ELEMENTS OF UNIVERSAL ALGEBRA AND MODAL COMPANIONS OF INTERMEDIATE LOGICS

1. Varieties and logics

1.1. **Varieties.** For any class K of (universal) algebras, let $\mathbf{H}(\mathsf{K})$, $\mathbf{S}(\mathsf{K})$, and $\mathbf{P}(\mathsf{K})$, denote the classes of homomorphic images of algebras in K, subalgebras of algebras in K and products of algebras in K, respectively.

Definition 1.1. *A class K of algebras is called a* variety *if $\mathbf{S}(\mathsf{K}) \subseteq \mathsf{K}$, $\mathbf{P}(\mathsf{K}) \subseteq \mathsf{K}$, and $\mathbf{H}(\mathsf{K}) \subseteq \mathsf{K}$.*

Theorem 1.2 (Tarski)**.** *A class of algebras K is a variety iff $\mathbf{HSP}(\mathsf{K}) = \mathsf{K}$.*

We say that an equation $\varphi(x_1, \dots, x_n) \approx \psi(y_1, \dots, y_m)$ *holds* or *is valid* on an algebra A and write $A \models \varphi \approx \psi$ if for every $a_1, \dots, a_n, b_1, \dots, b_m \in A$ we have $\varphi(a_1, \dots, a_n) = \psi(b_1, \dots, b_m)$.

Theorem 1.3 (Birkhoff)**.** *A class of algebras V is a variety iff V is equationally definable. That is, there is a set of equations Σ such that for each $\varphi \approx \psi$ in Σ and for each algebra A we have*

$$A \in \mathsf{V} \quad \textit{iff} \quad A \models \varphi \approx \psi.$$

Definition 1.4. *We say that a variety V is generated by a class K if $\mathsf{V} = \mathbf{HSP}(\mathsf{K})$.*

If for each $A \in \mathsf{K}$ we have $A \models \varphi \approx \psi$, we will write $\mathsf{K} \models \varphi \approx \psi$. It is a corollary of (the proof) of Birkhoff's theorem that a class K generates a variety V iff for each equation $\varphi \approx \psi$ we have

$$\mathsf{K} \models \varphi \approx \psi \quad \text{iff} \quad \mathsf{V} \models \varphi \approx \psi$$

Exercise 1.5. Show that K generates V iff $\mathsf{K} \subseteq \mathsf{V}$ and for any equation $\varphi \approx \psi$, we have that $\mathsf{V} \not\models \varphi \approx \psi$ implies $\mathsf{K} \not\models \varphi \approx \psi$.

If a variety V is generated by a class K we write $\mathsf{Var}(\mathsf{K}) = \mathsf{V}$.

Lemma 1.6 (Jónsson's Lemma)**.** *Let V be a congruence distributive variety*[1] *such that $\mathsf{V} = \mathsf{Var}(\mathsf{K})$. Then $\mathsf{V} = \mathbf{PHSP_U}(\mathsf{K})$, where $\mathbf{P_U}$ stands for ultraproducts.*

1.2. **Subdirectly irreducible algebras.** In this section we discuss subdirectly irreducible algebras and the second variety theorem of Birkhoff.

Definition 1.7. *An algebra A is a* subdirect product *of an indexed family $\{A_i\}i \in I$ of algebras if*

(1) *A is a subalgebra of the product $\prod_{i\in I} A_i$*
(2) *$\pi_i(A) = A_i$, where $\pi_i : \prod_{i\in I} A_i \to A_i$ is the i-th projection.*

An embedding $\alpha : A \to \prod_{i\in I} A_i$ is subdirect *if $\alpha(A)$ is a subdirect product of $\{A_i\}i \in I$.*

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Definition 1.8. *An algebra A is* subdirectly irreducible *if for every subdirect embedding*

$$\alpha : A \to \prod_{i\in I} A_i$$

there is an $i \in I$ such that

$$\pi_i \circ \alpha : A \to A_i$$

is an isomorphism.

