Varieties of Heyting algebras and superintuitionistic logics

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Heyting algebras

A Heyting algebra is a bounded distributive lattice $(A, \wedge, \vee, 0, 1)$ equipped with a binary operation $\to$, which is a right adjoint of $\wedge$. This means that for each $a, b, x \in A$ we have

$$a \wedge x \leq b \ \text{ iff } \ x \leq a \to b.$$

Heyting algebras

Heyting algebras pop up in different areas of mathematics.

1. **Logic**: Heyting algebras are algebraic models of intuitionistic logic.
2. **Topology**: opens of any topological space form a Heyting algebra.
3. **Geometry**: open subpolyhydra of any polyhedron form a Heyting algebra.
4. **Category theory**: subobject classifier of any topos is a Heyting algebra.
5. **Universal algebra**: lattice of all congruences of any lattice is a Heyting algebra.

Outline

The goal of the tutorial is to give an insight into the complicated structure of the lattice of varieties of Heyting algebras.

The outline of the tutorial:

1. Heyting algebras and superintuitionistic logics
2. Representation of Heyting algebras
3. Hosoi classification of the lattice of varieties of Heyting algebras
4. Jankov formulas and splittings
5. Canonical formulas

Part 1: Heyting algebras and superintuitionistic logics

Constructive reasoning

One of the cornerstones of classical reasoning is the law of excluded middle $p \vee \neg p$.

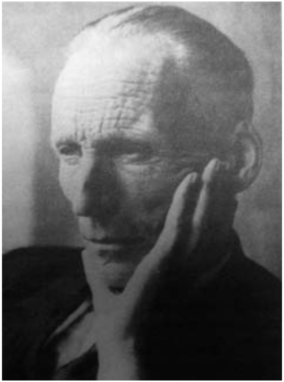
Constructive viewpoint: Truth = Proof.

The law of excluded middle $p \vee \neg p$ is constructively unacceptable.

For example, we do not have a proof of Goldbach's conjecture nor are we able to show that this conjecture does not hold.

Constructive reasoning

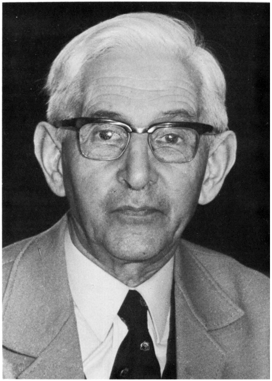
On the grounds that the only accepted reasoning should be constructive, the dutch mathematician L. E. J. Brouwer rejected classical reasoning.



Luitzen Egbertus Jan Brouwer (1881 - 1966)

Intuitionistic logic

In 1930's Brouwer's ideas led his student Heyting to introduce intuitionistic logic which formalizes constructive reasoning.



Arend Heyting (1898 - 1980)

Intuitionistic logic

Roughly speaking, the axiomatization of intuitionistic logic is obtained by dropping the law of excluded middle from the axiomatization of classical logic.

CPC = classical propositional calculus
IPC = intuitionistic propositional calculus.

The law of excluded middle is not derivable in intuitionistic logic. So **IPC** $\subsetneq$ **CPC**.

In fact,

$$\mathbf{CPC} = \mathbf{IPC} + (p \vee \neg p).$$

There are many logics in between **IPC** and **CPC**

Superintuitionistic logics

A superintuitionistic logic is a set of formulas containing **IPC** and closed under the rules of substitution and Modus Ponens.

Superintuitionistic logics contained in **CPC** are often called intermediate logics because they are situated between **IPC** and **CPC**.

As we will see, intermediate logics are exactly the consistent superintuitionistic logics.

Since we are interested in consistent logics, we will mostly concentrate on intermediate logics.

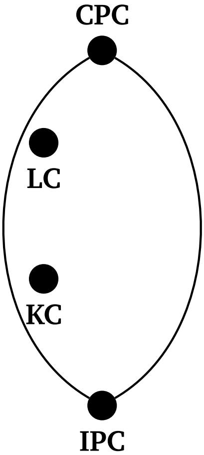
Intermediate logics

$\mathbf{LC} = \mathbf{IPC} + (p \to q) \vee (q \to p)$

Gödel-Dummett calculus

$\mathbf{KC} = \mathbf{IPC} + (\neg p \vee \neg\neg p)$

weak law of excluded middle



Equational theories of Heyting algebras

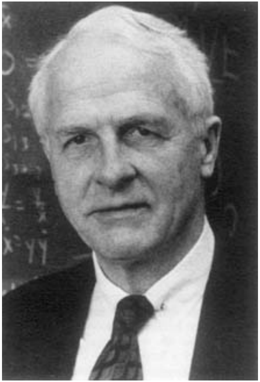
Each formula φ in the language of **IPC** corresponds to an equation $\varphi \approx 1$ in the theory of Heyting algebras.

Conversely, each equation $\varphi \approx \psi$ can be rewritten as $\varphi \leftrightarrow \psi \approx 1$, which corresponds to the formula $\varphi \leftrightarrow \psi$.

This yields a one-to-one correspondence between superintuitionistic logics and equational theories of Heyting algebras.

Varieties of Heyting algebras

By the celebrated Birkhoff theorem, equational theories correspond to varieties; that is, classes of algebras closed under subalgebras, homomorphic images, and products.



Garrett Birkhoff (1911 - 1996)

Varieties of Heyting algebras

Thus, superintuitionistic logics correspond to varieties of Heyting algebras, while intermediate logics to non-trivial varieties of Heyting algebras.

Heyt = the variety of all Heyting algebras.

Bool = the variety of all Boolean algebras.

$\Lambda(\mathbf{IPC})$ = the lattice of superintuitionistic logics.

$\Lambda(\mathbf{Heyt})$ = the lattice of varieties of Heyting algebras.

Theorem. $\Lambda(\mathbf{IPC})$ is dually isomorphic to $\Lambda(\mathbf{Heyt})$.

Consequently, we can investigate superintuitionistic logics by means of their corresponding varieties of Heyting algebras.