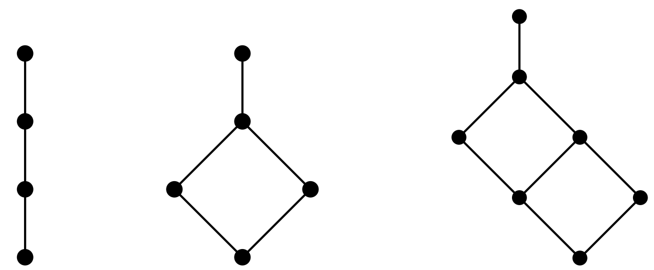
Mathematical structures in logic
Exercise classes 3-4

Heyting algebras, Boolean algebras

February 20-27, 2017

(1) Let A_1, A_2 and A_3 be the following posets



(a) Convince yourself that A_1, A_2 and A_3 are all Heyting algebras.

(b) Identify the joins and the pseudo-complements in A_1, A_2 and A_3.

(c) Is A_1 isomorphic to a bounded sublattice (Heyting subalgebra) of A_2 or A_3?

(2) (Atoms and co-atoms)

(a) Describe atoms and co-atoms on a Boolean algebra of the form $\wp(X)$.

(b) Show that in every Boolean algebra, if a is an atom, then $\neg a$ is a co-atom.

(c) Find a Heyting algebra A with an atom a such that $\neg a$ is not a co-atom

(3) Show that the lattice $(\text{Cofin}(\mathbb{N}) \cup \{\emptyset\}, \subseteq)$ of cofinite subsets of $\mathbb{N}$ (together with $\emptyset$) is a bounded distributive lattice. Is L a complete lattice? Is it a Heyting algebra?

(4) Show that not every bounded distributive lattice is isomorphic to the lattice of upsets of some poset.

(5) We abbreviate $a \to 0$ with $\neg a$. Show that in every Heyting algebra

(a) $a \wedge \neg a = 0$ but not necessarily $a \vee \neg a = 1$;

(b) $a \leq b$ iff $a \to b = 1$;

(c) $a \leq \neg\neg a$;

(d) $\neg a \wedge \neg b = \neg(a \vee b)$;

(e) $\neg a \vee \neg b \leq \neg(a \wedge b)$ but not necessarily $\neg(a \wedge b) \leq \neg a \vee \neg b$

(f) $a \to (b \to c) = (a \wedge b) \to c$;

(g) $b \leq c \implies a \to b \leq a \to c$;

(h) $b \leq c \implies c \to a \leq b \to a$.

Additional exercises

(6) Let A be a Boolean algebra. Show that $a \wedge b = \neg(\neg a \vee \neg b)$ and $a \vee b = \neg(\neg a \wedge \neg b)$

(7) Let A_2 and A_3 be as in exercise 1.

(a) Is A_2 isomorphic to a bounded sublattice (Heyting subalgebra) of A_3?

(b) Is there a surjective bounded lattice (Heyting algebra) homomorphism from A_3 to A_2?

(8) Let L be a bounded distributive lattice. Show that pairs of complemented elements of L are in one-to-one correspondence with decompositions of the form $L \cong L_1 \times L_2$. (Hint: Try to understand first what this means for powerset lattices.)

(9) *For people who know some category theory:* Given a poset $(P, \leq)$ we can see it as a category having P as objects and there is a morphism from p to q iff $p \leq q$. Try to connect the notions of lattice theory that we encountered so far (suprema, infima, bounds, Heyting implications, complements, ...) to categorical structure (such as products, coproducts, ...).