[1]This means that for each $A \in \mathsf{V}$ $Con(A)$ is distributive.

Theorem 1.9. *An algebra A is subdirectly irreducible iff A is trivial or there exists a least non-diagonal congruence of A, i.e., $Con(A) \setminus \{\Delta\}$ has a least congruence.*

Proof. For the proof consult Theorem 8.4 in the Universal Algebra book. □

For each variety V let V_{SI} denote the class of all subdirectly irreducible algebras of V.

Theorem 1.10 (Birkhoff)**.** *Every variety V is generated by V_{SI}.*

Corollary 1.11.

(1) *A Boolean algebra B is subdirectly irreducible iff there is a least non-unital filter of B.*
(2) *A Heyting algebra H is subdirectly irreducible iff there is a least non-unital filter of H.*

Proof. Exercise. □

Corollary 1.12.

(1) *A Boolean algebra B is subdirectly irreducible iff in its dual Stone space X_B there is a greatest closed set $C \neq X_B$.*
(2) *A distributive lattice D is subdirectly irreducible iff in its dual Priestley space X_D there is a greatest closed set $C \neq X_D$.*
(3) *A Heyting algebra B is subdirectly irreducible iff in its dual Esakia space X_H there is a greatest closed upset $C \neq X_H$.*

Proof. Exercise. □

Corollary 1.13.

(1) *A Boolean algebra B is subdirectly irreducible iff B is isomorphic to the two element Boolean algebra $\mathbf{2}$ iff X_B is a singleton set*
(2) *A distributive lattice D is subdirectly irreducible iff D is isomorphic to the two element Boolean algebra $\mathbf{2}$ iff X_D is a singleton set*
(3) *A Heyting algebra H is subdirectly irreducible iff H has the second greatest element iff X_H is strongly rooted.*

Proof. Exercise. □

Corollary 1.14.

(1) $\mathsf{BA} = \mathsf{Var}(\mathbf{2})$.
(2) $\mathsf{DL} = \mathsf{Var}(\mathbf{2})$.
(3) $\mathsf{HA} = \mathsf{Var}(\{H : H \text{ has a second greatest element}\})$.

Note that in (1) and (2) the signature is different. Boolean algebras have implication (negation) in their signature. So bounded and distributive lattices on the one hand and Boolean and Heyting algebras on the other have a different signature.

1.3. **Lattices of varieties and extensions of logics.** A *superintuionistic logic* L is a set of formulas containing intuitonistic propositional calculus **IPC** and closed under the rules of substitution and Modus Ponens. An *intermediate logic* is a superintuionistic logic L such that $\mathbf{IPC} \subseteq L \subseteq \mathbf{CPC}$.

Theorem 1.15. *Every consistent superintuionistic logic is an intermediate logic.*

Let $\mathrm{Ext}(L)$ denote the set of extensions of an intermediate logic L, and let $\Lambda(\mathsf{V})$ denote the set subvarieties of a variety of Heyting algebras V.

Theorem 1.16.

(1) $(\mathrm{Ext}(L), \subseteq)$ *forms a lattice.*
(2) $(\Lambda(\mathsf{V}), \subseteq)$ *forms a lattice.*

Proof. Exercise. □

We will denote the lattices $(\mathrm{Ext}(L), \subseteq)$ and $(\Lambda(\mathsf{V}), \subseteq)$, by $\mathrm{Ext}(L)$ and $\Lambda(\mathsf{V})$, respectively.

Theorem 1.17.

(1) *Every superintuionistic logic is sound and complete wrt a variety of Heyting algebras.*
(2) $\mathrm{Ext}(\mathsf{IPC})$ *and* $\Lambda(\mathsf{HA})$ *are dually isomorphic* (*that is, the lattice* $(\mathrm{Ext}(\mathsf{IPC}), \subseteq)$ *is isomorphic to* $(\Lambda(\mathsf{HA}), \supseteq)$).

