Mathematical structures in logic
Exercise classes 5-6

Alexandroff topologies and Universal algebra

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(1) Let $(P, \leq)$ be a poset. Then $\mathrm{Up}(P)$, the upsets on P, form a topology on P.

   (a) Show that this is an Alexandroff topology, i.e. show that the open sets are closed under arbitrary intersections.

   (b) Show that for every $P' \subseteq P$, $\mathrm{Cl}(P') = {\downarrow}P'$.

   (c) Describe the interior of a set $P' \subseteq P$.

   Here $\mathrm{Cl}(P')$ denotes the closure of P' and as usual ${\downarrow}P' := \{q \in P \mid \exists p \in P', q \leq p\}$.

(2) Show that every equational class is a variety, i.e. show that validity of equations is preserved by the operators **H**, **S** and **P**.

(3) (a) Find an example of a Heyting algebra A and a subalgebra A' of A such that A' is not a homomorphic image of A.

   (b) Find an example of a Heyting algebra that has a homomorphic image B such that B is not isomorphic to a subalgebra of A.

   (*Hint:* finite such examples exist).

(4) Let R be a pre-order, i.e. transitive and reflexive relation on a set X. Define $\Box_R\colon \wp(X) \to \wp(X)$ by

$$\Box_R(U) = \{x \in X\colon R[x] \subseteq U\},$$

   where $R[x] = \{x' \in X\colon xRx'\}$.

   (a) Show that $(\wp(X), \Box_R)$ is an **S4**-algebra[1] .

   (b) Determine the fixed points of the operator $\Box_R$.

   (c) Can you define a finite join preserving function $\Diamond_R\colon \wp(X) \to \wp(X)$ in a similar way? What are the fixed points for this operator.

[1] Also know as an *interior algebra*.

Additional exercises

(1) (More on Alexandroff spaces and posets)

(a) Let $f\colon P \to Q$ be a function between posets $(P, \leq_P)$ and $(Q, \leq_Q)$. Show that f is order-preserving iff f is continuous with respect to the topologies $\mathrm{Up}(P)$ and $\mathrm{Up}(Q)$.

(b) Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be posets. Characterise the order-preserving maps $f\colon P \to Q$ with the property that f is an open map as a function between the induced Alexandroff spaces.

(2) Which of the class operations **H**, **S** and **P** preserves quasi-equations, i.e., clauses of the form

$$s_1 \approx t_1, \ldots, s_n \approx t_n \implies s \approx t.$$