Mathematical structures in logic
Exercise class 4

Logics and varieties of algebras, filters and congruences

March 19, 2017

Recall that a *superintuitionistic logic* is a set L of propositional formulas such that $L \supseteq \mathbf{IPC}$ and L is closed under the rules of (MP) and (Subs).

(1) Show that the set of superintuionistic logics $\Lambda(\mathbf{IPC})$ forms a bounded lattice with respect to $\subseteq$. Describe the bounds, meets and joins in this lattice.

(2) Show that the subvarieties of the variety of Heyting algebras $\Lambda(\mathsf{HA})$ form a lattice with respect to $\subseteq$. Describe the bounds, meets and joins in this lattice.

(3) Let L be a superintuitionistic logic. Recall that V_L is the variety axiomatized by $\{\varphi \approx 1 : L \vdash \varphi\}$. Show that for superintuionistic logics L, L' the following holds

$$L \subseteq L' \text{ implies } \mathsf{V}_L \supseteq \mathsf{V}_{L'}.$$

and

$$L \not\subseteq L' \text{ implies } \mathsf{V}_L \not\supseteq \mathsf{V}_{L'}.$$

(Hint: use that L is complete wrt V_L. That is, for each formula φ we have

$$L \vdash \varphi \text{ iff } \mathsf{V}_L \models \varphi \approx 1.)$$

(4) Let V be a variety of Heyting algebras. Recall that L_V is the least superintuionistic logic containing the set $\{\varphi \leftrightarrow \psi : \mathsf{V} \models \varphi \approx \psi\}$. Show that for each variety of Heyting algebras V and each superintuionistic logic L we have

$$\mathsf{V}_{L_\mathsf{V}} = \mathsf{V} \text{ and } L_{\mathsf{V}_L} = L.$$

(5) Deduce that the lattice of superintuionistic logics $(\Lambda(\mathbf{IPC}), \subseteq)$ is dually isomorphic to the lattice $(\Lambda(\mathsf{HA}), \subseteq)$ of varieties of Heyting algebras.

(6) Let A be a Boolean algebra.

(a) Show that for each $a \in A$, the set ${\uparrow}a = \{b \in A : a \leq b\}$ is a filter.

(b) Show that $F \subseteq A$ is a filter iff $I = \{\neg a : a \in F\}$ is an ideal.

(7) Let A be an 8-element Boolean algebra. Find filters $F, F' \in A$ such that A/F is isomorphic to **2** and A/F' is isomorphic to **4**, where **2** and **4** are 2 and 4 element Boolean algebras, respectively.

(8) Let A be a finite linear Heyting algebra (i.e., for each $a, b \in A$ we have $a \leq b$ or $b \leq a$).

(a) Characterize all (maximal, prime) filters and ideals of A.

(b) Describe all the homomorphic images of A.

(9) Let $(A, \Box)$ be a modal algebra. A filter $F \subseteq A$ is called a *modal filter* if for each $a \in A$

$$a \in F \text{ implies } \Box a \in F.$$

Show that congruences of $(A, \Box)$ are in one-to-one correspondence with modal filters of A. (You may assume the correspondence between Boolean congruences and filters.)

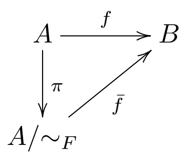
(10) Let A be a Heyting algebra. Let $D = \{a \in A : \neg a = 0\}$.

(a) Show that D is a filter. Elements of this filter are called *dense elements.*

(b) Show that A/D is isomorphic to the Boolean algebra $\mathrm{Rg}(A)$ of all regular elements of A.

(11) Let A be a Heyting algebra, $f : A \to B$ a homomorphism of Heyting algebras, F a filter on A, $\sim_F$ the congruence associated to F, and $\pi : A \to A/{\sim_F}$ the quotient mapping. Show that the following are equivalent:

- There is a morphism $\bar{f}$ such that



  commutes.
- For each $a \in F$, $f(a) = 1_B$.

Furthermore, show that the morphism $\bar{f}$ is unique if it exists.