Proof. (Sketch) Each formula φ in the language of IPC corresponds to an equation $\varphi \approx 1$ in the theory of Heyting algebras. Conversely, each equation $\varphi \approx \psi$ can be rewritten as $\varphi \leftrightarrow \psi \approx 1$, which corresponds to the formula $\varphi \leftrightarrow \psi$. This yields a one-to-one correspondence between superintuitionistic logics and equational theories of Heyting algebras. By the Birkhoff theorem (Theorem 1.3), equational theories correspond to varieties. Thus, superintuitionistic logics correspond to varieties of Heyting algebras, while intermediate logics to non-trivial varieties of Heyting algebras. (Exercise: check the remaining details).

□

2. Modal companions of intermediate logics

2.1. Closure algebras and their duality.

Definition 2.1. *An* $\mathsf{S4}$*-algebra* (*alternatively, a* closure algebra *or an* interior algebra) *is a pair* $(B, \Box)$ *where* B *is a Boolean algebra and* $\Box : B \to B$ *a modal operator such that for each* $a, b \in B$ *we have*

(1) $\Box 1 = 1$,
(2) $\Box(a \wedge b) = \Box a \wedge \Box b$,
(3) $\Box a \leq \Box\Box a$,
(4) $\Box a \leq a$.

If we let $\Diamond a = \neg\Box\neg a$, then the S4-axioms can be rewritten as:

(1) $\Diamond 0 = 0$,
(2) $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$,
(3) $\Diamond\Diamond a \leq \Diamond a$,
(4) $a \leq \Diamond a$.

Let CA denote the variety of all $\mathsf{S4}$-algebras (e.g., closure algebras, justifying the notation CA).

Theorem 2.2.

(1) *Every* (*normal*[2]) *extension of* $\mathbf{S4}$ *is sound and complete wrt a variety of* $\mathsf{S4}$*-algebras.*
(2) *The lattice* $\mathsf{NExt}(\mathbf{S4})$ *of normal extensions of* $\mathsf{S4}$ *is dually isomorphic to the lattice* $\Lambda(\mathsf{CA})$ *of subvarieties of* $\mathsf{S4}$*-algebras.*

Proof. Follows from Theorem 1.3. The only extra fact to note is that the Necessitation rule is equivalent to the axiom $\Box 1 = 1$. □

Definition 2.3 ($\mathsf{S4}$-spaces)**.** *A pair* (X, R) *is called an* $\mathsf{S4}$-space *or an* Esakia quasi-order *if* X *is a Stone space and* $R \subseteq X^2$ *a quasi-order* (*reflexive and transitive relation*) *such that:*

(1) $R[x] = \{y \in X : xRy\}$ *is a closed set.*
(2) *For each* $U \in \mathsf{Clop}(X)$ *we have that* $\Diamond_R(U) \in \mathsf{Clop}(U)$,

where

$$\Diamond_R(U) = \{x \in X : R[x] \cap U \neq \emptyset\}.$$

[2] Normal extensions are the extensions closed under the Necessitation rule $\varphi/\Box\varphi$, which corresponds to the axiom $\Box 1 = 1$.

Let

$$\Box_R(U) = \{x \in X : R[x] \subseteq U\}.$$

Then for each $U \subseteq X$ we have $\Box_R(U) = X \setminus \Diamond_R(X \setminus U)$ (Exercise: verify this). Note that every Esakia space is an $\mathsf{S4}$-space (Exercise: verify this).

Lemma 2.4. *For each* $\mathsf{S4}$*-space* (X, R) *the algebra* $(\mathsf{Clop}(X), \Box_R)$ (*alt.* $(\mathsf{Clop}(X), \Diamond_R)$) *is an* $\mathsf{S4}$*-algebra.*

Proof. Exercise. □

Theorem 2.5 (Representation of $\mathsf{S4}$-algebras)**.** *Every* $\mathsf{S4}$*-algebra* $(B, \Box)$ *is isomorphic to the algebra* $(\mathsf{Clop}(X), \Box_R)$ *for some* $\mathsf{S4}$*-space* (X, R).

