Mathematical structures in logic
Exercise sheet 5

Congruence extension property, Stone duality and free (Boolean) algebras

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1 Congruence extension property

Let Ω be a signature. A class $\mathcal{K}$ of Ω-algebras is said to enjoy *the congruence extension property (CEP)* if for all $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ with $\mathbf{A}$ a subalgebra of $\mathbf{B}$ then for all congruence θ on $\mathbf{A}$ there exists a congruence θ' on $\mathbf{B}$ such that $\theta' \cap A^2 = \theta$.

(1) Show that a variety $\mathbb{V}$ of Ω-algebras has the congruence extension property iff for every diagram

$$\begin{tikzcd} \mathbf{A} \arrow[r, hook] \arrow[d, two heads] & \mathbf{B} \\ \mathbf{C} & \end{tikzcd}$$

of $\mathbb{V}$-algebras there exist a $\mathbb{V}$-algebra $\mathbf{D}$ such that the following diagram

$$\begin{tikzcd} \mathbf{A} \arrow[r, hook] \arrow[d, two heads] & \mathbf{B} \arrow[d, two heads] \\ \mathbf{C} \arrow[r, hook] & \mathbf{D} \end{tikzcd}$$

commutes.

(2) Show that any variety of Heyting algebras enjoy the CEP. (*Hint: use the correspondence between congruences and filters.*)

(3) Does the variety $\mathbb{L}$ of (bounded) lattices enjoy the CEP?

2 Some consequences of Stone duality

In the following let $\mathbf{B}$ be a Boolean algebra and $X_{\mathbf{B}}$ its dual Stone space.

(1) Let $(\mathsf{Fil}(\mathbf{B}), \subseteq)$ be the poset of filters on $\mathbf{B}$ and let $(\mathsf{Cl}(X_{\mathbf{B}}), \subseteq)$ be the poset of closed subsets of $X_{\mathbf{B}}$. Show that there is an order-reversing bijection between $(\mathsf{Fil}(\mathbf{B}), \subseteq)$ and $(\mathsf{Cl}(X_{\mathbf{B}}), \subseteq)$. Can you say something similar about the poset of ideals on $\mathbf{B}$?

(2) Show that there is a one-to-one correspondence between the atoms of $\mathbf{B}$ and the isolated points of $X_{\mathbf{B}}$.

(Recall that in a topological space $\mathfrak{X} = (X, \tau)$ a point $x \in X$ is *isolated* if $\{x\}$ is open).

(3) Let $\mathcal{I}_{X_{\mathbf{B}}} := \{x \in X_{\mathbf{B}} \mid x \text{ is an isolated point}\}$. Show that $\mathbf{B}$ is atomic iff $\mathcal{I}_{X_{\mathbf{B}}}$ is dense in $X_{\mathbf{B}}$.

(Recall that in a topological space $\mathfrak{X} = (X, \tau)$ a subset $Y \subseteq X$ is *dense* in X iff the topological closure of Y is X. Equivalently, Y is dense if for every $x \in X$ and every open U with $x \in U$ it follows that $U \cap Y \neq \emptyset$. Recall further that a Boolean algebra $\mathbf{B}$ is *atomic* if below every non-zero element of $\mathbf{B}$ there is an atom.)

(4) Brouwer's Theorem states that there is exactly one (up to homeomorphism) second countable*A topological space is second countable if it has a countable basis.* Stone space without isolated points, namely the Cantor space. Translate this into an equivalent statement about Boolean algebras.

(5) Show that for every Boolean algebra $\mathbf{B}$ the underlying set of $X_{\mathbf{B}}$ is isomorphic to the set $\mathrm{Hom}_{\mathbb{BA}}(\mathbf{B}, \mathbf{2})$ of Boolean algebra homomorphism from $\mathbf{B}$ to the two element Boolean algebra $\mathbf{2}$.

(6) Show that for every Boolean algebra $\mathbf{B}$ the dual Stone space $X_{\mathbf{B}}$ is (homeomorphic) to a closed subspace of the set 2^B equipped with the product topology.

(7) Conclude that the Stone spaces are precisely the topological spaces which are homeomorphic to a closed subspace of 2^X for some set X.

3 Free algebras

Let Ω be a signature and let $\mathcal{K}$ be a class of Ω-algebras. For X any set a $\mathcal{K}$-algebra $\mathbf{F}$ is said to be *freely generated by* X (or *free over* X) if there exists an injection of sets $\iota \colon X \hookrightarrow F$ such that for any $\mathcal{K}$-algebra $\mathbf{A}$ and any function of sets $h \colon X \to A$ there exists a unique Ω-homomorphism $\overline{h} \colon \mathbf{F} \to \mathbf{A}$ such that $\overline{h} \circ \iota = h$.

(1) Show that for any cardinal κ and any sets X, X' of cardinality κ if $\mathbf{F}, \mathbf{F}'$ are $\mathcal{K}$-algebra free over X and X' respectively then $\mathbf{F}$ and $\mathbf{F}'$ must be isomorphic. Thus if it exists we may speak about *the* free $\mathbf{K}$-algebra on κ-many generators.

(2) Convince yourself that if $\mathbb{V}$ is a (non-trivial) variety then the free-algebras on X always exists for any non-emptyset set X. (*Hint: consider the set of terms in X modulo the equational theory of* $\mathbb{V}$.) What about the case where $X = \emptyset$?

(3) Let $\mathbb{V}$ be a variety. Show that any algebra in $\mathbb{V}$ is a homomorphic image of some free algebra in $\mathbb{V}$.

(4) Show that any variety $\mathbb{V}$ is generated by the collection of free $\mathbb{V}$-algebras on finitely many generators.

(5) Show that any variety $\mathbb{V}$ is generated by the free $\mathbb{V}$-algebra of countably many generators.

(6) Give an example of a non-trivial (universal) class of algebras for which free algebras do not exists.

4 Free Boolean algebras

(1) Show that the dual Stone space of the free Boolean algebra on X is (homeomorphic to) 2^X.

(2) Conclude that the free Boolean algebra on n-generators is finite and of cardinality 2^{2^n}.

(3) Conclude that the dual Stone space of the free Boolean algebra on countably many generators is (homeomorphic to) the Cantor space.

(4) Can you show that the free Boolean algebra on countably many generators is atomless without using duality?