Mathematical structures in logic
Exercise class 7

March 17, 2016

1. Let R be a quasi-order, i.e. transitive and reflexive, on a set X. Define $\Box_R \colon \wp(X) \to \wp(X)$ by
$$\Box_R(U) = \{x \in X \colon R[x] \subseteq U\},$$
where $R[x] = \{x' \in X \colon xRx'\}$.
   (a) Show that $(\wp(X), \Box_R)$ is an **S4**-algebra[1] .
   (b) Determine the fixed points of the operator $\Box_R$.
   (c) Can you define a finite join preserving function $\Diamond_R \colon \wp(X) \to \wp(X)$ in a similar way? What are the fixed points for this operator.

2. Let $\mathfrak{X} = (X, \tau)$ be a topological space show $(\wp(X), \mathrm{Int}_\tau)$ is an **S4**-algebra where $\mathrm{Int}_\tau \colon \wp(X) \to \wp(X)$ is the interior operation. What are the fixed points for Int_τ? Let $(P, \leq)$ be a poset and let $(P, \tau_\leq)$ be the corresponding Alexandroff space. What is the relationship between the **S4**-algebras $(\wp(P), \Box_\leq)$ and $(\wp(P), \mathrm{Int}_{\tau_\leq})$?

3. (Gödel translation)
   (a) Let $\mathfrak{A} = (A, \Box)$ be an **S4**-algebra. Show that
$$H(\mathfrak{A}) := \{a \in A \colon \Box a = a\}$$
is a sublattice of A having a Heyting implication given by
$$a \to_{H(\mathfrak{A})} b := \Box(a \to_A b).$$
   (b) Show that for every finite Heyting algebra A there exists a finite **S4**-algebra $\mathfrak{B} = (B, \Box)$ such that $H(\mathfrak{B}) \simeq A$ as Heyting algebras.
   (c) Convince yourself that modal logic **S4** is sound and complete with respect to **S4**-algebras.
   (d) Show that $\vdash_{\mathbf{IPC}} \phi$ if and only $\vdash_{\mathbf{S4}} \phi^\Box$, where $\phi^\Box$ denotes the Gödel translation of ϕ.[2]

4. Is the variety of **S4**-algebras locally finite?

[1] Also know as an *interior algebra*.

[2] *Hint:* For one direction use a) to show that if $\phi^\Box$fails on some **S4** algebra then it also fails on a Heyting algebra. For the other direction use that **IPC** has the finite model property show that if ϕ fails on some finite HA, say A, then $\phi^\Box$ fails on some **S4**-algebra of the form $(\wp(X_A), \Box)$.

Additional exercises

1. Let $\mathbb{V}$ be a variety of algebras for some fixed signature. Show, using Birkhoff's Theorem[3], that a class of $\mathbb{V}$-algebras K generates $\mathbb{V}$ iff

$$\mathbb{V} \models s \approx t \iff K \models s \approx t,$$

for all equations $s \approx t$ in the appropriate language.

2. Let $f\colon (P, \leq) \to (P', \leq')$ be an order preserving function between partial orders. Show that the following are equivalent
   (a) The function f is a p-morphism;
   (b) $f^{-1}({\downarrow}U') = {\downarrow}f^{-1}(U')$, for all $U' \subseteq P'$;
   (c) $f^{-1}({\downarrow}p') = {\downarrow}f^{-1}(\{p'\})$, for all $p' \in P'$;
   (d) ${\uparrow}f(p) = f[{\uparrow}p]$ for all $p \in P$.
   (e) the function f is open as a continuous map between the induced Alexandroff spaces $(P, \tau_{\leq})$ and $(P', \tau_{\leq'})$.

3. (For those familiar with the canonical model construction) Recall the definition of the Lindenbaum-Tarski algebra $\mathrm{Form}(P)/{\sim_{\mathbf{IPC}}}$ for **IPC** and the canonical frame $\mathfrak{F}_C$. Show (in detail) that the dual Esakia space of $\mathrm{Form}(P)/{\sim_{\mathbf{IPC}}}$ is $\mathfrak{F}_C$ topologized via the topology generated by the subbasis

$$\{V(\psi), V(\psi)^c \mid \psi \in \mathrm{Form}(P)\},$$

where V is the canonical valuation on $\mathfrak{F}_C$.

4. Find a modal algebra A for which the posets $(\mathrm{Con}(A), \subseteq)$ and $(\mathrm{Fil}(A), \subseteq)$ are not isomorphic.[4] What kind of filters correspond to congruences on modal algebras?

5. Given a finite **S4**-algebra $(A, \Box)$ can you find a quasi-order $R_\Box$ on $\mathrm{At}(A)$ such that $(A, \Box)$ and $(\wp(\mathrm{At}(A)), \Box_{R_\Box})$ are isomorphic as **S4**-algerbras? What happens in the case where A is infinite?

6. Let X be a set. Show that there is a one-to-one correspondence between relations R on X and functions $\xi_R\colon X \to \wp(X)$. Let (X, R) and (X', R') be Kripke frames and let $f\colon X \to X'$be a relation preserving map show that f is a p-morphism iff the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & X' \\ \big\downarrow{\scriptstyle \xi_R} & & \big\downarrow{\scriptstyle \xi_{R'}} \\ \wp(X) & \xrightarrow{\ f[-]\ } & \wp(X') \end{array}$$

commutes

7. (For whose who know some category theory) Show that the category of Kripke frames and p-morphisms is isomorphic to the category of coalgebras for the (covariant) power-set functor $\wp\colon \mathbf{Set} \to \mathbf{Set}$.

[3]Note that Birkhoff's Theorem say that if $\mathbb{V}$ is a variety then $\mathbb{V}$ are precisely then set of algebras validating the equational theory of $\mathbb{V}$.

[4]*Hint:* It suffices to find a finite algebra A such that $\mathrm{Con}(A)$ and $\mathrm{Fil}(A)$ are not of the same cardinality.