Proof. (Sketch) Let X be the Stone dual of B (i.e., the space of all ultrafilters of B) we define R on X by

$$xRy \text{ iff } \Box a \in x \;\Rightarrow\; a \in y, \quad \text{for any } a \in B,$$

alternatively we can define

$$xRy \text{ iff } b \in y \;\Rightarrow\; \Diamond b \in x, \quad \text{for any } b \in B.$$

Then $(B, \Box)$ is isomorphic to the algebra $(\mathsf{Clop}(X), \Box_R)$ (Exercise: verify this). □

Exercise 2.6. Formulate a duality theorem for the category of $\mathsf{S4}$-algebras and the category of $\mathsf{S4}$-spaces. How do you define these categories?

2.2. **The Gödel translation and skeletons.** We consider the following translation from the propositional language to the modal language.

Definition 2.7 (The Gödel Translation)**.**

(1) $(\bot)^* = \bot$,
(2) $(p)^* = \Box p$, *where* $p \in \mathsf{Prop}$,
(3) $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$,
(4) $(\varphi \vee \psi)^* = \varphi^* \vee \psi^*$,
(5) $(\varphi \to \psi)^* = \Box(\varphi^* \to \psi^*)$.

Let $\mathcal{B} = (B, \Box)$ be an $\mathsf{S4}$-algebra. Let $\rho(\mathcal{B}) = \{\Box a : a \in B\}$. It is easy to see that $\rho(\mathcal{B}) = \{a \in B : \Box a = a\}$. (Exercise: verify this.) Moreover, $\rho(\mathcal{B})$ forms a Heyting algebra where $0, 1, \wedge_{\rho(\mathcal{B})}, \vee_{\rho(\mathcal{B})}$ are the same operations as on B and for $a, b \in \rho(\mathcal{B})$, the implication $\to_{\rho(\mathcal{B})}$ is defined by

$$a \to_{\rho(\mathcal{B})} b = \Box(a \to_{\mathcal{B}} b).$$

Lemma 2.8. *For each* $\mathsf{S4}$*-algebra* $\mathcal{B} = (B, \Box)$*, the algebra* $\rho(\mathcal{B})$ *is a Heyting algebra.*

Proof. Exercise. □

As mentioned earlier when we write $A \models \varphi$ we mean that $A \models \varphi \approx 1$.

Lemma 2.9 (Key lemma)**.** *Let* $\mathcal{B}$ *be an* $\mathsf{S4}$*-algebra, and* φ *a propositional formula. then*

$$\rho(\mathcal{B}) \models \varphi \quad \textit{iff} \quad \mathcal{B} \models \varphi^*.$$

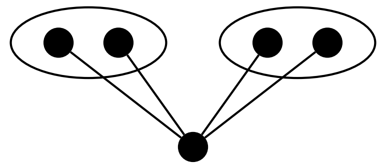
Proof. Exercise. □

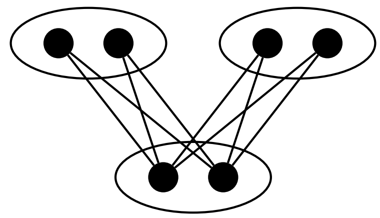
Let $\mathfrak{G} = (X, R)$ be a quasi-ordered (reflexive and transitive) set. A *cluster* is an equivalence class of the relation:

$$x \sim y \text{ if } (xRy \text{ and } yRx).$$

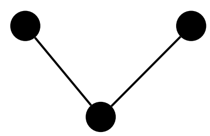
We say that a poset (reflexive, transitive, anti-symmetric) $\mathfrak{F}$ is the *skeleton of* $\mathfrak{G}$ if by identifying all the clusters in $\mathfrak{G}$ we obtain $\mathfrak{F}$.

Example 2.10. It is easy to see that the two quasi-ordered sets drawn below





Have the same skeleton drawn below



Thus we can think of a quasi-order as a poset of clusters.

