Mathematical structures in logic
Exercise class 1

Posets and (distributive) lattices

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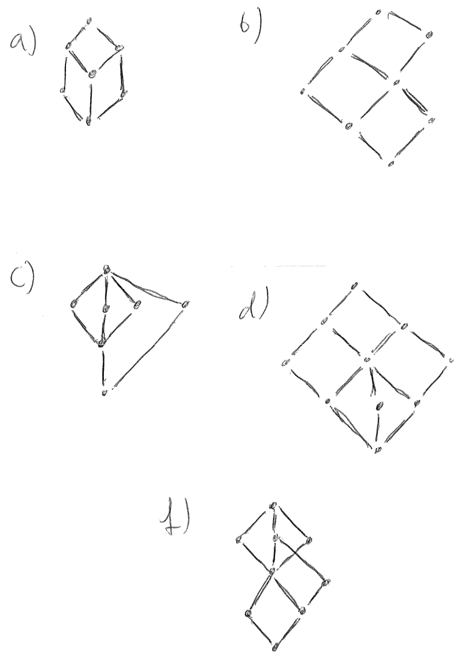
(1) Which of the following lattices are modular and which of them are distributive?



(2) Suppose $(L, \vee, \wedge)$ is a lattice. Recall that we defined a relation $a \leq b$ iff $a \wedge b = a$. Now define a relation $\leq'$ on L via $a \leq' b$ iff $a \vee b = b$. Show that $\leq = \leq'$.

(3) Show that every lattice satisfies:

$$(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$$

(4) Recall that in class we defined on a lattice $(L, \leq)$ $a \wedge b := \inf\{a, b\}$ and $a \vee b := \sup\{a, b\}$. Show that these operations satisfy

- $x \vee (y \vee z) = (x \vee y) \vee z$
- $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

(5) Recall that if $(X, \leq)$ is a partial order then the *covering relation* $\prec$ on X is defined as

$$x \prec y \iff x < y \,\&\, \forall z \in X (x < z \leq y \implies z = y).$$

Given two partial orders $(P, \leq_P)$ and $(Q, \leq_Q)$ we define a relation $\leq$ on $P \times Q$ via $(p, q) \leq (p', q')$ iff $p \leq_P p'$ and $q \leq_Q q'$.

- Prove that $\leq$ is a partial order on $P \times Q$.
- Prove that $(p, q) \prec (p', q')$ iff

$$(p = p' \text{ and } q \prec_Q q') \text{ or } (p \prec_P p' \text{ and } q = q')$$

(6) Funny examples:

- Give an example of a lattice $(L, \leq)$ such that no infinite subset $X \subseteq L$ has a least upper bound.
- Consider the poset $(\mathbb{N}, \leq)$. Is this a lattice? Is it complete?
- Find an example of a lattice $(L, \leq)$ that contains a subset $A \subseteq L$ such that $\inf A$ and $\sup A$ exist but $\sup A \neq \inf A$ and $\sup A \leq \inf A$.
- Find an example of a poset where $\inf \emptyset$ does not exist.

Additional exercises

- Find all posets with 4 elements. (Hint: There are 16)
- Let $f : (L, \leq) \to (L', \leq')$ and $g : (L', \leq') \to (L, \leq)$ be order-preserving maps between the lattices $(L, \leq)$ and $(L', \leq')$ such that $g(f(x)) = x$ for all $x \in L$ and $f(g(y)) = y$ for all $y \in L'$. Show that f and g establish a lattice isomorphism between L and L'.
- Prove that the absorption laws imply $a \wedge a = a$ and $a \vee a = a$.