Mathematical structures in logic
Exercise class 3

Filters, congruences, atoms

February 18, 2016

(1) Suppose $\sim$ is a congruence of a bounded distributive lattice L. Show that $\{a \in L \mid a \sim 1\} \subseteq L$ is a filter on L.

(2) (Prime filters and maximal filters.) Let B be a Boolean algebra. Show that:

  (a) If F is a filter on B, then $I := \{\neg a \in B \mid a \in F\}$ is an ideal on B.

  (b) If F is a filter, then $B \setminus F$ may not be an ideal on B.

  (c) If F is a prime filter, then $B \setminus F$ is a prime ideal.

  (d) Which of these statements are true for Heyting algebras?

(3) (Atoms and co-atoms)

  (a) Describe atoms and co-atoms on a Boolean algebra of the form $\wp(X)$.

  (b) Show that in every Boolean algebra, if a is an atom, then $\neg a$ is a co-atom.

  (c) Find a Heyting algebra A with an atom a such that $\neg a$ is not a co-atom

(4) Let B be a Boolean algebra. Show that maximal filters on B are in one-to-one correspondence with surjective homomorphisms $f : B \to \mathbf{2}$, where $\mathbf{2}$ denotes the two element Boolean algebra.

Additional exercises

(5) Show that there is a one-to-one correspondence between filters and ideals on a Boolean algebra. Can you find a similar correspondence in the case of Heyting algebras?

(6) Describe all Heyting algebras where $\{1\}$ is a prime filter.

(7) Let $\mathfrak{X} = (X, \tau)$ be a topological space. Recall that a subset $X' \subseteq X$ is *dense* iff $U \cap X' \neq \emptyset$, for every non-empty open set $U \in \tau$. Show that the dual space of every countable Boolean algebra has a countable dense subset.