Theorem 2.11. *Let $\mathcal{B} = (B, \Box)$ be an* $\mathsf{S4}$*-algebra and (X, R) its dual* $\mathsf{S4}$*-space, then the dual Esakia space of $\rho(\mathcal{B})$ is the skeleton of (X, R).*

Proof. Exercise. It might be easier to verify this claim for a finite $\mathcal{B}$. □

Corollary 2.12. *For every Esakia space (X, R) (i.e., when R is reflexive, transitive and anti-symmetric). We have $\rho((\mathsf{Clop}(X), \Box_R)) = \mathsf{ClopUp}(X)$.*

Theorem 2.13 (Gödel-McKinsey-Tarski)**.** *For each formula φ in the propositional language we have*

$$\mathbf{IPC} \vdash \varphi \quad \textit{iff} \quad \mathbf{S4} \vdash \varphi^*.$$

Proof. Suppose $\mathbf{S4} \nvdash \varphi^*$. By Theorem 2.2, there exists an $\mathsf{S4}$-algebra $\mathcal{B}$ such that $\mathcal{B} \not\models \varphi^*$. Then, by the Key Lemma $\rho(\mathcal{B}) \not\models \varphi$. So by Theorem 1.17, $\mathbf{IPC} \nvdash \varphi$.

Conversely, suppose $\mathbf{IPC} \nvdash \varphi$. By Theorem 1.17, there is a Heyting algebra A such that $A \not\models \varphi$. By the Esakia duality A is isomorphic to $\mathsf{ClopUp}(X)$ for some Esakia space X. By Corollary 2.12, there exists an $\mathsf{S4}$-algebra $\mathcal{B}$ such that $\rho(\mathcal{B}) = A$. Therefore, by the Key Lemma again we have that $\mathcal{B} \not\models \varphi^*$ and by Theorem 2.2, $\mathbf{S4} \nvdash \varphi^*$.

□

2.3. **Modal companions.** We will now attempt to lift the correspondence between **IPC** and **S4** to the extension of **IPC** and **S4**.

Definition 2.14. *A modal logic $M \supseteq \mathbf{S4}$ is a* modal companion *of an intermediate logic $L \supseteq \mathbf{IPC}$ if for any propositional formula φ we have*

$$L \vdash \varphi \quad \textit{iff} \quad M \vdash \varphi^*.$$

Example 2.15. *In case you are familiar with the logics below.*

(1) **S4** *is a modal companion of* **IPC***.*

(2) **S5** *is a modal companion of* **CPC**.

(3) **S4.2** *is a modal companion of* **KC**.

(4) **S4.3** *is a modal companion of* **LC**.

Recall that

- $\mathbf{S4.2} = \mathbf{S4} + \Diamond\Box p \to \Box\Diamond p$ is the logic of directed quasi-orders.
- $\mathbf{S4.3} = \mathbf{S4} + \Box(\Box p \to \Box q) \vee \Box(\Box q \to \Box p)$ is the logic of linear quasi-orders.

If M is the modal companion of L we will denote L by $\rho(M)$.

Lemma 2.16. *For $M \supseteq \mathbf{S4}$ we have $\rho(M) = \{\varphi : M \vdash \varphi^*\}$.*

Proof. Exercise. □

Therefore, we can think of ρ as a map $\rho : \mathsf{NExt}(\mathbf{S4}) \to \mathsf{Ext}(\mathbf{IPC})$. For each logic $L \supseteq \mathbf{IPC}$ we denote by V_L the corresponding variety of Heyting algebras. That is, $\mathsf{V}_L = \{A \in \mathsf{HA} : A \models L\}$ and for each $M \supseteq \mathbf{S4}$ we let $\mathsf{V}_M = \{\mathcal{B} \in \mathsf{CA} : \mathcal{B} \models M\}$.

