Mathematical structures in logic
Exercise class 6

March 10, 2016

(1) Redo exercise (1) from the tutorial sheet 2 (see blackboard) using duality and enjoy how much simpler it is!

(2) (Subdirectly irreducible algebras)

(a) Show that if A has a least non-trivial congruence θ then A is subdirectly irreducible;[1]

(b) Let A be a Heyting algebra. Convince yourself that the we have an isomorphism of poset

$$(\mathrm{Con}(A), \subseteq) \simeq (\mathrm{Fil}(A), \subseteq).$$

Conclude that a Heyting algebra is s.i. iff there exists a least non-unital filter on A;

(c) Show that Heyting algebra is subdirectly irreducible iff it has a second greatest element.

(3) Let $\mathfrak{P} = (X, \tau, \leq)$ be the dual Priestley space of a distributive lattice D. Show that for every clopen upset U of $\mathfrak{P}$ there exists $a \in D$ such that $U = \varphi(a)$, where $\varphi(a)$ is the set of prime filters on D containing a.[2]

(4) (a) Recall that we have an order-reversing one-to-one correspondence between filters on a Boolean algebra B and the closed subsets of X_B.

(b) Deduce that for every Boolean algebra its corresponding congruence lattice is distributive. (Varieties with this property are called *congruence distributive.*)

(5) (A bit harder:) Let A be a Heyting algebra and X_A its dual Esakia space.

(a) Show that all closed upsets of X_A are of the form $\bigcap_{a \in I} \varphi(a)$ for some $I \subseteq A$.

(b) Show that there is an order-reversing one-to-one correspondence between filters on A and closed upsets of X_A.

(c) Deduce that the variety of Heyting algebras is congruence distributive.

[1] *Hint:* Given a subdirect embedding $e\colon A \hookrightarrow \prod_{i \in I} A_i$, show that there is $a, b \in A$ such that $a \neq b$ and $(a, b) \in \theta$. Show that there must exists $i_0 \in I$ such that $(\pi_{i_0} \circ e)(a) \neq (\pi_{i_0} \circ e)(b)$ and conclude that $\pi_{i_0} \circ e\colon A \to A_i$ is injective by showing that $\ker(\pi_{i_0} \circ e)$ is a congruence on A not containing θ.

[2] *Hint:* You will most likely have to use compactness twice, first for a cover of U^c and then for a cover of U.

Additional exercises

(1) (For those familiar with the canonical model construction) Recall the definition of the Lindenbaum-Tarski algebra $\mathrm{Form}(P)/{\sim_{\mathbf{IPC}}}$ for **IPC** and the canonical frame $\mathfrak{F}_C$. Show (in detail) that the dual Esakia space of $\mathrm{Form}(P)/{\sim_{\mathbf{IPC}}}$ is $\mathfrak{F}_C$ topologized via the topology generated by $\{V(\psi), V(\psi)^c \mid \psi \in \mathrm{Form}(P)\}$, where V is the canonical valuation on $\mathfrak{F}_C$.

(2) Give an example of an ordered algebra A for which the posets $(\mathrm{Con}(A), \subseteq)$ and $(\mathrm{Fil}(A), \subseteq)$ are not isomorphic.[3]

[3] *Hint:* It suffices to find a finite algebra A such that $\mathrm{Con}(A)$ and $\mathrm{Fil}(A)$ are not of the same cardinality.