Mathematical structures in logic
Exercise class 5

Applications of duality and a bit of universal algebra

3 March 2016

(1) Let $(P, \leq)$ be a poset. Then $\mathrm{Up}(P)$, the upsets on P, form a topology on P.

  (a) Show that this is an Alexandrov topology, i.e. show that the open sets are closed under arbitrary intersections.
  (b) Show that for every $P' \subseteq P$, $\mathrm{Cl}(P') = {\downarrow}P'$.
  (c) Describe the interior of a set $P' \subseteq P$.

  Here $\mathrm{Cl}(P')$ denotes the closure of P' and as usual ${\downarrow}P' := \{q \in P \mid \exists p \in P', q \leq p\}$.

(2) Show that every equational class is a variety, i.e. show that validity of equations is preserved by the operators **H**, **S** and **P**.

(3) (a) Find an example of a Heyting algebra A and a subalgebra A' of A such that A' is not a homomorphic image of A.
  (b) Find an example of a Heyting algebra that has a homomorphic image B such that B is not isomorphic to a subalgebra of A.

(4) Let $(X, \tau, \leq)$ be a Priestley space show that

  (a) the set ${\uparrow}x$ is closed for each $x \in X$;
  (b) the sets ${\uparrow}F$ and ${\downarrow}F$ are closed for each closed subset F of (X, τ).

Additional exercises

(1) Let $\mathfrak{P} = (X, \tau, \leq)$ be a Priestley space. Show that for every clopen upset U of $\mathfrak{P}$ there exists $a \in X$ such that $U = \phi(a)$, where $\phi(a)$ is the set of prime filters on $(X, \leq)$ containing a (*Hint:* You will most likely have to use compactness twice, first for a cover of U^{c} and then for a cover of U.)

(2) (More on Alexandroff spaces and posets)

  (a) Let $\mathfrak{X} = (X, \tau)$ be an Alexandrov space. Can you find a partial order $\leq$ on X such that $\tau = \mathrm{Up}((X, \leq))$? Supply a proof are give a counter-example.

(b) Let $f\colon P \to Q$ be a function between posets $(P, \leq_P)$ and $(Q, \leq_Q)$. Show that f is order-preserving iff f is continuous with respect to the topologies $\mathrm{Up}(P)$ and $\mathrm{Up}(Q)$.

(c) Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be posets. Characterise the order-preserving maps $f\colon P \to Q$ with the property that f is an open map as a function between the induced Alexandrov spaces.

(3) Which of the class operations **H**, **S** and **P** preserves quasi-equations, i.e., clauses of the form

$$s_1 \approx t_1, \ldots, s_n \approx t_n \implies s \approx t.$$