Theorem 2.17. *For each $M \supseteq \mathbf{S4}$ we have that $\mathsf{V}_{\rho(M)} = \mathsf{Var}(\{\rho(\mathcal{B}) : \mathcal{B} \in \mathsf{V}_M\})$.*

Proof. For each $\mathcal{B} \in \mathsf{V}_M$, by the Key Lemma we have that $\rho(\mathcal{B}) \in \mathsf{V}_{\rho(M)}$. So $\{\rho(\mathcal{B}) : \mathcal{B} \in \mathsf{V}_M\} \subseteq \mathsf{V}_{\rho(M)}$. Now assume that $\mathsf{V}_{\rho(M)} \not\models \varphi$. Then by Theorem 1.17, $\rho(M) \nvdash \varphi$. So, as M is a modal companion of $\rho(M)$ we have $M \nvdash \varphi^*$. Thus, by Theorem 2.2, there is an algebra $\mathcal{B} \in \mathsf{V}_M$ such that $\mathcal{B} \not\models \varphi^*$. But then by the Key Lemma, $\rho(\mathcal{B}) \not\models \varphi$. Therefore, we found and algebra in $\{\rho(\mathcal{B}) : \mathcal{B} \in \mathsf{V}_M\}$ which refutes φ. By Exercise 1.5 this means that $\{\rho(\mathcal{B}) : \mathcal{B} \in \mathsf{V}_M\}$ generates $\mathsf{V}_{\rho(M)}$. □

We now define $\tau, \sigma : \mathrm{Ext}(\mathbf{IPC}) \to \mathrm{NExt}(\mathbf{S4})$ by

$$\tau(L) = \mathbf{S4} + \{\varphi^* : \varphi \in L\}$$

and

$$\sigma(L) = \mathbf{Grz} + \{\varphi^* : \varphi \in L\}$$

where

$$\mathbf{Grz} = \mathbf{S4} + (\Box(\Box(p \to \Box p) \to p) \to p)).$$

Theorem 2.18. *For each $L \supseteq \mathbf{IPC}$:*

(1) *$\tau(L)$ is a modal companion of L,*
(2) *$\sigma(L)$ is a modal companion of L.*

Proof. (1) By Lemma 2.16, $\rho(\tau(L)) = \{\varphi : \tau(L) \vdash \varphi^*\}$. It is easy to see that $L \subseteq \rho(\tau(L))$. Conversely, suppose $L \nvdash \varphi$. Then there is $A \in \mathsf{V}_L$ such that $A \not\models \varphi$. Then for each $\mathcal{B}$ such that $\rho(\mathcal{B}) = A$, by the Key Lemma we have $\mathcal{B} \models \tau(L)$ and $\mathcal{B} \not\models \varphi^*$. (How do we know that at least one such $\mathcal{B}$ exists?) By Theorem 2.2, $\tau(L) \nvdash \varphi^*$. Therefore, $\varphi \notin \rho(\tau(L))$.

(2) is similar to (1) but uses some facts about **Grz** so we skip it. □

Corollary 2.19.

(1) $\mathsf{V}_{\tau(L)} = \mathsf{Var}(\{\mathcal{B} \in \mathsf{CA} : \rho(\mathcal{B}) \in \mathsf{V}_L\})$.
(2) $\mathsf{V}_{\sigma(L)} = \mathsf{Var}(\{(\mathsf{Clop}(X), \Box_R) : \mathsf{ClopUp}(X) \in \mathsf{V}_L\})$.

Proof. (1) Exercise. The proof of (2) requires some facts about **Grz**. □

Theorem 2.20. *For each intermediate logic L we have $\rho^{-1}(L) = [\tau(L), \sigma(L)]$. That is, if $\rho(M) = L$, then $\tau(L) \subseteq M \subseteq \sigma(L)$.*

Proof. (1) Let M be such that $\rho(M) = L$. Then by Theorem 2.17, $\mathsf{V}_{\rho(M)} = \mathsf{Var}(\{\rho(\mathcal{B}) : \mathcal{B} \in \mathsf{V}_M\})$. Thus, $\{\rho(\mathcal{B}) : \mathcal{B} \in \mathsf{V}_M\} \subseteq \mathsf{V}_L$. Therefore, $\mathsf{V}_M \subseteq \{\mathcal{B} \in \mathsf{CA} : \rho(\mathcal{B}) \in \mathsf{V}_L\}$. But then $\mathsf{Var}(\mathsf{V}_M) = \mathsf{V}_M \subseteq \mathsf{Var}(\{\mathcal{B} \in \mathsf{CA} : \rho(\mathcal{B}) \in \mathsf{V}_L\}) = \mathsf{V}_{\tau(L)}$. Thus, $\mathsf{V}_M \subseteq \mathsf{V}_{\tau(L)}$ and by Theorem 2.2, we have that $\tau(L) \subseteq M$.

