest element.

(f) Show that if $\mathbf{A}$ is a Heyting algebra then $\max(X_\mathbf{A})$ is closed. Conclude that the set of maximal points of any Esakia space is closed. Is this also the case for all Priestley spaces? (*Hint:* Show that $\max(X_\mathbf{A}) = \bigcap\{\phi(a) \colon a \in D(\mathbf{A})\}$, where $D(\mathbf{A})$ is the filter of dense elements of $\mathbf{A}$. For this you might want to look at elements of the form $a \vee \neg a$.)

(g) By the above we may for any Heyting algebra $\mathbf{A}$ consider $\max(X_\mathbf{A})$ as an Esakia space determined by the subspace topology. Can you describe the dual algebra of the Esakia space $\max(X_\mathbf{A})$ in terms of $\mathbf{A}$? (*Hint:* Show that if C_F is a closed upset of $\mathfrak{X}_\mathbf{A}$ corresponding to the filter F on $\mathbf{A}$ then dual algebra of C_F is (isomorphic to) the quotient $\mathbf{A}/F$).

2 Adjoints between posets

Let $h \colon P \to Q$ be an order-preserving map between posets. A *right* or *upper adjoint of* h is an order-preserving map $h^\sharp \colon Q \to P$ satisfying

$$h(a) \leq b \iff a \leq h^\sharp(b).$$

Similarly, a *left* or *lower adjoint of* h is an order-preserving map $h_\flat \colon Q \to P$ satisfying

$$b \leq h(a) \iff h_\flat(b) \leq a.$$

(a) Let $h \colon P \to Q$ be an order-preserving map between posets with P and Q complete posets. Show that h has a right (left) adjoint iff h preserves all suprema (infima);

(b) Let $h \colon \mathbf{L} \to \mathbf{L}'$ be a complete lattice homomorphism[1] between complete lattices $\mathbf{L}, \mathbf{L}'$. Show that h has a left adjoint $h_\flat \colon \mathbf{L}' \to \mathbf{L}$ which maps completely join-prime elements of $\mathbf{L}'$ to completely join-prime elements of $\mathbf{L}$.

3 Discrete duality for Boolean algebras

Let **Set** be the category of sets and functions and let **CABA** be the category of complete atomic Boolean algebras and complete Boolean algebra homomorphisms. Prove that the correspondence between **Set** and **CABA** from HW 5, exercise 1, is part of a dual equivalence $\mathbf{Set}^{\mathrm{op}} \cong \mathbf{CABA}$, i.e.,

(a) Show that $\wp : \mathbf{Set} \to \mathbf{CABA}$ and $\mathcal{A} : \mathbf{CABA} \to \mathbf{Set}$ are contravariant functors. (What are the actions on morphisms?)

(b) Show that the isomorphisms from HW 5, exercise 1 are natural, i.e., show that for complete atomic Boolean algebra $\mathbf{B}$ the isomorphims $\eta_{\mathbf{B}} : \mathbf{B} \to \wp(\mathcal{A}(\mathbf{B}))$ are components of a natural transformation $\eta : \mathrm{Id}_{\mathbf{CABA}} \Rightarrow \wp \circ \mathcal{A}$ and similarly, for every set X, the bijections $\mu_X : X \to \mathcal{A}(\wp(X))$ are components of a natural transformation $\mu : \mathrm{Id}_{\mathbf{Set}} \Rightarrow \mathcal{A} \circ \wp$.

So you need to show that for every complete Boolean homomorphism $f \in \mathrm{Hom}_{\mathbf{CABA}}(\mathbf{B}, \mathbf{C})$ and every map $g \in \mathrm{Hom}_{\mathbf{Set}}(X, Y)$ the following diagrams commute.

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{\quad f \quad} & \mathbf{C} \\ \downarrow{\scriptstyle \eta_{\mathbf{B}}} & & \downarrow{\scriptstyle \eta_{\mathbf{C}}} \\ \wp((\mathcal{A}(\mathbf{B})) & \xrightarrow{\wp(\mathcal{A}(f))} & \wp(\mathcal{A}(\mathbf{C})) \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\quad g \quad} & Y \\ \downarrow{\scriptstyle \mu_X} & & \downarrow{\scriptstyle \mu_Y} \\ \mathcal{A}(\wp(X)) & \xrightarrow{\mathcal{A}(\wp(g))} & \mathcal{A}(\wp(Y)) \end{array}$$

(c) What happens when you restrict this duality to the finite objects?

[1] That is a function of the underlying sets preserving arbitrary joins and meets.

4 Discrete duality for modal algebras

For this exercise we need a few additional definitions. A modal algebra $(\mathbf{A}, \Diamond)$ is called *perfect* if $\mathbf{A}$ is a complete and atomic Boolean algebra and $\Diamond \colon \mathbf{A} \to \mathbf{A}$ is a completely additive operator, i.e.,

$$\Diamond \bigvee S = \bigvee \Diamond[S].$$

Finally, a relation preserving map $f \colon \mathfrak{S} \to \mathfrak{S}'$ between Kripke frames $\mathfrak{S} = (W, R)$ and $\mathfrak{S}' = (W', R')$ is a *p-morphism* if it satisfies

$$\forall w \in W \forall v' \in W' (f(w) R' v' \implies \exists v \in W \ (wRv \text{ and } f(v) = v')).$$

(a) Let $\mathfrak{S} = (W, R)$ be a Kripke frame, i.e., a set wit a binary relation. Show that $\mathfrak{S}^+ := (\wp(W), \Diamond_R)$ is a perfect modal algebra;

(b) Let $\mathbf{A}$ be a perfect modal algebra and let $\mathbf{A}_+ := (\mathcal{A}(\mathbf{A}), R_{\Diamond_\mathbf{A}})$ be the Kripke frame defined by

$$a R_{\Diamond_\mathbf{A}} a' \iff a \leq \Diamond_\mathbf{A} a'.$$

Show that $(\mathfrak{S}^+)_+ \cong \mathfrak{S}$ as Kripke frames and that $(\mathbf{A}_+)^+ \cong \mathbf{A}$ as modal algebras, for all Kripke frames $\mathfrak{S}$ and all perfect modal algebras $\mathbf{A}$.

