ELEMENTS OF UNIVERSAL ALGEBRA

1. Varieties

For any class K of (universal) algebras, let $\mathbf{H}(\mathsf{K})$, $\mathbf{S}(\mathsf{K})$, and $\mathbf{P}(\mathsf{K})$, denote the classes of homomorphic images of algebras in K, subalgebras of algebras in K and products of algebras in K, respectively.

Definition 1.1. *A class* K *of algebras is called a* variety *if* $\mathbf{S}(\mathsf{K}) \subseteq \mathsf{K}$, $\mathbf{P}(\mathsf{K}) \subseteq \mathsf{K}$, *and* $\mathbf{H}(\mathsf{K}) \subseteq \mathsf{K}$.

Theorem 1.2 (Tarski)**.** *A class of algebras* K *is a variety iff* $\mathbf{HSP}(\mathsf{K}) = \mathsf{K}$.

We say that an equation $\varphi(x_1, \dots, x_n) \approx \psi(y_1, \dots, y_m)$ *holds* or *is valid* on an algebra A and write $A \models \varphi \approx \psi$ if for every $a_1, \dots, a_n, b_1, \dots, b_m \in A$ we have $\varphi(a_1, \dots, a_n) = \psi(b_1, \dots, b_m)$.

Theorem 1.3 (Birkhoff)**.** *A class of algebras* V *is a variety iff* V *is equationally definable. That is, there is a set of equations* Σ *such that for each* $\varphi \approx \psi$ *in* Σ *and for each algebra* A *we have*

$$A \in \mathsf{V} \quad \textit{iff} \quad A \models \varphi \approx \psi.$$

Definition 1.4. *We say that a variety* V *is generated by a class* K *if* $\mathsf{V} = \mathbf{HSP}(\mathsf{K})$.

If for each $A \in \mathsf{K}$ we have $A \models \varphi \approx \psi$, we will write $\mathsf{K} \models \varphi \approx \psi$. It is a corollary of (the proof) of Birkhoff's theorem that a class $\mathsf{K} \subseteq \mathsf{V}$ generates a variety V iff for each equation $\varphi \approx \psi$ we have

$$\mathsf{K} \models \varphi \approx \psi \quad \text{implies} \quad \mathsf{V} \models \varphi \approx \psi$$

Obviously, K generates V iff $\mathsf{K} \subseteq \mathsf{V}$ and for any equation $\varphi \approx \psi$, we have that $\mathsf{V} \not\models \varphi \approx \psi$ implies $\mathsf{K} \not\models \varphi \approx \psi$. If a variety V is generated by a class K, we write $\mathsf{Var}(\mathsf{K}) = \mathsf{V}$.

Lemma 1.5 (Jónsson's Lemma)**.** *Let* V *be a congruence distributive variety*[1] *such that* $\mathsf{V} = \mathsf{Var}(\mathsf{K})$. *Then* $\mathsf{V} = \mathbf{PHSP_U}(\mathsf{K})$, *where* $\mathbf{P_U}$ *stands for ultraproducts.*

1.1. Subdirectly irreducible algebras.

In this section we discuss subdirectly irreducible algebras and the second variety theorem of Birkhoff.

Definition 1.6. *An algebra* A *is a* subdirect product *of an indexed family* $\{A_i\}_{i \in I}$ *of algebras if*

1. A *is a subalgebra of the product* $\prod_{i \in I} A_i$
2. $\pi_i(A) = A_i$, *where* $\pi_i : \prod_{i \in I} A_i \to A_i$ *is the i-th projection.*

An embedding $\alpha : A \to \prod_{i \in I} A_i$ *is* subdirect *if* $\alpha(A)$ *is a subdirect product of* $\{A_i\}_{i \in I}$.

Definition 1.7. *An algebra* A *is* subdirectly irreducible *if for every subdirect embedding*

$$\alpha : A \to \prod_{i \in I} A_i$$

there is an $i \in I$ *such that*

$$\pi_i \circ \alpha : A \to A_i$$

is an isomorphism.

Theorem 1.8. *An algebra* A *is subdirectly irreducible iff* A *is trivial or there exists a least non-diagonal congruence of* A, *i.e.,* $Con(A) \setminus \{\Delta\}$ *has a least congruence.*

Proof. For the proof consult Theorem 8.4 in the Universal Algebra book. ☐

[1] This means that for each $A \in \mathsf{V}$ the lattice $Con(A)$ of its congruences is distributive.

For each variety V let V_{SI} denote the class of all subdirectly irreducible algebras of V.

Theorem 1.9 (Birkhoff)**.** *Every variety* V *is generated by* V_{SI}*.*