(2) is a bit more tricky and we will skip it for now.

□

Example 2.21. Recall that **CPC** $= Log(\mathfrak{F}_1)$, where

$\mathfrak{F}_1$

Which modal logics are modal companions of **CPC**?

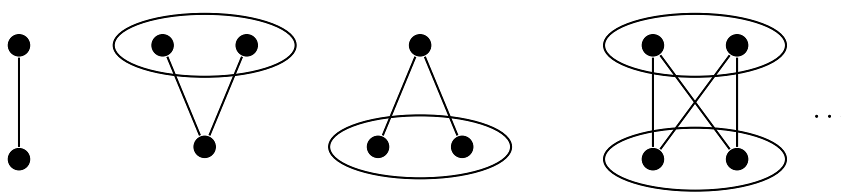
$\mathfrak{G}_1$ $\mathfrak{G}_2$ $\mathfrak{G}_3$

$$Log(\mathfrak{G}_1) \supsetneq Log(\mathfrak{G}_2) \supsetneq Log(\mathfrak{G}_3) \supsetneq \cdots \supsetneq \mathbf{S5}$$

Exercise 2.22. Only for those who are familiar with modal logic. Verify these inclusions. Find formulas showing that the inclusions are strict.

$$Log(\mathfrak{G}_1) \supsetneq Log(\mathfrak{G}_2) \supsetneq Log(\mathfrak{G}_3) \supsetneq \cdots \supsetneq \mathbf{S5}$$

Example 2.23. We see that $Log(\mathfrak{G}_1)$ is the *greatest* modal companion of **CPC** and **S5** is the *least* one. For the intermediate logic of the two-chain we have modal companions given by the following frames.



Exercise 2.24. Do these modal companions form a chain?

Theorem 2.25.

(1) $\tau(\mathbf{IPC}) = \mathbf{S4}$ *and* $\sigma(\mathbf{IPC}) = \mathbf{Grz}$.
(2) $\tau(\mathbf{CPC}) = \mathbf{S5}$ *and* $\sigma(\mathbf{CPC}) = Log(\mathfrak{G}_1) = \mathbf{S5} \cap \mathbf{Grz}$.
(3) $\tau(\mathbf{KC}) = \mathbf{S4.2}$ *and* $\sigma(\mathbf{KC}) = \mathbf{Grz.2}$
(4) $\tau(\mathbf{LC}) = \mathbf{S4.3}$ *and* $\sigma(\mathbf{LC}) = \mathbf{Grz.3}$

We finish by mentioning the classical theorem about modal companions. Let $\mathsf{NExt}(\mathbf{Grz})$ denote the *lattice of normal extensions* of **Grz**.

Theorem 2.26.

(1) $\tau, \sigma : \mathsf{Ext}(\mathbf{IPC}) \to \mathsf{NExt}(\mathbf{S4})$ *are lattice homomorphisms.*
(2) $\tau : \mathsf{Ext}(\mathbf{IPC}) \to \mathsf{NExt}(\mathbf{S4})$ *is an embedding of the lattice of intermediate logics into the lattice of normal extensions of* **S4**.
(3) (*Blok-Esakia*) $\sigma : \mathsf{Ext}(\mathbf{IPC}) \to \mathsf{NExt}(\mathbf{Grz})$ *is an isomorphism from the lattice of intermediate logics onto the lattice of normal extensions of* **Grz**.

We can picture the connection between intermediate logics and normal extensions of **S4** by the following diagram.