Mathematical structures in logic
Exercise class 4

Dualities for Boolean algebras

February 25, 2016

In the following let B be a Boolean algebra and X_B its dual Stone space.

(1) Let $(\mathsf{Fil}(B), \subseteq)$ be the poset of filters on B and let $(\mathsf{Cl}(X_B), \subseteq)$ be the poset of closed subsets of X_B. Show that there is an order-reversing bijection between $(\mathsf{Fil}(B), \subseteq)$ and $(\mathsf{Cl}(X_B), \subseteq)$. Can you say something similar about the poset of ideals on B?

(2) Show that there is a one-to-one correspondence between the atoms of B and the isolated points of X_B.

(Recall that in a topological space $\mathfrak{X} = (X, \tau)$ a point $x \in X$ is *isolated* if $\{x\}$ is open).

(3) Let $\mathcal{I}_{X_B} := \{x \in X_B \mid x \text{ is an isolated point}\}$. Show that B is atomic iff $\mathcal{I}_{X_B}$ is dense in X_B.

(Recall that in a topological space $\mathfrak{X} = (X, \tau)$ a subset $Y \subseteq X$ is *dense* in X iff the topological closure of Y is X. Equivalently, Y is dense if for every $x \in X$ and every open U with $x \in U$ it follows that $U \cap Y \neq \emptyset$.)

Additional exercises

(4) (Atoms and join-irreducibles)

(a) Show that in a bounded lattice all atoms are necessarily join-irreducible.

(b) Does atoms and join-irreducible elements coincide for Boolean algebras?

(5) (*For people how know (or want to learn) a bit more topology*)

(a) Use (2) to show that there exists *atomless* Boolean algebras, i.e., Boolean algebras without any atoms.

(b) Determine (up to isomorphism) the number of countable atomless Boolean algebras.

(6) *For people who know (want to learn a bit more) category theory.* Let **Set** be the category of sets and functions and let **CABA** be the category of complete atomic Boolean algebras and complete Boolean algebra homomorphisms. Prove that the correspondence between **Set** and **CABA** from HW 4, exercise 1, is part of a dual equivalence $\mathbf{Set}^{\mathrm{op}} \cong \mathbf{CABA}$, i.e.

(a) Show that $\mathcal{P} : \mathbf{Set} \to \mathbf{CABA}$ and $\mathcal{A} : \mathbf{CABA} \to \mathbf{Set}$ are contravariant functors. (What are the action on morphisms?)

(b) Show that the this isomorphisms from HW 4, exercise 1 are natural, i.e. show that for complete atomic Boolean algebra B the isomorphims $\eta_B : B \to \mathcal{P}(\mathcal{A}(B))$ are components of a natural transformation $\eta : \mathrm{Id}_{\mathbf{CABA}} \Rightarrow \mathcal{P} \circ \mathcal{A}$ and similarly, for every set X, the bijections $\mu_X : X \to \mathcal{A}(\mathcal{P}(X))$ are components of a natural transformation $\mu : \mathrm{Id}_{\mathbf{Set}} \Rightarrow \mathcal{A} \circ \mathcal{P}$.

So you need to show that for every complete Boolean homomorphism $f \in \mathrm{Hom}_{\mathbf{CABA}}(B, C)$ and every map $g \in \mathrm{Hom}_{\mathbf{Set}}(X, Y)$ the following diagrams commute.

$$\begin{array}{ccc} B & \xrightarrow{\quad f \quad} & C \\ {\scriptstyle \eta_B}\downarrow & & \downarrow{\scriptstyle \eta_C} \\ \mathcal{P}(\mathcal{A}(B)) & \xrightarrow[\mathcal{P}(\mathcal{A}(f))]{} & \mathcal{P}(\mathcal{A}(C)) \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\quad g \quad} & Y \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle \mu_Y} \\ \mathcal{A}(\mathcal{P}(X)) & \xrightarrow[\mathcal{P}(\mathcal{A}(g))]{} & \mathcal{A}(\mathcal{P}(Y)) \end{array}$$