(c) Show that the category of Kripke frames and p-morphism is dually equivalent to the category of perfect modal algebras and complete modal algebra homomorphisms.

(d) What happens when you restrict this duality to the finite objects?

5 Discrete duality for bounded distributive lattices

The aim of this exercise is to generalise the duality between **Set** and **CABA** to the setting of bounded distributive lattices. Let **Pos** be the category of posets and order-preserving maps. Furthermore, let $\mathbf{CbDL_{cJg}}$ denote the category of complete and completely join-generated bounded distributive lattices and complete bounded lattice homomorphisms. Recall that a lattice L is completely join-generated if every element in L is the join of the completely join-prime elements below it.

(a) Let $\mathsf{P} = (P, \leq)$ be a poset. Show that $\mathsf{P}^+ = \mathsf{Up}(P)$ is a complete and completely join-generated bounded distributive lattice. Extend this to a functor $(-)^+ \colon \mathbf{Pos} \to \mathbf{CbDL_{cJg}}$.

(b) Define a functor $(-)_+ \colon \mathbf{CbDL_{cJg}} \to \mathbf{Pos}$

(c) Show that that **Pos** and $\mathbf{CbDL_{cJg}}$ are dually equivalent.

(d) Find a full subcategory of **Pos** which is dually equivalent to the category of complete and completely join-generated Heyting algebras.

(e) What happens when you restrict these dualities to the finite objects?

6 Discrete duality and logic

(a) Let $\mathfrak{G} = (W, \leq)$ be an intuitionistic Kripke frame, i.e., a poset and let ϕ be a formula in the language of propositional intuitionistic logic. Show that $\mathfrak{G} \Vdash \phi$ iff $\mathfrak{G}^+ \models \phi \approx 1$, where $\mathfrak{G}^+$ is the Heyting algebra of up-sets of $\mathfrak{G}$.

(b) Let $\mathfrak{G} = (W, R)$ be a Kripke frame, i.e., a set wit a binary relation and let ϕ be a formula in the language of propositional modal logic. Show that $\mathfrak{G} \Vdash \phi$ iff $\mathfrak{G}^+ \models \phi \approx 1$, where $\mathfrak{G}^+$ is the complex modal algebra $(\wp(W), \Diamond_R)$.

7 The Freyd glueing construction

Let $h\colon \mathbf{L} \to \mathbf{L}'$ be bounded meet-semi lattice homomorphism between bounded meet-semi lattices $\mathbf{L} = (L, \wedge, 1)$ and $\mathbf{L} = (L', \wedge, 1)$ and define

$$\mathbf{L} \times_{\gamma(h)} \mathbf{L}' := \{(a, b) \in L \times L' \colon b \leq h(a)\}.$$

Note that this is not standard notation.

(a) Show that if $\mathbf{L}$ and $\mathbf{L}'$ are bounded lattices and $h\colon L \to L'$ is a homomorphism of the underlying bounded meet-semi lattices then $\mathbf{L} \times_{\gamma(h)} \mathbf{L}'$ is a bounded sublattice of the direct product $\mathbf{L} \times \mathbf{L}'$

(b) Show that if $\mathbf{A}$ and $\mathbf{B}$ are Heyting algebras and $h\colon A \to B$ is a homomorphism of the underlying bounded meet-semi lattices then $\mathbf{A} \times_{\gamma(h)} \mathbf{B}$ is in fact a Heyting algebra and the projection $\pi\colon \mathbf{A} \times_{\gamma(h)} \mathbf{B} \twoheadrightarrow \mathbf{A}$ is a Heyting algebra homomorphism. (*Hint:* This is no longer necessarily a sub-Heyting algebra of the direct product).

(c) Let $c_1\colon A \to \mathbf{2}$ be the bounded semi-lattice homomorphism determined by $c_1(a) = 1$ iff $a = 1$. Show that $\mathbf{A} \times_{\gamma(c_1)} \mathbf{2}$ is a prime. Recall that a Heyting algebra is *prime* if the top element 1 is join-irreducible. (*Hint:* Show that $\mathbf{A} \times_{\gamma(c_1)} \mathbf{2}$ has a second greatest element.)

(d) Show that all free Heyting algebras are prime. (*Hint:* Show that the the unique map from $\mathscr{F}_{\mathbb{HA}}(X) \to \mathscr{F}_{\mathbb{HA}}(X) \times_{\gamma(c_1)} \mathbf{2}$ extending $x \mapsto (x, 0)$ is an injection)

(e) Conclude that **IPC** enjoys *the disjunction property*, i.e., if $\vdash_{\mathbf{IPC}} \phi \vee \psi$ then either $\vdash_{\mathbf{IPC}} \phi$ or $\vdash_{\mathbf{IPC}} \psi$.

(f) Let $c_1\colon A \to \mathbf{2}$ be as in (c). Compute the dual Esakia space of the algebra $\mathbf{A} \times_{\gamma(c_1)} \mathbf{2}$.